Frames and stable bases for subspaces of $L_2(\mathbb{R}^d)$: the duality principle of Weyl-Heisenberg sets*

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Abstract

We announce selected results concerning Weyl-Heisenberg (=WH) frames and WH stable bases in d dimensions, $d \geq 1$. Of particular interest is the notion of the *adjoint* of a WH set, and the *duality principle* which characterizes a WH (tight) frame in term of the stability (orthonormality) of its adjoint. The actions of taking adjoint and taking dual commute, hence the dual WH frame can be computed via the dual basis of the adjoint. Finally, for a large collection of WH frames a method for an exact computation of the frame bounds is presented.

1 Frames, stable bases, and Weyl-Heisenberg sets defined

Let X be a countable subset of $L_2(\mathbb{R}^d)$. Let $T := T_X$ be the operator

$$T : \ell_2(X) \to L_2(\mathbb{R}^d) : c \mapsto \sum_{x \in X} c(x)x.$$

If T is well-defined and bounded, X is **a Bessel set**. If, in addition, $H := \operatorname{ran} T$ is closed, X is **a frame**. If T is also injective, X is **a Riesz (or stable) basis**. X is **fundamental** whenever $H = L_2(\mathbb{R}^d)$. Whenever X is a frame, there exists a well-defined linear bounded operator from H onto the orthogonal complement of $\ker T$, denoted here by T^{-1}. The numbers $\|T\|^2$, $\|T^{-1}\|^2$ are known as **the Riesz (frame) bounds**. The formal adjoint of T, T^*, has the form

$$T^* : H \to \ell_2(X) : f \mapsto (\langle f, x \rangle)_{x \in X}.$$

Frames can be used to decompose and reconstruct functions in H. For this purpose, one needs to complement the given frame by another one, the so-called **dual frame** which is defined as $\mathrm{R}X$, with $\mathrm{R} := \mathrm{R}_X := (T_X T_X^*)^{-1}$. In this regard, computing $c_f := T_{\mathrm{R}X}^* f$ is the **decomposition** of f, and computing $T_X c_f$ is the **reconstruction** of f. A frame whose two frame bounds coincide is a **tight frame**. General references on these topics are [1-3] which contain specific discussions of Weyl-Heisenberg sets (in one variable), and which contain also an extensive list of references. The results here are based on the methods developed in [4] for the analysis of shift-invariant frames and Riesz bases.

A **lattice** K is the image $A_K \mathbb{Z}^d$ of an invertible linear map $A_K : \mathbb{R}^d \to \mathbb{R}^d$. The **volume** $|K| := |\det A_K|$ of the lattice K depends only on K. The **dual lattice** $\widetilde{K}$ of K is the lattice

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defined by $\widetilde{K} := \{\ell \in \mathbb{R}^d : \ \ell \cdot k \in 2\pi\mathbb{Z}, \ \text{all}\, k \in K\}$. For example, the dual of $h\mathbb{Z}^d$ is $\frac{2\pi}{h}\mathbb{Z}^d$. Always, $|K|\,|\widetilde{K}| = (2\pi)^d$.

Let (K, L) be a pair of lattices, and let Φ be a finite subset of $L_2(\mathbb{R}^d)$. The **Weyl-Heisenberg set** (WHS) $X := (K, L, \Phi)$ is the collection $(K, L, \Phi) := \{e_\ell \varphi(\cdot - k) : \ \varphi \in \Phi, \ k \in K, \ \ell \in L\}$, with $e_\ell : x \mapsto e^{i\ell \cdot x}$. In this announcement we always assume that Φ is a singleton and refer then to X as a PWH set. Some of the stated results extend to general WH sets.

2 The adjoint of a PWH set and the duality principle

Given a PWH set $X = (K, L, \varphi)$, we define its **adjoint** as the PWH set $X^* := (\widetilde{L}, \widetilde{K}, \varphi)$. Defining the **density** of X as $\rho X := \frac{(2\pi)^d}{|K||L|}$, we see that $\rho X \rho X^* = 1$. We call $X = (K, L, \varphi)$ **self-adjoint** whenever $K = \widetilde{L}$ (which implies $\rho X = 1$), since then $X = X^*$.

EXAMPLE 1. A univariate PWH set is of the form $X = (K, L, \varphi) = (p\mathbb{Z}, 2\pi q\mathbb{Z}, \varphi)$, $p, q > 0$. Here, $\rho X = (pq)^{-1}$. The adjoint set is $X^* = (\mathbb{Z}/q, 2\pi\mathbb{Z}/p, \varphi)$, and its density is, indeed, pq. A conclusion: a univariate PWH X is self-adjoint iff its density is 1.

THEOREM 2. *(The duality principle of WH sets).*
Let $\varphi \in L_2(\mathbb{R}^d)$ with $\|\varphi\| = 1$, and let X be a PWH set generated by φ. Then:
(a) T_X is injective if and only if T_{X^*} is surjective, i.e., if and only if X^* is fundamental.
(b) X is a Bessel set if and only if X^* is one. In that case, $\|T_{X^*}\|^2 = \rho X^* \|T_X\|^2$.
(c) X is a frame if and only if X^* is a frame. In that case, $\|T_{X^*}{}^{-1}\|^2 = \rho X^* \|T_X{}^{-1}\|^2$.
(d) X is a tight frame if and only if X^* is a tight frame.
(e) X is a Riesz basis if and only if X^* is a fundamental frame.
(f) X is an orthonormal basis if and only if X^* is a fundamental tight frame.

Several extensions/improvements of known univariate results follow directly from Theorem 2. We mention here one of these: from (e), a self-adjoint X is a Riesz basis iff it is a fundamental frame. A less obvious consequence of the duality principle is the following:

THEOREM 3. *Let X be a PWH frame, X^* its adjoint. Then the dual of its adjoint is the adjoint of its dual; more precisely, $\mathrm{R}_{X^*}X^* = (\rho X)^{\frac{1}{2}}(\mathrm{R}_X X)^*$.*

The virtue of the last theorem is the following. Suppose that we seek the dual of a fundamental PWH set $X = (K, L, \varphi)$. It is (essentially) known (and easy to see) that the dual has the form (K, L, ψ), still, finding/approximating ψ is in general a non-trivial task. On the other hand, by the duality principle, the adjoint X^* is a *Riesz basis* whose dual is characterized by standard bi-orthogonality relations hence is easier to find.

Theorem 2 (f) can be written explicitly as follows:

COROLLARY 4. *Let X be as in Theorem 2. Then X is a tight frame for $L_2(\mathbb{R}^d)$ if and only if the set $\{e_\ell \varphi(\cdot - k) : (k, \ell) \in (\widetilde{L}, \widetilde{K})\}$ is orthonormal.*

3 Zak transform analysis of compressible WH sets

The *Zak transform* is known to be an important tool in the analysis of univariate self-adjoint (and related) WH sets. We define the multivariate analog of the transform as follows. First, let K be some lattice in $\mathbb{R}^d$. Let $K \ni k \mapsto \widetilde{k} \in \widetilde{K}$ be *any* linear isomorphism. Then

$$U_Z^K f(w, t) := \sum\nolimits_{k \in K} f(w - k) e_{\widetilde{k}}(t), \quad f \in L_2(\mathbb{R}^d).$$

A lattice pair (K, L) **compressible** if the group $\widetilde{K} \cap L$ has a finite index in L, i.e., if $\widetilde{K} \cap L$ is d-dimensional. (The univariate $(p\mathbb{Z}, 2\pi q\mathbb{Z})$ is compressible iff its density $(pq)^{-1}$ is rational.) We define Γ (resp. Δ) to be any set of representers for the quotient group $L/(L \cap \widetilde{K})$ (resp. $\widetilde{K}/(L \cap \widetilde{K})$). The order of Γ is the **compression factor** of X, and the order of Δ is the

decomposition factor of X. We then introduce a matrix $\widetilde{\mathcal{Z}}_X(w,t)$, $(w,t) \in \mathbb{R}^d \times \mathbb{R}^d$, indexed by $\Delta \times \Delta$, and referred to as the **dual Zak matrix** of X, with its (δ,δ')-entry being defined as

$$\sum_{\gamma\in\Gamma} (U_Z^{L\cap\widetilde{K}}\hat{\varphi})(w+\gamma-\delta,t)\overline{U_Z^{L\cap\widetilde{K}}\hat{\varphi}(w+\gamma-\delta',t)}.$$

EXAMPLE 5. Set $X := (K,L,\varphi) := (2\pi m\mathbb{Z}/n, \mathbb{Z}, \varphi)$, $\|\varphi\| = 1$, m,n positive integers. Here, $\widetilde{K} \cap L = n\mathbb{Z}$, and $\widetilde{K} = n\mathbb{Z}/m$, and hence we may choose $\Delta = \frac{n}{m}, 2\frac{n}{m}, ..., m\frac{n}{m}$. Also, $L = \mathbb{Z}$, hence we may choose $\Gamma = 1, 2, ..., n$. The compression factor here is n and the decomposition factor is m. The dual Zak matrix is then an $m \times m$ matrix with (j,j')-entry

$$\sum_{r=1}^{n} (\sum_{s\in\mathbb{Z}} \hat{\varphi}(w+ns+r+j\frac{n}{m})e_{2\pi s/n}(t))\overline{(\sum_{s\in\mathbb{Z}} \hat{\varphi}(w+ns+r+j'\frac{n}{m})e_{2\pi s/n}(t))}.$$

THEOREM 6. *Let $X = (K,L,\varphi)$ be a compressible WH set, and let $\widetilde{\mathcal{Z}}$ be its associated dual Zak matrix (defined a.e.) Let $\widetilde{\Lambda}(w,t)$ (resp. $\widetilde{\lambda}(w,t)$) be the largest (resp. smallest) eigenvalue of $\widetilde{\mathcal{Z}}(w,t)$. Then, for some constant c that depends only on $|K|$ and $|L|$,*

(a) X is a Bessel set if and only if $\widetilde{\Lambda}$ is essentially bounded. Also, $\|T_X\|^2 = c\|\widetilde{\Lambda}\|_{L_\infty}$.

(b) A Bessel set X is a fundamental frame if and only if $\widetilde{\lambda}$ is essentially bounded away from zero. Also, $\|T_X{}^{-1}\|^2 = c\|1/\widetilde{\lambda}\|_{L_\infty}$.

(c) X is a fundamental tight frame if and only if $\widetilde{\mathcal{Z}}$ is the identity a.e.

THEOREM 7. *Let $X = (K,L,\varphi)$ be a fundamental compressible PWH frame. Given $(w,t) \in \mathbb{R}^d \times \mathbb{R}^d$, let $V = (v_\delta)_{\delta\in\Delta}$ be the vectors (in $\mathbb{C}^\Gamma$) $v_\delta(\gamma) = U_Z^{L\cap\widetilde{K}}\hat{\varphi}(w+\gamma-\delta,t)$. Let $\widetilde{V}$ be the dual basis of V, i.e., $\operatorname{span}\widetilde{V} = \operatorname{span} V$, and the Gram matrix $\langle V,\widetilde{V}\rangle$ is the identity. Let ψ be the generator of the dual frame of X. Then $U_Z^{L\cap\widetilde{K}}\hat{\psi}(w,t) = \widetilde{v}_0(0)$.*

Theorem 7 implicitly claims that the vectors in V are linearly independent, making, thus, that condition necessary for X to be a fundamental frame. Also, one notices that V is linearly independent only if $\#\Delta \le \#\Gamma$. Thus, we obtain an analogue of a well-known univariate result (cf. [1]): if X is a compressible fundamental frame, then

$$\rho X = \frac{\#\Gamma}{\#\Delta} \ge 1.$$

4 Remarks

The results of [4] are also useful for the analysis of *wavelet frames*. Details will appear elsewhere. It was brought to our attention that a univariate version of Corollary 4 is stated in [5], and that univariate results similar to Theorem 6 can be found in [6].

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