Frames and Stable Bases
for Shift-Invariant Subspaces of $L_2(\mathbb{R}^d)$

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ABSTRACT

Let $X$ be a countable fundamental set in a Hilbert space $H$, and let $T$ be the operator

$$T : \ell_2(X) \to H : c \mapsto \sum_{x \in X} c(x)x.$$  

Whenever $T$ is well-defined and bounded, $X$ is said to be a Bessel sequence. If, in addition, ran $T$ is closed, then $X$ is a frame. Finally, a frame whose corresponding $T$ is injective is a stable basis (also known as a Riesz basis).

This paper considers the above three properties for subspaces $H$ of $L_2(\mathbb{R}^d)$, and for sets $X$ of the form

$$X = \{ \phi(\cdot - \alpha) : \phi \in \Phi, \alpha \in \mathbb{Z}^d \},$$

with $\Phi$ either a singleton, a finite set, or, more generally, a countable set. The analysis is performed on the Fourier domain, where the two operators $TT^*$ and $T^*T$ are decomposed into a collection of simpler “fiber” operators. The main theme of the entire analysis is the characterization of each of the above three properties in terms of the analogous property of these simpler operators.

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1. Introduction

1.1. General

We study in this paper certain types of "bases" for shift-invariant subspaces of $L_2(\mathbb{R}^d)$. Our primary objective is to connect among three important families of "basis" sets: shift-invariant sets, Weyl-Heisenberg sets, and affine (wavelet) sets. The present paper is the first in a series of three, and is concerned with the basic theory of shift-invariant bases for shift-invariant spaces. The two other papers, [RS1] and [RS2], will focus on the applications of the theory developed here to Weyl-Heisenberg and affine sets.

Given $X \subset L_2(\mathbb{R}^d)$, we say that $X$ is a shift-invariant (SI, for short) set if it is invariant under all possible shifts, i.e., invariant under all integer translations. A shift-invariant subspace $S$ of $L_2(\mathbb{R}^d)$ is a closed subspace which is also a shift-invariant set. Such spaces play an important role in the areas of Multivariate Splines, Wavelets, Radial Function Approximation and Sampling Theory.

The following terminology is commonly used in the context of shift-invariant spaces. First, for a given $\Phi \subset L_2(\mathbb{R}^d)$, the space generated by $\Phi$, denoted by

$$ S(\Phi), $$

is the smallest (closed) shift-invariant space that contains $\Phi$. The set of shifts of $\Phi$

$$ (1.1.1) \quad E_\phi := \{ E^\alpha \phi : \phi \in \Phi, \alpha \in \mathbb{Z}^d \}, $$

with

$$ (1.1.2) \quad E^\alpha f \mapsto f(\cdot - \alpha), $$

is then clearly fundamental in $S(\Phi)$, and is a natural candidate for the previously discussed $X$. The space $S$ is a principal shift-invariant (PSI) space in case $S = S(\Phi)$ for a singleton $\Phi$, and, more generally, is a finitely generated shift-invariant (FSI) space if $\Phi$ above is finite. Many articles are devoted, wholly or in part, to the study of Riesz (=unconditional=stable) bases for PSI and FSI spaces (cf. e.g. [JM], [BDR1]). In particular, a complete characterization of such bases is given in [BDR1], which, further, introduces and analyses the more general notion of quasi-stable bases. These results form the starting point of the present paper.

We provide here a complete characterization of frames and tight frames in FSI spaces, and draw interesting connections between these notions and the notions of quasi-stability and quasi-orthogonality of [BDR1]. We further give a comprehensive analysis of infinitely generated SI spaces, and employ in that course two complementary approaches termed here as "Gramian Analysis" and "dual Gramian Analysis".
1.2. Notations

The Fourier transform of a tempered distribution $f$ is denoted here by $\hat{f}$, and is defined, for $f \in L_1(\mathbb{R}^d)$, by

$$\hat{f}(w) := \int_{\mathbb{R}^d} f(t) e^{-iw \cdot t} \, dt,$$

where

$$e_w : t \mapsto e^{iw \cdot t}.$$

The inverse Fourier transform of $f$ is denoted by $f^\vee$.

We frequently discuss in this paper functions that are defined on $\mathbb{T}^d$, the $d$-dimensional torus. Those functions can be viewed as $2\pi$-periodic functions, via the standard transformation $\mathbb{R}^d \ni w \mapsto e^{iw} := (e^{iw_1}, \ldots, e^{iw_d}) \in \mathbb{T}^d$. Though we may refer to such functions as being defined on $\mathbb{T}^d$, we always treat their argument as real. Thus, “multiplying a function defined on $\mathbb{T}^d$ by a function defined on $\mathbb{R}^{dn}$ simply means “multiplying a $2\pi$-periodic function by $\ldots$”.” Following this slight abuse of terminology, we write “$\Omega \subset \mathbb{T}^{dn}$” and mean “$\Omega \subset [-\pi d\pi]^d$”. The $2\pi$-periodic extension, $\Omega + 2\pi \mathbb{Z}^d$, of $\Omega$ is denoted by $\Omega^0$.

The inner product (norm) of any Hilbert space $H$ discussed in this paper is denoted by $\langle \cdot, \cdot \rangle_H$ ($\| \cdot \|_H$, respectively). The default inner product and norm are these of $L_2(\mathbb{R}^d)$. We may also suppress the subscripts in $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$ if they are clear from the context.

Given a set $X$, the notation $\ell_2(X)$ stands (as usual) for the space of square-summable sequences on $X$, with the standard inner product. Also, if $Y \subset X$, we embed $\ell_2(Y)$ canonically in $\ell_2(X)$ (i.e., by defining each $c \in \ell_2(Y)$ to be zero on $X \setminus Y$). The space $\ell_0(X)$ is the space of all finitely supported sequences in $\ell_2(X)$, and is considered as a subspace of the latter (i.e., equipped with the same norm).

Vectors in $\mathbb{R}^d$ are considered as either row vectors or column vectors, and the exact meaning should be clear from the context.

For a countable $\Phi \subset L_2(\mathbb{R}^d)$, we define the Hilbert space of $L_2(\mathbb{T}^d)$-valued $\Phi$-vectors as follows

$$L_2^\Phi := \{ \langle \cdot, \cdot \rangle_{L_2^\Phi} : \tau_{\phi} \in L_2(\mathbb{T}^d) ; \sum_{\phi \in \Phi} \| \tau_{\phi} \|_{L_2(\mathbb{T}^d)}^2 < \infty \}.$$

The inner product here is

$$\langle \tau, \tau' \rangle_{L_2^\Phi} := \sum_{\phi \in \Phi} \langle \tau_{\phi}, \tau'_{\phi} \rangle_{L_2(\mathbb{T}^d)}.$$

If $\tau \in L_2^\Phi$, then $\tau(w) \in \ell_2(\Phi)$, for almost all $w \in \mathbb{T}^d$. 

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The space \( L^2_\Phi \) enters the discussion in this paper as the image under the Fourier transform of the sequence space \( \ell_2(\mathbb{Z}^d \times \Phi) \). Indeed, given \( c \in \ell_2(\mathbb{Z}^d \times \Phi) \), we denote by \( c_{\phi}, \phi \in \Phi \), the restriction of \( c \) to \( \mathbb{Z}^d \times \langle \phi \rangle \). The Fourier series \( \hat{c}_\phi \) of \( c_{\phi} \) is defined as

\[
\hat{c}_\phi := \sum_{\alpha \in \mathbb{Z}^d} c_{\phi}(\alpha)e^{-\alpha}.
\]

Accordingly, the Fourier transform of \( c \in \ell_2(\mathbb{Z}^d \times \Phi) \) is defined as the element

\[
\hat{c} := (\hat{c}_\phi)_{\phi \in \Phi} \in L^2_\Phi.
\]

Note that this Fourier transformation is an isometry between \( \ell_2(\mathbb{Z}^d \times \Phi) \) and \( L^2_\Phi \).

The following **bracket product** plays an important role in the analysis of shift-invariant spaces: given \( f \) and \( g \) in \( L_2(\mathbb{R}^d) \), the bracket product is defined as

\[
[f,g] := \sum_{\alpha \in 2\pi \mathbb{Z}^d} f(\cdot + \alpha)\overline{g(\cdot + \alpha)}.
\]

Then, \([f,g] \) is a well-defined element of \( L_1(\mathbb{R}^d) \), and satisfies

\[
\|[f,f]\|_{L_1(\mathbb{R}^d)} = \|f\|^2_{L_2(\mathbb{R}^d)}.
\]

Also, a standard periodization argument yields that

\[
\langle f, g(\cdot - \alpha) \rangle = 0, \quad \forall \alpha \in \mathbb{Z}^d \quad \iff \quad ([f,g] = 0, \quad a.e.).
\]

Finally, we find it convenient to define \( g/f \) as follows:

\[
g/f : x \mapsto \begin{cases} 
g(x)/f(x), & x \in \text{supp } f \cap \text{supp } g, \\
0, & \text{otherwise}. \end{cases}
\]

1.3. Preliminaries

In this section we briefly recall some elementary facts concerning fundamental sets in Hilbert spaces. While most of the material here can be found in [C], [D1,2], [DS], [HW] and in several other references, it makes the paper more self-contained, and allows us to introduce the basic terminology in its natural setup. Only occasional proofs are given here.

Let \( H \) be a separable Hilbert space and \( X \) a countable subset of \( H \). We attempt to introduce the operator

\[
T := T_X : \ell_2(X) \rightarrow H : c \mapsto \sum_{x \in X} c(x)x.
\]

\( T \) is certainly well-defined on the finitely supported elements of \( \ell_2(X) \). \( X \) is said to be a **Bessel sequence/set** if \( T \) is bounded on the subspace of finitely supported sequences. In such a case, it is continuously extended to a bounded operator on \( \ell_2(X) \).

Associated with \( T_X \) is the map \( T^* := T_X^* : H \) defined by

\[
T^* : h \mapsto \{\langle h,x \rangle_H \}_{x \in X}.
\]
Proposition 1.3.2. $T^*$ is a bounded map from $H$ into $\ell_2(X)$ if and only if $X$ is a Bessel set. In such a case $T^*$ is the adjoint of $T$ and $\|T\| = \|T^*\|$.

Now, let $T$ be any bounded operator from a Hilbert space $H'$ into a Hilbert space $H$. Then the set

$$C_T := H' \ominus \ker T.$$  

(i.e., the orthogonal complement of $\ker T$ in $H'$) is well-defined, $T$ is injective on $C_T$, $\text{ran} T = \text{ran}(T|_{C_T})$, and $\text{ran} T^*$ is dense in $C_T$. In this paper, we use the notation $T^-1$ to denote the inverse map from $\text{ran} T$ to $C_T$ and, similarly, denote by $T^*^-1$ the inverse map from $\text{ran} T^*$ to $H \ominus \ker T^*$. These maps are usually referred to as partial (or pseudo) inverses. With these conventions, we have the following result.

Proposition 1.3.4. Let $X$ be a Bessel set, and $T := T_X$, $T^* := T^*_X$ as before. Then the following conditions are equivalent:
(a) $\text{ran} T$ is closed.
(b) $T$ is bounded below on $C_T$.
(c) $T^*$ is onto $C_T$.
(d) $T^*$ is bounded below on $H \ominus \ker T^*$.

When one (hence all) of these conditions holds, we have $\|T^*^-1\| = \|T^-1\|$.

Definition 1.3.5. Let $H$ be a Hilbert space and $X$ a fundamental Bessel set in $H$. We say that $X$ is a frame for $H$ if one (hence all) of the conditions of Proposition 1.3.4 holds. A frame $X$ is called tight if $\|T\|\|T^-1\| = 1$. We call a frame for $H := L_2(\mathbb{R}^d)$ a fundamental frame.

Thus, $X$ is a frame if and only if there exist constants $C_1, C_2$ such that the inequalities

$$C_1 \|h\|^2 \leq \sum_{x \in X} |\langle h, x \rangle_H|^2 \leq C_2 \|h\|^2$$

hold (for all $h \in H$). The sharpest possible constants are $C_2 = \|T\|^2 = \|T^*\|^2$ and $C_1 = 1/\|T^-1\|^2 = 1/\|T^*^-1\|^2$ and are usually referred to as the frame bounds. A frame is tight if and only if its frame bounds coincide.

A notion closely related to frames is that of a stable basis for $H$ (also known as a Riesz or unconditional basis) defined as follows:

Definition 1.3.6. A stable basis $X$ for $H$ is a frame for $H$ whose corresponding $T_X$ is injective. Equivalently, it is a frame whose corresponding $T^*_X$ is onto $\ell_2(X)$.

Given a frame $X$ for $H$, the map

$$TT^* : H \rightarrow H : h \mapsto \sum_{x \in X} \langle h, x \rangle_H x$$

is called the frame operator. $TT^*$ is continuously invertible, and we use

$$R := R_X$$
for its inverse. Since the map $R$ maps $X$ 1-1 onto $RX$, we may identify canonically the spaces $\ell_2(X)$ and $\ell_2(RX)$, as we do hereafter, without further notice.

Since $R$ is self-adjoint, $T^*_X R = T^*_RX$, and hence (i): $T^*_RX$ is a right inverse of $T_X$, and (ii): $RX$ is a frame (the latter since $T^*_RX$ is composed of two continuously invertible maps). The frame $RX$ is known as the dual frame of $X$, and some basic facts concerning dual frames are collected in the following proposition. *

**Proposition 1.3.7.** Let $RX$ be the dual frame of the frame $X$. Then:

(a) $X$ is the dual frame of $RX$ (i.e., duality is reflexive).

(b) $T_X T^*_RX = T^*_RX T_X = I_H$, with $I_H$ the identity map on $H$.

(c) $\ker T_X = \ker T^*_RX$ and $C_T X = C_{T^*_R} X$.

(d) The dual frame $RX$ is the only Bessel set $R'X$ in $H$ that satisfies $T_X T^*_R X = I_H$ and $\ker T_X = \ker T^*_RX$.

**Proof.** Since $R T_X = T^*_RX$, we have $T^*_RX T^*_RX = R T_X T^*_RX = R$, hence the dual of the frame $RX$ is $R^{-1} RX = X$, which shows (a).

For (b), we already know that $T_X T^*_RX = I_H$. Taking adjoints (or, alternatively, interchanging the roles of $X$ and $RX$, which is possible thanks to (a)), we get that $T^*_RX T_X = I_H$.

The relation $T^*_RX = R T_X$ shows also that $\ker T_X = \ker T^*_RX$, and hence $C_T X = C_{T^*_R} X$, which proves (c).

Finally, assume $R' : X \to H$ satisfies the conditions in (d). Define (on $X$) a map $K := R - R'$. Then $K X$ is Bessel, and $T_X T^*_R X = T_X (T^*_RX - T^*_R X) = 0$, showing that $\ker T_X \supset C_{T^*_R} X$. Further, since $\ker T^*_R X = \ker T^*_RX = \ker T_X$ (by assumption), we have $\ker T_k X \supset \ker T_X$. Thus, $\ker T_k X$ contains its orthogonal complement $C_{T^*_R} X$. This implies that $T_k X = 0$, hence, $K X = 0$. ♠

The above proposition allows us to represent the orthogonal projector onto $H$ with the aid of a frame and its dual:

**Proposition 1.3.8.** Let $S$ be a closed subspace of a Hilbert space $H$. Suppose that $X$ is a frame of $S$ with a dual frame $RX$. Then $T_X T^*_R X$ is the orthogonal projector $P_S : H \to S$, i.e.,

$$P_S h = \sum_{x \in X} \langle h, RX \rangle x.$$  

**Proof.** The definition of $T_X T^*_R X$ directly implies that its range lies in $S$, and hence, by (b) of Proposition 1.3.7, it is, indeed, a projector. It is also orthogonal, since $T^*_RX$, hence $T_X T^*_R X$, obviously vanish on the orthogonal complement of $S$ in $H$. ♠

Part (d) of Proposition 1.3.7 provides a criterion for checking whether a certain Bessel set $RX$ is the dual frame of $X$, or not. However, that criterion might be hard to implement, since it requires the identification of $\ker T_X$ and $\ker T^*_RX$. The following corollary provides us with partial remedy to that difficulty.

* The symbol $\hat{x}$ which is commonly used in the literature to denote the dual frame is used in this paper for a totally different purpose. In any case, the use of $\hat{x}$ to denote the dual of $x$ is an abuse of mathematical notations, since it suppresses the dependence of $R X x$ on $X \backslash x$. The notation $\hat{x}$ for the dual has many other drawbacks. To see one of them, try to rewrite the discussion here on dual frames using it instead of $R$. 

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Corollary 1.3.9. Let $H$ be a Hilbert space, $H'$ a closed subspace of $H$, $X$ a frame for $H'$, and $R$ a map from $X$ to $H'$. Assume that $RX$ is a Bessel set which is fundamental in $H'$. Then the following conditions are equivalent:

(a) RX is the dual frame of $X$.
(b) $T^*_R T_X, T^*_X T_R, T_X T^*_R, T^*_X T^*_R$ are orthogonal projectors.
(c) $T^*_R T_X, T^*_X T^*_R$ are orthogonal projectors.

Proof. The equivalence of (b) and (c) follows from the fact that every orthogonal projector is, in particular, self-adjoint, and hence, assuming (c), we get that $T^*_R T_X = T^*_X T_R$, and $T_X T^*_R = T^*_R T^*_X$ verifying thereby (b).

Assume (a). The fact that $T_X T^*_R$ is then an orthogonal projector is the statement of Proposition 1.3.8. This implies that $T^*_X T_R$ is a projector. Since RX is a frame, $T_R$ maps $\ell_2(X)$ onto $H'$, and since $X$ is a frame, $T^*_X$ maps $H'$ onto $C_T$. Hence, $T^*_X T_R$ must be the identity on $C_T$. The orthogonal complement of $C_T$ is $\ker T_X = \ker T_R$ (the equality by (c) of Proposition 1.3.7), and $T^*_X T_R$ certainly vanishes on $\ker T_R$. Hence it is orthogonal.

Now, assume (b). By statement (d) of Proposition 1.3.7, in order to prove that RX is the dual frame of $X$, we only need show that $C_T = C_{T_R}$. For that, we first observe that, since both $X$ and RX are fundamental in $H'$, $T^*_X T_R$ maps $C_{T_R}$ 1-1 densely into $C_T$. Since that operator certainly vanishes on $\ker T_R$ and is assumed to be orthogonal, we must have $C_T = C_{T_R}$. ♠

For a shift-invariant set $X = E_\phi$ (with $E_\phi$ as in (1.1.1)), we use the abbreviated notations

$$T_\phi := T_{E_\phi}, \quad T^*_\phi := T^*_{E_\phi}.$$ 

For this case, the search for the dual frame is simpler due to the following proposition.

Proposition 1.3.10. The dual $R(E_\phi)$ of a shift-invariant frame $E_\phi$ is the shift-invariant frame $E_{R\phi}$ generated by $R\Phi$. In particular, the dual of a principal (respectively, finite) shift-invariant frame is also a principal (finite) shift-invariant frame.

Proof. We need to show that $R$ commutes with shifts $E^\alpha : f \mapsto f(\cdot - \alpha), \alpha \in \mathbb{Z}^d$. For that, it suffices to show that the map

$$T_\phi T^*_\phi : f \mapsto \sum_{x \in E_\phi} \langle f, x \rangle x$$

commutes with shifts $E^\alpha$ (and use the fact that $R$ is the inverse of that map). Indeed, for $\alpha \in \mathbb{Z}^d$,

$$(T_\phi T^*_\phi)(E^\alpha f) = \sum_{x \in E_\phi} \langle E^\alpha f, x \rangle x = \sum_{x \in E_\phi} \langle f, E^{-\alpha} x \rangle x = \sum_{x \in E_\phi} \langle f, x \rangle E^\alpha x = E^\alpha T_\phi T^*_\phi f,$$

with the fact that $E^\alpha E_\phi = E_\phi$ being used in the penultimate equality. ♠
The central notions in this paper are the **pre-Gramian matrix**, **the Gramian matrix**, and the **dual Gramian matrix**. In principle, the objective is to decompose the involved operators \( T \) and \( T^* \) into a collection of simpler operators (“fibers”), indexed by \( w \in \mathbb{T}^d \). Each one of the “fiber” operators acts from a sequence space to (the same or another) sequence space and its matrix representation can be explicitly described in terms of the Fourier transforms of the generators \( \Phi \).

The main theme of the entire analysis is as follows: **every property of the set** \( E_\Phi \) (such as being a Bessel set, a frame, a stable basis etc.) **is equivalent to the “fiber” operators satisfying an analogous property in a uniform way** (here “uniformity” refers to the norms of the underlying operators).

The **pre-Gramian operator** \( T_\Phi \) is simply the Fourier transform analog of the operator \( T \). If \( c \in \ell_2(E_\Phi) \) is finitely supported, we see that

\[
(T_\Phi c)^\wedge = \sum_{\phi \in \Phi} \hat{c}_\phi \hat{\phi}.
\]

Hence, we may introduce an operator \( J_\Phi \), which is defined, at least, on the space

\[
L^\Phi_0 := \{ \hat{c} : c : E_\Phi \mapsto \Phi \text{ is finitely supported}\},
\]

by the rule

\[
J_\Phi : \tau \mapsto \sum_{\phi \in \Phi} \tau_\phi \hat{\phi}.
\]

Since the Fourier transform is an isometry, the boundedness, invertibility, and other properties of \( T_\Phi \) can be equally studied via \( J_\Phi \).

The definition of \( J_\Phi \) extends naturally to spaces larger than \( L^\Phi_0 \); for instance, if \( \Phi \) is finite, the rule in (1.4.3) can be extended to the entire \( L^\Phi_2 \) (In such a case, \( J_\Phi \tau \) need not be a \( L_2(\mathbb{R}^d) \)-function, but is always defined a.e.).

More relevant to our purposes, the pre-Gramian can be “evaluated” on \( \mathbb{T}^d \) in the following way: we define the **value** \( J_\Phi(w) \) of \( J_\Phi \) at \( w \in \mathbb{T}^d \) as the \((2\pi \mathbb{Z}^d \times \Phi)\)-matrix

\[
J_\Phi(w) := (\hat{\phi}(w + \alpha))_{\alpha, \phi}.
\]

Since each \( \hat{\phi} \) is well-defined only up to a null-set, so is the function \( w \mapsto J_\Phi(w) \). In a natural way, the matrix \( J_\Phi(w) \) can be viewed as a densely defined operator on \( \ell_2(\Phi) \). In any case, (1.4.1) together with (1.4.3) show that, for \( c \in \ell_2(E_\Phi) \),

\[
((T_\Phi c)^\wedge(w + \alpha))_{\alpha \in 2\pi \mathbb{Z}^d} = J_\Phi(w)\hat{c}(w).
\]

In summary, we have decomposed \( T_\Phi \), on the Fourier domain, into a collection of operators \( \{ J_\Phi(w) : w \in \mathbb{T}^d \} \), defined for almost every \( w \), each of which acts on a dense subspace of \( \ell_2(\Phi) \) and represents the action of \( J_\Phi \) on the coset \( w + 2\pi \mathbb{Z}^d \). Because of the explicit matrix representation of each \( J_\Phi(w) \), questions like its boundedness, invertibility etc., are far more accessible than their \( T_\Phi \) counterparts. Thus, our goal is to study \( T_\Phi \) via the behaviour of the “fibers” \( J_\Phi(w), w \in \mathbb{T}^d \).
The spectrum of the space $S(\Phi)$ generated by $\Phi$ is defined (up to a null-set) as
\[ \sigma \Phi := \sigma S(\Phi) := \{ w \in \mathbb{T}^d : J_\Phi(w) \neq 0 \}. \]

An equivalent definition of the spectrum is:
\[ (1.4.5) \quad \sigma \Phi := \{ w \in \mathbb{T}^d : \langle \hat{\phi}, \hat{\phi} \rangle(w) \neq 0, \text{ for some } \phi \in \Phi \}. \]

For a FSI space, it was proved in [BDR1] that the spectrum of $S$ only relies on the space and is independent of any particular selection of the generators of the space. That proof can be carried on to infinitely generated SI spaces.

Next, we want to decompose the operator $T_\Phi$. Since the Fourier transform is an isometry, the (formal, say) relation $J_\Phi = \overline{T_\Phi}$, leads to the relation
\[ \overline{T_\Phi} = J_\Phi^* \]

In §2 (cf. (2.1.1)) we show that, given $\phi \in \Phi$ and $f \in L_2(\mathbb{R}^d)$, the sequence $T_\Phi f$, though need not be in $\ell_2(E_\phi)$, is always in the Wiener algebra of $E_\phi$, more precisely, consists of the Fourier coefficients of the $L_1(\mathbb{T}^d)$-function $[\hat{f}, \hat{\phi}]$. This leads to the conclusion that $J_\Phi^*$, the Fourier transform analogue of $T_\Phi^*$, has the form
\[ (1.4.6) \quad J_\Phi^* : f \mapsto ([f, \widehat{\phi}], \phi \in \Phi, \]
and allows us to introduce “point evaluation” with respect to $J_\Phi^*$: we define $J_\Phi^*(w)$ to be the following operator acting on $\ell_2(2\pi \mathbb{Z}^d)$:
\[ (1.4.7) \quad J_\Phi^*(w) : c \mapsto \left( \sum_{\alpha \in 2\pi \mathbb{Z}^d} c(\alpha)\overline{\phi(w + \alpha)} \right), \phi \in \Phi. \]

(Note that $J_\Phi^*(w)$ and $J_\Phi^*(w)$ are not the same. For example, if $\phi \in \Phi$ and $f \in L_2(\mathbb{R}^d)$, then $J_\Phi^*(w)(f) = \int_{\mathbb{R}^d} f(w + \alpha) \overline{\phi(w + \alpha)} d\mu_\alpha$.

As expected, the analysis above reveals that the matrix representation of the operator $J_\Phi^*(w)$ is the adjoint of the matrix representation of the operator $J_\Phi(w)$. I.e., we had verified that “evaluation” commutes with taking adjoints. After making that observation, and with only very few necessary exceptions, we will identify $J_\Phi$ with its matrix representation $(J_\Phi(w))_{w \in \mathbb{T}^d}$.

The following lemma collects two useful facts that were just observed.

**Lemma 1.4.8.** Let $\Phi \subset L_2(\mathbb{R}^d)$ be a countable set. Then for any $c \in L_0(E_\phi)$ and $f \in L_2(\mathbb{R}^d)$, and for a.e. $w \in \mathbb{T}^d$,
\[ (1.4.9) \quad \overline{T_\Phi c(w + \cdot)_{2\pi \mathbb{Z}^d}} = J_\Phi(w)\hat{c}(w), \]

and
\[ (1.4.10) \quad \overline{T_\Phi f(w)} = J_\Phi^*(w)(\hat{f}_{\mathbb{Z}^d}). \]

Two self-adjoint operators can be constructed from $J_\Phi$. The first is the Gramian $G := G_\Phi$, which is defined by
\[ G := J_\Phi^*J_\Phi. \]

Previous considerations imply that $G_\Phi$ is the Fourier transform representer of $T_\Phi^*T_\Phi$. This fact allows us to draw the following immediate conclusions.
Proposition 1.4.11. For the densely defined linear operators $T_\Phi$ and $G$:

(i) $T_\Phi$ is bounded if and only if $G$, considered as an endomorphism of $L^2_\Phi$, is well-defined and bounded. Also, $\|G\| = \|T_\Phi\|^2$.

(ii) Assume $T_\Phi$ (hence, $G$) is bounded. Then, $T_\Phi$ is partially invertible if and only if $G$ is partially invertible. Also, $\|G^{-1}\| = \|T_\Phi^{-1}\|^2$.

(iii) Assume $T_\Phi$ is bounded. Then, $T_\Phi$ is invertible if and only if $G$ is invertible. Also, $\|G^{-1}\| = \|T_\Phi^{-1}\|^2$.

We define the value $G(w)$ of $G$ at $w \in \mathbb{T}^d$ as

\begin{equation}
G(w) := J_\Phi^*(w)J_\Phi(w) = \langle \tilde{\phi}, \tilde{\phi}'(w) \rangle_{\phi, \phi' \in \Phi}.
\end{equation}

In general, for a.e. $w \in \mathbb{T}^d$, the Gramian $G(w)$ is a densely defined self-adjoint operator on $\ell_2(\Phi)$ (hopefully into itself). In order to make any good use of $G(w)$, one needs to make sure that, at least on $L^2_\Phi$, evaluation commutes with the application of $G$, i.e., that

\[(G\tau)(w) = G(w)\tau(w), \quad \text{for } \tau \in L^2_\Phi, \text{ and for a.e. } w \in \mathbb{T}^d.\]

This is actually obtained by summation-by-parts, whose straightforward justification is omitted here. Hence:

Lemma 1.4.13. For every $c \in \ell_0(E_\Phi)$, and for a.e. $w \in \mathbb{T}^d$,

\[((T_\Phi^*T_\Phi)c)(w)\rangle_{\phi \in \Phi} = G(w)c(w).\]

The notation

\[\Lambda(w) := \|G(w)\|\]

stands for the operator norm of $G(w)$, and is assumed to be $\infty$ whenever $G(w)$ is not well-defined or is unbounded. In case $G(w)$ is also boundedly invertible, we denote its bounded inverse by $G(w)^{-1}$, and set

\[\lambda(w) := \|G(w)^{-1}\|^{-1}.
\]

Also, we set

\[\lambda^+(w) := \|G(w)^{-1}\|^{-1}.
\]

In case $\Phi$ is finite, $\Lambda(w)$ and $\lambda(w)$ are clearly the largest and smallest eigenvalues of the finite-order matrix $G(w)$. A closer look may reveal that $\lambda^+(w)$ is, in such a case, the smallest non-zero eigenvalue of $G(w)$.

Typical results concerning the Gramian analysis can be found in Theorem 2.2.7 (PSI spaces), Theorem 2.2.14 (PSI spaces, several generators), Theorem 2.3.6 (FSI spaces), and Theorems 3.2.3 and 3.4.1 (infinitely generated SI spaces).

The Gramian approach is efficient for the study of those properties of $E_\Phi$ which are “visible” via the operator $T_\Phi$, primarily orthogonality and stability properties. In contrast, other properties such as $E_\Phi$ being a fundamental frame or a fundamental tight frame are better analysed with the aid of the adjoint $T_\Phi^*$. For the analysis of this adjoint operator, we introduce another self-adjoint operator which we call the dual Gramian $\tilde{G}$. It is obtained by multiplying the pre-Gramians, but in reverse order, namely,

\begin{equation}
\tilde{G} := \tilde{G}_\Phi := J_\Phi J_\Phi^*.
\end{equation}
Problems of well-definedness are more subtle here than in the Gramian case. Fully detailed discussions of that point are given in §3.3, and we mention here only two facts: first, if \( E_\Phi \) is a Bessel set, then \( \widetilde{G} \) is a well-defined self-adjoint bounded endomorphism of \( L_2(\mathbb{R}^d) \). Second, if \( E_\Phi \) is not a Bessel set, the definition (1.4.14) may not make sense, and it is safer to view \( \widetilde{G} \) as a quadratic form, i.e., to define it by

\[
\widetilde{G} : f \mapsto \| J_\Phi^* f \|_{L_2}^2 = \sum_{\phi \in \Phi} \| J_\Phi^* f \|_{L_2(\mathbb{T}^d)}^2 = \sum_{\phi \in \Phi} \| [f, \phi] \|_{L_1(\mathbb{T}^d)}^2.
\]

The evaluation \( \widetilde{G}(w) \) of the dual Gramian is the \((2\pi \mathbb{Z}^d \times 2\pi \mathbb{Z}^d)\)-matrix whose \((\alpha, \alpha')\)-entry has the form

\[
\sum_{\phi \in \Phi} \widetilde{\phi}(w + \alpha) \overline{\phi}(w + \alpha'). \tag{1.4.15}
\]

The argument \( w \) may be restricted to \( \mathbb{T}^d \). For a general \( E_\Phi \), the entries of \( \widetilde{G}(w) \) may not be well-defined (in the sense that the sum in their definition needs not converge absolutely). Nevertheless, we will show (in §3.3) that, whenever \( E_\Phi \) is a Bessel set, the sum in (1.4.15) converges absolutely for every \( \alpha, \alpha' \in 2\pi \mathbb{Z}^d \) and for a.e. \( w \). Thus, for a Bessel set \( E_\Phi \), \( \widetilde{G}(w) \) is well-defined a.e., and can viewed as a densely defined operator from \( \ell_2(2\pi \mathbb{Z}^d) \) (hopefully into itself). Moreover, we will show then that the basic relation

\[
(\widetilde{G} f)(w) = \widetilde{G}(w) f_w
\]

(with \( f_w \) the restriction of \( f \) to \( w + 2\pi \mathbb{Z}^d \)) holds a.e. A similar relation is drawn in §3.3 even in the non-Bessel case, under the assumption that the entries of \( \widetilde{G}(w) \) are well-defined, and with the interpretation of \( \widetilde{G} \) and \( \widetilde{G}(w) \) as quadratic forms.

Analogously to the Gramian case, we define here the following functions

\[
\begin{align*}
\widetilde{\lambda}(w) & := \| \widetilde{G}(w) \|, \\
\lambda(w) & := \| \widetilde{G}(w)^{-1} \|^{-1}, \\
\widetilde{\lambda}^+(w) & := \| \widetilde{G}(w)^{-1} \|^{-1},
\end{align*}
\]

and attempt to study properties of \( E_\Phi \) in terms of the behaviour of these functions. Our main results in this regard are Theorem 3.3.5, and Theorem 3.4.1.

The Gramian/dual Gramian analyses are also efficient for studying the connection between a frame and its dual: given two sets \( \Phi, \Psi \subset L_2(\mathbb{R}^d) \), and some bijection \( R : \Phi \to \Psi \), this is done via the study of the matrices \( J_\Phi(w)J_\Phi^*(w) \), and \( J_\Psi(w)J_{R\Phi}(w) \), as discussed in §4.

1.5. An example

We provide here an example, which is taken from [RS1], (and is a specific type of what we call there “self-adjoint Weyl-Heisenberg sets”) that illustrates the potential power of the results to be developed in this paper.
Let $\phi \in L_2(\mathbb{R}^d)$. Let

$$\Phi := (e_{\alpha} \phi)_{\alpha \in 2\pi \mathbb{Z}^d}.$$ 

Indexing $\Phi$ by $2\pi \mathbb{Z}^d$, the pre-Gramian $J_\Phi(w)$ is found to be

$$J_\Phi(w) = (\hat{\phi}(w + \alpha + \beta))_{\alpha, \beta \in 2\pi \mathbb{Z}^d}.$$ 

Therefore, $J_\Phi(w) = \overline{J_\Phi(w)}$, and hence

$$G_\Phi(w) = \overline{G_\Phi(w)}.$$ 

Now, Theorem 3.2.3 characterizes the stability property of $E_\Phi$ in terms of the Gramian fibers $G_\Phi(w), w \in \mathbb{T}^d$. On the other hand, the same criterion when applied to $\tilde{G}_\Phi(w), w \in \mathbb{T}^d$, is shown to be equivalent to $E_\Phi$ being the fundamental frame (Theorem 3.3.5). This recovers the following well-known fact (cf. e.g. [D1,2]):

**Corollary 1.5.1.** With $\Phi$ as above, $E_\Phi$ is a stable basis if and only if it is a fundamental frame.

1.6. An application: estimating the frame bounds

The main results of this paper are concerned with the connections between the spectrum of the operators $G$ and $\tilde{G}$ and the spectra of the operators $G(w)$ and $\tilde{G}(w), w \in \mathbb{T}^d$. As we mentioned before, information about the fiber operators $G(w)$ and $\tilde{G}(w)$ is more readily available as compared to similar information concerning $G$ and $\tilde{G}$. Still, computing exactly, e.g., the norm of $G(w)$ (considered as a linear map from $\ell_2(\Phi)$ into itself) might appear as a hard task. However, estimating this norm (either from below or from above) in terms of the Fourier transforms of the functions in the generating set $\Phi$ is quite easy. This subsection is devoted to the discussion of such estimates.

To this end, we let $I$ be a countable (or finite) index set, and let $M$ be a complex-valued non-negative Hermitian matrix with rows and columns indexed by $I$, and considered as an operator from $\ell_2(I)$ into itself. We use the following estimates of $\|M\|:

$$\sup_{i \in I} \left( \sum_{j \in I} |M(i, j)|^2 \right)^{1/2} \leq \|M\| \leq \sup_{i \in I} \sum_{j \in I} |M(i, j)|.$$ 

Combining these estimates with Theorem 3.2.3, we obtain our first estimate for $\|T_\Phi\|:

**Corollary 1.6.2.** Let $\Phi$ be a countable (or finite) subset of $L_2(\mathbb{R}^d)$.

(a) If the function

$$B_1 : \mathbb{T}^d \times \Phi \to \mathbb{R} : (w, \phi) \mapsto \sum_{\phi \in \Phi} \left| \sum_{\alpha \in 2\pi \mathbb{Z}^d} \hat{\phi}(w + \alpha) \overline{\phi}(w + \alpha) \right|$$

is essentially bounded, then $E_\Phi$ is a Bessel set, and $\|T_\Phi\|^2 \leq \|B_1\|_{L_\infty(\mathbb{T}^d \times \Phi)}$.

(b) If $E_\Phi$ is a Bessel set, then the function

$$B_2 : \mathbb{T}^d \times \Phi \to \mathbb{R} : (w, \phi) \mapsto \left( \sum_{\phi \in \Phi} \left| \sum_{\alpha \in 2\pi \mathbb{Z}^d} \hat{\phi}(w + \alpha) \overline{\phi}(w + \alpha) \right|^2 \right)^{1/2}$$

is essentially bounded, and $\|T_\Phi\|^2 \geq \|B_2\|_{L_\infty(\mathbb{T}^d \times \Phi)}$.

On the other hand, combining (1.6.1) with Theorem 3.3.5, we obtain different estimates:
Corollary 1.6.3. Let $\Phi$ be a countable (or finite) subset of $L_2(\mathbb{R}^d)$.

(a) If the function

$$\tilde{B}_1 : \mathbb{R}^d \rightarrow \mathbb{R} : w \mapsto \sum_{\alpha \in 2\pi \mathbb{Z}^d} \left| \sum_{\phi \in \Phi} \hat{\phi}(w) \hat{\phi}(w + \alpha) \right|$$

is essentially bounded, then $E_\Phi$ is a Bessel set, and $\|T_\Phi\| \leq \|\tilde{B}_1\|_{L_\infty(\mathbb{R}^d)}$.

(b) If $E_\Phi$ is a Bessel set, then the function

$$\tilde{B}_2 : \mathbb{R}^d \rightarrow \mathbb{R} : w \mapsto \left( \sum_{\alpha \in 2\pi \mathbb{Z}^d} \left| \sum_{\phi \in \Phi} \hat{\phi}(w) \hat{\phi}(w + \alpha) \right|^2 \right)^{\frac{1}{2}}$$

is bounded and $\|T_\Phi\|^2 \geq \|\tilde{B}_2\|_{L_\infty(\mathbb{R}^d)}$.

For the estimation of the other frame bound, we need a bound on $\|M^{-1}\|$. In what follows we employ the estimate

$$\|M^{-1}\| \leq \sup_{i \in I} \left( |M(i,i)| - \sum_{j \in I \setminus i} |M(i,j)| \right)^{-1},$$

which is valid for any Hermitian diagonally dominant $M$. An application of this estimate to Theorem 3.2.3 yields the following:

Corollary 1.6.5. Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable (or finite), and assume that $E_\Phi$ is a Bessel set. Then $E_\Phi$ is a stable basis if the function

$$b_1 : \mathbb{R}^d \times \Phi \rightarrow \mathbb{R} : (w,\phi) \mapsto \left( \sum_{\alpha \in 2\pi \mathbb{Z}^d} \left| \hat{\phi}(w + \alpha) \right|^2 - \sum_{\phi \in \Phi \setminus \Phi} \left| \sum_{\alpha \in 2\pi \mathbb{Z}^d} \hat{\phi}(w + \alpha) \phi(w + \alpha) \right| \right)^{-1}$$

is positive and essentially bounded. Furthermore, in this case

$$\|T_\Phi^{-1}\|^2 \leq \|b_1\|_{L_\infty(\mathbb{R}^d \times \Phi)},$$

Finally, an application of (1.6.4) to Theorem 3.3.5 yields the following:

Corollary 1.6.6. Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable (or finite), and assume that $E_\Phi$ is a Bessel set. Then $E_\Phi$ is a fundamental frame if the function

$$\tilde{b}_1 : \mathbb{R}^d \rightarrow \mathbb{R} : w \mapsto \left( \sum_{\phi \in \Phi} \left| \hat{\phi}(w) \right|^2 - \sum_{\alpha \in 2\pi \mathbb{Z}^d \setminus 0} \left| \sum_{\phi \in \Phi} \hat{\phi}(w) \phi(w + \alpha) \right| \right)^{-1}$$

is positive and essentially bounded. Furthermore,

$$\|T_\Phi^{-*}\|^2 \leq \|\tilde{b}_1\|_{L_\infty(\mathbb{R}^d)}.$$
Example 1.6.7. Suppose that, for every \( \phi \in \Phi \), for every \( \alpha \in 2\pi \mathbb{Z}^d \), and for almost every \( w \in \mathbb{R}^d \), 
\( \hat{\phi}(w) \hat{\phi}(w + \alpha) = 0 \) (e.g., each \( \hat{\phi} \) is supported in some cube \( t_\phi + [0, 2\pi)^d \), \( t_\phi \in \mathbb{R}^d \)). Then, the (square root of the) function \( \tilde{B}_1 \) can be replaced by the function

\[
g : \mathbb{R}^d \to \mathbb{R} : w \mapsto \left( \sum_{\phi \in \Phi} |\hat{\phi}(w)|^2 \right)^{\frac{1}{2}}.
\]

Similarly, the function \( \tilde{b}_1 \) can be replaced by \( 1/g \). Consequently, we obtain \( E_\Phi \) is a fundamental frame if the two functions \( g \) and \( 1/g \) are essentially bounded. In fact, the results of this paper will show that the converse of this last statement is valid as well.

2. Finitely generated SI spaces

2.1. General

While general SI spaces are best analysed with simultaneous use of the Gramian and dual Gramian matrices, this is not the case for FSI spaces. The reason is easy to inspect: for a finitely generated SI space, the dual Gramian matrix is infinite, while the Gramian matrix is finite. This explains to a large extent the prevalence of Gramian analysis in the study of FSI spaces. Moreover, in the principal case, the Gramian matrix is reduced to a single function, providing thereby a further significant simplification in the course of study of such spaces. Therefore, we will first present (in the next subsection) a detailed analysis of bases for PSI spaces, and only then discuss the FSI counterpart of that theory. The present subsection is devoted to some simple initial observations and estimates.

In the PSI case, the generating set \( \Phi \) is a singleton \((\phi)\), and the operator \( T^*_{\phi} := T^*_{(\phi)} \) then takes the particularly simple form

\[
T^*_{\phi} : f \mapsto \{ \langle f, E^n \phi \rangle \}_{n \in \mathbb{Z}^d}.
\]

From Parseval’s identity, and the \( 2\pi \)-periodicity of the exponentials \( e_n, \alpha \in \mathbb{Z}^d \), we obtain that

\[
(1.1) \quad \langle f, E^n \phi \rangle = (2\pi)^{-d} \langle \hat{f}, \hat{n}, e_n \rangle = (2\pi)^{-d} \langle \hat{f}, \hat{\phi}, e_n \rangle_{L_2(\mathbb{T}^d)}.
\]

Therefore, \( T^*_{\phi} f \) is the set of Fourier coefficients of the \( L_1(\mathbb{T}^d) \)-function \( \hat{f}, \hat{\phi} \), that is

\[
(1.2) \quad T^*_{\phi} f = \{ \hat{f}, \hat{\phi} \}.
\]

In particular,

Proposition 2.1.3. Given \( \phi, f \in L_2(\mathbb{R}^d) \),

\[
|| T^*_{\phi} f ||_{L_2(\mathbb{T}^d)} = (2\pi)^{-d/2} || \hat{f}, \hat{\phi} ||_{L_2(\mathbb{T}^d)}.
\]

Some coarse estimates can be derived directly from the above. By Schwartz inequality,

\[
|| \hat{f}, \hat{\phi} ||^2 \leq || \hat{f}, \hat{f} || || \hat{\phi}, \hat{\phi} ||.
\]

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Thus, for $\Phi \subset L_2(\mathbb{R}^d)$,
\[
\|T_\Phi f\|^2_{\ell_2(E_\Phi)} \leq (2\pi)^{-d} \|\hat{f}, \hat{f}\| L_1(\mathbb{T}^d) \sum_{\phi \in \Phi} \|\hat{\phi}, \hat{\phi}\|_{L_1(\mathbb{T}^d)}.
\]
Since $\|f\|^2 = (2\pi)^{-d} \|\hat{f}, \hat{f}\|_{L_1(\mathbb{T}^d)}$, we conclude that
\[
\|T_\Phi f\|_{\ell_2(E_\Phi)} \leq \|f\| \sum_{\phi \in \Phi} \|\hat{\phi}, \hat{\phi}\|^\frac{1}{2}_{L_\infty(\mathbb{T}^d)}.
\]
Denoting
\[
\tilde{\Phi} := (\sum_{\phi \in \Phi} \|\hat{\phi}, \hat{\phi}\|^\frac{1}{2})^*,
\]
we have proved the following result.

**Proposition 2.1.4.** Given $\Phi \subset L_2(\mathbb{R}^d)$, $E_\Phi$ is a Bessel set in case $\tilde{\Phi} \in L_\infty(\mathbb{T}^d)$, and we then have
\[
\|T_\Phi\| \leq \|\tilde{\Phi}\|_{L_\infty(\mathbb{T}^d)}.
\]

We will show later that equality holds in the above in case $\Phi$ is taken from some PSI subspace of $L_2(\mathbb{R}^d)$. Further, we will show that for a finite $\Phi$ the boundedness of $\tilde{\Phi}$ is not only sufficient for $E_\Phi$ to be a Bessel sequence, but also necessary. However, the bound provided by $\|\tilde{\Phi}\|_{L_\infty(\mathbb{T}^d)}$ is, in general, not sharp.

### 2.2. Frames in PSI spaces

Throughout this subsection, $S$ is a PSI subspace of $L_2(\mathbb{R}^d)$ generated by some (fixed) function.

Motivated by the search for an explicit representation for the orthogonal projection onto shift-invariant spaces, [BDR1] introduces and studies the notions of quasi-stable and quasi-orthogonal bases for FSI spaces. For PSI spaces, in the terminology used in the present paper, its definitions are as follows:

**Definition 2.2.5.** ([BDR1]) Let $\phi \in L_2(\mathbb{R}^d)$, and let $T_\phi$ be the operator
\[
T_\phi : \ell_2(\mathbb{Z}^d) \to S(\phi) : c \mapsto \sum_{\alpha \in \mathbb{Z}^d} E^\alpha \phi c(\alpha).
\]
Then $\phi$ is called a **quasi-stable generator** if $T_\phi$ is a well-defined bounded map, and provides an isomorphism between $C_{T_\phi} := (\ker T_\phi)^\perp$ and $S(\phi)$. If, further, that isomorphism is an isometry, $\phi$ is termed a **quasi-orthogonal generator**.

In view of (b) of Proposition 1.3.4, and Definition 1.3.5 of frames and tight frames we obtain the following Corollary.

**Corollary 2.2.6.** Let $\phi \in L_2(\mathbb{R}^d)$. Then $E_\phi$ is a frame if and only if $\phi$ is a quasi-stable generator of $S(\phi)$. Further, this frame is tight if and only if $\phi$ is a scalar multiple of a quasi-orthogonal generator of $S(\phi)$.

Thus, implicitly, [BDR1] contains an extensive discussion of frames in PSI spaces. Furthermore, as we had learnt from the referee of this paper, frames for PSI spaces were (explicitly) studied by Benedetto and Li [BL]. Indeed, Theorem 7.7 of [BW] (which is attributed there to [BL]) is essentially equivalent to Theorem 2.2.7.
We recall the definition of the spectrum $\sigma S$ given in (1.4.5), and recall the notation
\[
\tilde{\phi} = [\tilde{\phi}, \tilde{\phi}] = \left( \sum_{\beta \in \mathbb{Z}^d} |\tilde{\phi}(\cdot + \beta)|^2 \right)^{\frac{1}{2}}.
\]

**Theorem 2.2.7.** ([BDR1], [BL]) Let $\phi \in L_2(\mathbb{R}^d)$ be given, and let $S$ be the PSI space generated by $\phi$.

(a) The shifts $E_\phi$ of $\phi$ form a Bessel sequence in $S$ if and only if $\tilde{\phi}$ is essentially bounded.

(b) The shifts $E_\phi$ of $\phi$ form a frame for $S$ if and only if $\tilde{\phi}$ and $1/\tilde{\phi}$ are essentially bounded on $\sigma S$. Furthermore,
\[
\|T_\phi\| = \|\tilde{\phi}\|_{L_{\infty}(\mathbb{T}^d)} = \|\tilde{\phi}\|_{L_{\infty}(\sigma S)},
\]
and
\[
\|T_\phi^{-1}\| = \|1/\tilde{\phi}\|_{L_{\infty}(\sigma S)}.
\]

Therefore, for a frame $E_\phi$, the inequalities
\[
\|f\|/\|1/\tilde{\phi}\|_{L_{\infty}(\sigma S)} \leq \left( \sum_{\alpha \in \mathbb{Z}^d} \|\langle f, E^n \phi \rangle\|^2 \right)^{\frac{1}{2}} \leq \|\tilde{\phi}\|_{L_{\infty}(\sigma S)} \|f\|, \quad f \in S,
\]
are valid and sharp.

(c) $E_\phi$ is a tight frame if and only if $\tilde{\phi} = \text{const \ (a.e.)}$ on its support.

(d) With $\psi := (\hat{\phi}/\tilde{\phi})^\vee$, the set $E_\phi$ is a tight frame for $S(\phi)$ (and hence every PSI space is generated by some PSI tight frame).

(e) The frame (tight frame) $E_\phi$ is a stable (orthogonal) basis for $S$ if and only if $\sigma S = \mathbb{T}^d$.

**Proof.** By Corollary 2.2.6, the shifts of $\phi$ form a frame (tight frame) if and only if $\phi$ is a quasi-stable (quasi-orthogonal) generator of $S(\phi)$. Therefore, the theorem follows from the corresponding results in section 2 of [BDR1].

We observe that the above (d) and (e) imply that $S$ contains an orthonormal basis $E_\phi$ if and only if $\sigma S = \mathbb{T}^d$. That case was termed regular in [BDR1]. Thus (e) above shows that the notions of a stable basis and a frame coincide for a principal shift-invariant $E_\phi$, provided that $S(\phi)$ is regular. It is worth mentioning that, in case $\phi$ is compactly supported, $S(\phi)$ is always regular.

The spaces $\ker T_\phi$ and $C_{T_\phi}$ were described explicitly in [BDR1] as follows:
\[
\ker T_\phi := \left\{ c \in \ell_2(\mathbb{Z}^d) : \supp \hat{c} \subset (\mathbb{T}^d \setminus \sigma S) \right\},
\]
and hence
\[
\ker T_\phi := \left\{ c \in \ell_2(\mathbb{Z}^d) : \supp \hat{c} \subset \sigma S \right\},
\]

\[(2.2.8) \quad C_{T_\phi} := \left\{ c \in \ell_2(\mathbb{Z}^d) : \supp \hat{c} \subset \sigma S \right\}.
\]

Next, we need the following characterization of the Fourier transforms of the elements of $S(\phi)$:

**Result 2.2.9.** ([BDR2]) Let $\phi, f \in L_2(\mathbb{R}^d)$. Then $f \in S(\phi)$ if and only if $\hat{f} = \tau \hat{\phi}$ for some $2\pi$-periodic function $\tau$. 

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Corollary 2.2.10. Let $S := S(\phi)$ be a PSI space, and assume that $E_\phi$ forms a frame for $S$. Then, given $c \in \ell_2(\mathbb{Z}^d)$, there exists $f \in S$ such that

$$c(\alpha) = \langle f, E^\alpha \phi \rangle, \quad \alpha \in \mathbb{Z}^d$$

if and only if $\hat{c}$ is supported in the spectrum of $S$. The unique solution $f$ has the form

$$(2.2.11) \quad f = \sum_{\alpha \in \mathbb{Z}^d} E^\alpha \phi c_f(\alpha),$$

with the sequence $c_f \in C_{T_\phi}$ being the solution of the discrete convolution equation

$$[\widehat{\phi}, \widehat{\phi}]^\vee \ast c_f = c.$$ 

Proof. By the definition of $T_\phi^*$, a solution $f$ exists if and only if $c$ lies in the range of $T_\phi^*$, i.e., if and only if $c \in C_{T_\phi}$. Therefore, in view of (2.2.8), we only need to prove the statements concerning the nature of the solution $f$. Since $E_\phi$ is a frame for $S$, then, given any $f \in S$, there exists a unique $c_f \in C_{T_\phi}$ that satisfies (2.2.11). Taking Fourier transforms, we obtain that $\hat{f} = \hat{c}_f \hat{\phi}$. Invoking (2.1.2), we see that

$$(2.2.12) \quad \hat{c} = T_\phi^* f = [\hat{f}, \hat{\phi}] = \hat{c}_f [\widehat{\phi}, \widehat{\phi}],$$

where, in the last equality, the periodicity of $\hat{c}_f$ was used. The desired result then follows by inversion. ♣

Given a frame $E_\phi$, Proposition 1.3.10 asserts that there exists a function $R_\phi \in S(\phi)$, such that $E_{R_\phi}$ is the dual frame of $E_\phi$. Further, we can compute $R_\phi$ as follows: first, we seek $c_\phi \in C_{T_\phi}$ such that $T_\phi c_\phi = \phi$. Applying Fourier transform, then multiplying by $\overline{\phi}$, and periodizing over $2\pi \mathbb{Z}^d$, we obtain the equation $\widehat{c}_\phi [\widehat{\phi}, \widehat{\phi}] = [\widehat{\phi}, \widehat{\phi}]$. Since $c_\phi$ is in $C_{T_\phi}$, it is supported on $\text{supp} \widehat{\phi} = \sigma S$, and so $\widehat{c}_\phi$ is the characteristic function $\chi$ of $\sigma S$. Let $c$ be the solution of $[\widehat{\phi}, \widehat{\phi}]^\vee \ast ? = c_\phi$, and $R_\phi := \hat{c}_\phi$. Then $E_{R_\phi}$ is the dual basis of $E_\phi$ by the fact $T_{R_\phi}^* \phi = c_\phi$ and by Corollary 2.2.10. Hence $\hat{c}$ is defined by

$$\hat{c} = \frac{\hat{c}_\phi}{[\phi, \phi]} = \frac{1}{[\phi, \phi]},$$

and $R_\phi$ is given by

$$(2.2.13) \quad R_\phi = \phi/[\phi, \phi].$$

This representation of $R_\phi$ is detailed in [BDR1] (using a different approach) and is well-known in the special regular case mentioned above (in which a frame becomes a stable basis).

The redundancy offered by frames does not really exist for principal shift-invariant ones. Yet, given a PSI space, one may use several functions from $S$ to generate a shift-invariant frame for $S$. The details of that case are given in the next theorem.
Theorem 2.2.14. Let $S$ be a PSI space, and $\Phi \subset S$ be a countable (or finite) set. Then

(a) $E_\Phi$ is a Bessel set if and only if the function

\[(2.2.15)\ \quad \tilde{\Phi} := \left( \sum_{\phi \in \Phi} \hat{\phi} \hat{\phi}^* \right)^{1/2}\]

is essentially bounded. Furthermore, $\|T_\Phi f\|_{\ell_2(E_\Phi)}^2 = \|\tilde{\Phi}\|_{L_\infty(\mathbb{R}^d)}^2$.

(b) $E_\Phi$ is a frame for $S$ if and only if $\tilde{\Phi}$ and $1/\tilde{\Phi}$ are essentially bounded on the spectrum $\sigma S$. In such a case, $\|T_\Phi\|^{-1} = \|1/\tilde{\Phi}\|_{L_\infty(\sigma S)}$.

(c) $E_\Phi$ is a tight frame if and only if $\tilde{\Phi}$ is constant a.e. on its support.

Proof. By Proposition 2.1.3, given $f \in L_2(\mathbb{R}^d)$,

$$\|T_\Phi f\|_{\ell_2(E_\Phi)}^2 = (2\pi)^{-d} \sum_{\phi \in \Phi} \|\hat{\phi}\|^2 \|\hat{\phi} \hat{\phi}^*\|_{L_1(\mathbb{R}^d)}.$$

Let $\psi$ be a generator of $S$. For $f \in S$ and $\phi \in \Phi$, Result 2.2.9 implies the existence of $2\pi$-periodic $\tau_\phi, \tau_f$ such that

$$\hat{f} = \tau_f \hat{\psi}, \quad \hat{\phi} = \tau_\phi \hat{\psi}, \quad \phi \in \Phi.$$

Therefore,

$$\|\hat{f}, \hat{\phi}\|^2 = |\tau_f|^2 |\omega_\phi|^2 \|\hat{\psi} \hat{\psi}^*\|^2 = \|\hat{f}, \hat{\phi}\| \|\hat{\phi} \hat{\phi}^*\|.$$

Consequently,

$$\|T_\Phi f\|_{\ell_2(E_\Phi)}^2 = (2\pi)^{-d} \|\hat{f}, \hat{\phi}\| \|\hat{\phi} \hat{\phi}^*\|_{L_1(\mathbb{R}^d)}.$$

Since $\|f\|_{\ell_2(\mathbb{R}^d)}^2 = (2\pi)^{-d} \|\hat{f}, \hat{\phi}\| \|\hat{\phi} \hat{\phi}^*\|_{L_1(\mathbb{R}^d)}$, and since $\hat{f}, \hat{\phi}$ is necessarily supported on $\sigma S$, the proof of the theorem relies on the comparison of

$$\|\hat{f}, \hat{\phi}\|_{L_1(\sigma S)}^2$$

and

$$\|\hat{f}, \hat{\phi}\| \|\hat{\phi} \hat{\phi}^*\|_{L_1(\sigma S)}^2.$$

Further, we note that Result 2.2.9 also implies that for any closed $\Omega \subset \sigma S$, there exists $f \in S$ for which $\hat{f}, \hat{\phi}$ is the characteristic function of $\Omega$. The proof can be then completed by a routine argument (cf. e.g., the proof of Theorem 2.16 in [BDR11]).

The final theorem of this subsection provides the details concerning the dual frame of the above $E_\Phi$ and a complete description of ker$T_\Phi$ and $C_{T_\Phi}$:

Theorem 2.2.16. Let $\Phi$ be a countable subset of a PSI space $S$, $E_\Phi$ its corresponding shift-invariant set. If $E_\Phi$ is a frame then:

(a) Let $\psi$ be any generator of $S$ (i.e., $S = S(\psi)$), and $c = (c_\phi)_{\phi \in \Phi} \in \ell_2(E_\Phi)$ (with $c_\phi$ the restriction of $c$ to $E_\Phi$). Then $c \in C_{T_\Phi}$ if and only if

$$(c_\phi)_\phi = \tau(\hat{\psi}, \hat{\phi} \hat{\phi}^*),$$

and

$$(\hat{\psi}, \hat{\phi} \hat{\phi}^*)_{\phi \in \Phi} \in \ell_2(E_\Phi).$$
for some $2\pi$-periodic function \( \tau \), that is supported on \( \sigma S \).
\( (b) \) The map \( R \) from the frame \( E_\Phi \) to its dual is given by
\[
R : f \mapsto (\hat{f}/\hat{\Phi}^2)^{\vee}.
\]
\( (c) \) The orthogonal projector \( P : L_2(\mathbb{R}^d) \to S \) can be written in the form
\[
P f = \sum_{\phi \in \Phi, \, a \in \mathbb{Z}^d} \langle f, E^a ((\hat{\phi}/\hat{\Phi}^2)^{\vee}) \rangle E^a \phi.
\]

**Proof.** Claim (c) is immediate from (b) and Proposition 1.3.8. To prove (b), we need to show that the map \( R \) inverts \( T_\Phi T_\Phi^* \), and this will follow as soon as we show that \( (T_\Phi T_\Phi^*)^{\vee} = \hat{\Phi}^2 \hat{f} \) on \( S \). For that, note first that Result 2.2.9 implies that, for every \( f, g \in S \),
\[
[\hat{f}, \hat{g}][\hat{g}] = [\hat{g}, \hat{g}][\hat{f}].
\]
Now, given \( f \in S \), we first recall that, by (2.1.2), for every \( \phi \in \Phi \),
\[
(\hat{T}_\Phi T_\Phi^* f)^{\vee} = [\hat{f}, \hat{\phi}] \hat{\phi}.
\]
This, together with (2.2.17) and the fact that \( T_\Phi T_\Phi^* = \sum_{\phi \in \Phi} T_\phi T_\phi^* \), implies that
\[
(\hat{T}_\Phi T_\Phi^* f)^{\vee} = \sum_{\phi \in \Phi} [\hat{f}, \hat{\phi}] \hat{\phi} = \sum_{\phi \in \Phi} \hat{\phi} \hat{f} = \hat{\Phi}^2 \hat{f}.
\]
This proves (b) and thereby (c).

To prove (a), we compute \( C_{T_\Phi} \) using the identity
\[
C_{T_\Phi} = \text{ran } T_\Phi^*.
\]
For \( f \in S \), there exists, by Result 2.2.9, a function \( \tau_f \) supported on \( \sigma S \), such that \( \hat{f} = \tau_f \hat{\Phi} \). By (2.1.2),
\[
\overline{T_\Phi^* f} = [\hat{f}, \hat{\phi}] = \tau_f [\hat{\psi}, \hat{\phi}].
\]
Since \( C_{T_\Phi} \) is the range of \( T_\Phi^* \), this shows that the Fourier transform of each \( c = (c_\phi)_{\phi \in \Phi} \in C_{T_\Phi} \) is of the form \( \hat{c}_\phi = \tau([\hat{\psi}, \hat{\phi}]) \), \( \forall \phi \in \Phi \), for some \( 2\pi \)-periodic \( \tau \) supported on \( \sigma S \), i.e., \( C_{T_\Phi} \) contains only sequences of the required form.

Conversely, assume that \( c = (c_\phi) \) satisfies \( \hat{c}_\phi = \tau([\hat{\psi}, \hat{\phi}]) \). We consider the nature of \( T_\phi c = \sum_{\phi \in \Phi} T_\phi c_\phi \). Applying Fourier transform, and invoking (2.2.17) once again, we obtain that
\[
\overline{T_\phi c} = \sum_{\phi \in \Phi} \hat{c}_\phi \hat{\phi} = \sum_{\phi \in \Phi} \tau([\hat{\psi}, \hat{\phi}]) \hat{\phi} = \sum_{\phi \in \Phi} \tau([\hat{\phi}, \hat{\phi}]) \hat{\psi} = \tau \hat{\Phi}^2 \hat{\psi}.
\]
Since \( T_\phi \) is bounded, \( \tau \hat{\Phi}^2 \hat{\psi} \in L_2(\mathbb{R}^d) \). On the other hand, since \( E_\Phi \) is a frame, then, by Theorem 2.2.14, \( \hat{\Phi} \) is bounded below on \( \sigma S^c \supset \text{supp } \hat{\psi} \), and therefore \( \tau \hat{\psi} \in L_2(\mathbb{R}^d) \). Thus, \( f := (\tau \hat{\psi})^{\vee} \) is in \( L_2(\mathbb{R}^d) \), and hence, by Result 2.2.9, is also in \( S \). Since the proof of the previous implication shows that \( T_\phi^* f = c \), we obtain that \( c \in \text{ran } T_\phi^* \), as needed. \( \diamondsuit \)
From (a) of Theorem 2.2.16, it easily follows that
\[
\ker \mathcal{T}_\Phi = \{ (c_\phi)_\phi \in \ell_2(E_\Phi) : \sum_{\phi \in \Phi} \hat{c}_\phi [\hat{\psi}, \hat{\phi}] = 0 \},
\]
with \( \psi \) some (any) generator of \( S \).

2.3. Frames in FSI spaces

In order to lift the results of the previous section from PSI spaces to FSI spaces, we need first the following FSI analog of Result 2.2.9 (cf. Theorem 1.7 in [BDR1]):

**Result 2.3.1.** Let \( \Phi \) be a finite subset of \( L_2(\mathbb{R}^d) \). A function \( f \in L_2(\mathbb{R}^d) \) is in \( S := S(\Phi) \) if and only if there exists \( \tau := (\tau_\phi)_{\phi \in \Phi} \), with each \( \tau_\phi \) a 2\( \pi \)-periodic function, such that
\[
\hat{f} = \sum_{\phi \in \Phi} \tau_\phi \hat{\phi}.
\]

Several different approaches are available for the analysis of frames in FSI spaces. We have chosen here the one which incorporates efficiently the results on PSI frames that were established in the previous subsection. We do that by studying first the straightforward case when the finite generating set \( \Phi \) of \( S \) induces an orthogonal decomposition of \( S \) into the sum \( \oplus_{\phi \in \Phi} S(\phi) \) of PSI spaces. We then reduce the general setup to that simple case.

Recall that, by (1.2.3), the space \( S(\phi) \) is orthogonal to the space \( S(\psi) \) if and only if \( [\hat{\phi}, \hat{\psi}] = 0 \), a.e. Thus, the sum \( \sum_{\phi \in \Phi} S(\phi) \) is orthogonal if and only if the Gramian matrix \( G \) is diagonal.

**Proposition 2.3.3.** If the Gramian matrix \( G \) is diagonal, then:
(a) \( E_\phi \) is a Bessel set if and only if, for each \( \phi \in \Phi \), \( \hat{\phi} \) is bounded on \( \sigma \phi = \sigma (S(\phi)) \). Furthermore,
\[
\| \mathcal{T}_\phi \| = \max_{\phi} \| \mathcal{T}_\phi \| = \max_{\phi} \| \hat{\phi} \|_{L_{\infty}(\mathbb{R}^d)}.
\]
(b) \( E_\phi \) is a frame for \( S(\Phi) \) if and only if, for each \( \phi \in \Phi \), \( \hat{\phi} \) and \( 1/\hat{\phi} \) are bounded on \( \sigma \phi \). The frame is tight if and only if, for every \( \phi, \hat{\phi} = \text{const} \) on \( \sigma \phi \) (with const independent of \( \phi \)). Furthermore,
\[
\| T_{\Phi}^{-1} \| = \max_{\phi} \| T_{\phi}^{-1} \| = \max_{\phi} \| 1/\hat{\phi} \|_{L_{\infty}(\sigma \phi)}.
\]

**Proof.** The orthogonal sum decomposition \( \oplus_{\phi} S(\phi) \) of \( S(\Phi) \) implies that \( T_\phi^* \) agrees with \( T_{\phi}^* \) on \( S(\phi) \) (recall that we naturally embed the target space \( \ell_2(E_\phi) \) of the latter into the target space \( \ell_2(E_\phi) \) of the former). Since \( \ell_2(E_\Phi) \) is (always) the orthogonal sum \( \oplus_{\phi} \ell_2(E_\phi) \), we conclude that, indeed,
\[
\| \mathcal{T}_\Phi \| = \| \mathcal{T}_\phi^* \| = \max_{\phi \in \Phi} \| \mathcal{T}_{\phi}^* \| = \max_{\phi \in \Phi} \| \mathcal{T}_\phi \|,
\]
and
\[
\| T_{\Phi}^{-1} \| = \| T_{\phi}^{-1} \| = \max_{\phi \in \Phi} \| T_{\phi}^{-1} \| = \max_{\phi \in \Phi} \| T_{\phi}^{-1} \|.
\]
The result then follows by an application of parts (a-c) of Theorem 2.2.7. ☐
In accordance with the definitions of §1.4, we define here
\[ \Lambda(w) \]
to be the largest eigenvalue of \( G(w) \),
\[ \lambda(w) \]
to be the smallest eigenvalue of \( G(w) \), and
\[ \lambda^+(w) \]
to be the smallest non-zero eigenvalue of \( G(w) \). Then, both \( \Lambda(w) \) and \( \lambda^+(w) \) are non-negative and well-defined on \( \sigma S \). Further, Proposition 2.3.3 can be stated as follows:

If \( G \) is diagonal, then \( E_\Phi \) is a Bessel set if and only if \( \|\Lambda\|_{L_\infty(\sigma S)} < \infty \). \( E_\Phi \) is a frame for \( S(\Phi) \) if and only if

\[ (2.3.4) \quad \Lambda \text{ and } 1/\lambda^+ \text{ are (essentially) bounded on the spectrum of } S, \]

and, moreover, the frame bounds of \( E_\Phi \) are \( \|\Lambda\|_{L_\infty(\sigma S)} \) and \( \|1/\lambda^+\|_{L_\infty(\sigma S)} \).

As Theorem 2.3.6 below asserts, the above characterizations are valid for general FSI spaces.

The proof of Theorem 2.3.6 is based on the following (technical) lemma:

**Lemma 2.3.5.** Given a finite order Hermitian matrix \( G \), whose entries are measurable functions defined on some domain \( \Omega \), there exists a matrix \( U := U_{\Phi \times \Phi} \) whose entries are measurable functions defined on \( \Omega \), such that \( U^*GU \) is a diagonal matrix, and \( U(w) \) is unitary for every \( w \in \Omega \).

Prior to proving the lemma, we state our theorem and show how it follows from that lemma. Part (d) of the theorem is due to [BDRI] (and was previously proved, under certain decay conditions on \( \Phi \), in [JM]). For the special case of quasi-regular FSI spaces (a notion that will be defined in the next subsection), Theorem 2.3.6 in its entirety was already proved in [BDRI] (cf. Corollary 3.30 there. In a quasi-regular FSI space \( S \), \( \lambda^+ = \lambda \) on \( \sigma S \), and hence the [BDRI]-analysis, which is based only on the functions \( \lambda \) and \( \Lambda \), can still go through).

**Theorem 2.3.6.** Let \( \Phi \subset L_2(\mathbb{R}^d) \) be finite with corresponding Gramian matrix \( G \), and corresponding eigenvalue functions \( \Lambda, \lambda, \) and \( \lambda^+ \). Then

(a) \( E_\Phi \) is a Bessel set if and only if \( \Lambda \) is essentially bounded. Furthermore,

\[ \|T_\Phi\|^2 = \|\Lambda\|_{L_\infty(\sigma S(\Phi))}. \]

(b) A Bessel set \( E_\Phi \) is also a frame if and only if \( 1/\lambda^+ \) is bounded on the spectrum of \( S(\Phi) \). In such a case,

\[ \|T_\Phi^{-1}\|^2 = \|1/\lambda^+\|_{L_\infty(\sigma S(\Phi))}. \]

(c) \( E_\Phi \) is a tight frame if and only if \( \Lambda = \lambda^+ = \text{const} \) on \( \sigma S(\Phi) \).
(d) The Bessel set $E_{\phi}$ is a stable basis for $S(\Phi)$ if and only if $1/\lambda$ is essentially bounded.

**Proof.** Let $U := (u_{\phi,\phi'})_{\phi,\phi' \in \Phi}$ be the unitary matrix from Lemma 2.3.5 (with respect to $G := G_{\Phi}$). Define

$$\Psi := \{ \tilde{\psi}_\phi : \tilde{\psi}_\phi := (U^{T} \hat{\Phi})_\phi := \sum_{\phi' \in \Phi} u_{\phi',\phi} \hat{\phi}, \ \phi \in \Phi \}.$$  

Since $U(w)$ is unitary for every $w \in T^d$, it follows that $U$, considered as an endomorphism of $L_2^2$, is also unitary. From that it easily follows that $\Psi \subset L_2(T^d)$ (in fact, $\sum_{\psi \in \Psi} ||\hat{\psi}||^2 = \sum_{\phi \in \Phi} ||\hat{\phi}||^2$). Thus, $\Psi \subset S(\Phi)$ by Result 2.3.1. Similarly, since $\hat{\Phi} = \mathcal{U} \hat{\Psi}$, $\Phi \subset S(\Psi)$, and, consequently, $S(\Psi) = S(\Phi)$. Further, $G_{\Psi} = U^* G_{\Phi} U$, hence $G_{\Phi}$ and $G_{\Psi}$ have the same eigenvalue functions.

To prove (a), we let $J_{\Phi}$ and $J_{\Psi}$ be defined as in (1.4.3). Then $J_{\Psi} = J_{\Phi} U$. Since $U$ is unitary, $J_{\Psi}$ is bounded if and only if $J_{\Phi}$ is, and the two maps have the same norm. Therefore, $E_{\Psi}$ is a Bessel set of $S(\Psi) = S(\Phi)$ if and only if $E_{\Phi}$ is so. Consequently, (a) follows from Proposition 2.3.3 and the fact that, for each $w$, $\{ \tilde{\psi}(w) \}_{\psi \in \Psi}$ are the eigenvalues of the diagonal matrix $G_{\Psi}(w)$.

The proofs of (b), (c) and (d) are similar. ♠

Now, we turn to proof of the Lemma.

**Proof of Lemma 2.3.5.** Since, for each $w \in \Omega$, the Hermitian matrix $G(w)$ can certainly be unitarily diagonalized, the actual goal of the proof is to achieve the required measurability.

Let $\Lambda_j(w)$, $w \in \Omega$, $j = 1,\ldots,n := \# \Phi$ denote the $j$th smallest eigenvalue of $G(w)$. Our first goal is to show that $\Lambda_j$ is a measurable function. For that we need the following claim.

**Claim 2.3.7.** Let $\{a_m\}_{m=0}^{n-1}$ be a set of convergent sequences $a_m : \mathbb{N} \to \mathbb{R}$. Let $a_m(0)$ denote the limit of $(a_m(k))_{k=1}^{\infty}$. For each non-negative integer $k$, let $q_k$ be the univariate polynomial

$q_k(t) := t^n + \sum_{m=0}^{n-1} a_m(k)t^m.$

Assume that each $q_k$ has only real roots, and let $\Lambda_{k,j}$ denotes the $j$th smallest root of $q_k$. Then $\Lambda_{k,j} \to_{k \to \infty} \Lambda_{0,j}$, for each $j = 1,\ldots,n$.

**Proof of Claim 2.3.7.** For each $k \geq 0$, let $\Lambda_k$ be the vector $(\Lambda_{k,j})_{j=1}^{n}$. It is clear that $(\Lambda_k)_{k \in \mathbb{N}}$ is bounded (in $\mathbb{R}^n$), hence it suffices to show that $\Lambda_0$ is the only limit point of $(\Lambda_k)_k$. In this regard, we note that, a limit point $\ell_j$ of the sequence $(\Lambda_{k,j})_k$, is a zero of $q_0$, since $\sum_{i=0}^{n} a_i(t^i)$ is a continuous function of $a_0,\ldots,a_n,t$.

To prove that the sequence $(\Lambda_k)_{k \in \mathbb{N}}$ has only one limit point, we let $l := (l_j)_{j=1}^{n}$ be a limit point of $(\Lambda_k)_k$. Then, it is clear that $(l_j)_j$ is non-decreasing, and, as observed above, all the $n$ entries of $l$ are roots of $q_0$. Since $q_0$ has only $n$ roots, $l$ will be proved to equal $\Lambda_0$ as soon as we show the following: “If $\theta$ occurs $m$ times in $l$, then its multiplicity as a root of $q_0$ is at least $m$”.  

Assume, therefore, that, $l_{s+1} = l_{s+2} = \ldots = l_{s+m} = \theta$, for some $s$ and $m$. Let $(k_i)_{i=1}^{\infty}$ be a set of increasing integers for which $(\Lambda_{k_i})_i$ converges to $l$. By Rolle’s theorem, for each fixed $r = 0,\ldots,m-1$, the $r$th order derivative $q_k^{(r)}$ of the polynomial $q_k$ would have a zero $z_k$ in the convex hull of $\{z_{k_i} \}_{i=0}^{m}$. Since, as $i \to \infty$, that convex hull shrinks to 0 (since each $(\Lambda_{k_i+r})_i$ converges to $l_{s+r} = \theta$), $z_k$ converges to $\theta$. Thus, $\theta$ is a limit point of roots of $(q_k^{(r)})_i$, $r = 0,\ldots,m-1$, hence $\theta$ is a root of $q_0$ of multiplicity $\geq m$, as claimed.
After establishing the claim, we can prove the measurability of the eigenfunctions $\Lambda_j$ as follows. We approximate the matrix $G$ by Hermitian matrices $G_k$ whose entries are simple measurable functions that converge (say, pointwise) to the entries of $G$. Let $q_0(w, \cdot)$ be the characteristic polynomial of $G(w)$, and $q_k(w, \cdot)$ the characteristic polynomial of $G_k(w)$, $k = 1, 2, \ldots$. Since the coefficients of $q_k(w, \cdot)$ are simple measurable functions, so is the $j$th smallest eigenvalue function $\Lambda_{k,j}(w)$ of $G_k(w)$. On the other hand, the coefficients of $q_k(w, \cdot)$ converge to the corresponding coefficients of $q_0(w, \cdot)$. Since $G(w)$ and $G_k(w)$, $k \in \mathbb{N}$ are Hermitian, their characteristic polynomials have only real roots. By the previous claim, this implies that, for every $j = 1, \ldots, n$, and for every $w$, the eigenvalue functions $(\Lambda_{k,j}(w))_k$ converge to $\Lambda_j(w)$. Thus, each $\Lambda_j$ is the pointwise limit of measurable functions, hence is measurable.

Finally, we construct the columns of $U$ inductively. Assume by induction that we already found $V = \{v_1, \ldots, v_{j-1}\}$ vectors whose entries are measurable functions, such that $Gv_i = \Lambda_i v_i$, for each $i = 1, \ldots, j-1$, and such that $\{v_1(w), \ldots, v_{j-1}(w)\}$ is an orthonormal set for every $w \in \Omega$.

For each $w$, let $k(w)$ be the largest integer that satisfies $\Lambda_j(w) = \Lambda_{j-k(w)}(w)$. For $k = 0, \ldots, n-1$, set $K_k := \{w \in \Omega : k(w) = k\}$. Then $(K_k)_k$ forms a measurable partition of $\Omega$. On each set $K_k$, we augment the matrix $\Lambda_j I - G$ by adding the row vectors $v_{j-k}, \ldots, v_{j-1}$ and obtain in this way a matrix $R$ with measurable entries, that satisfies rank $R(w) < n$, for every $w \in K_k$.

Precisely, rank $R(w) = n - m(w) + k$, where $m(w) > k$ is the multiplicity of $\Lambda_j(w)$. Applying the proof of Lemma 2.4 of [JS], we obtain a measurable vector $v_j$ such that $Rv_j = 0$ on $K_k$, and for every $w \in K_k$, $v_j(w)$ (considered as vector in $\mathbb{R}^n$) has norm 1. Since $R(w)v_j(w) = 0$, $w \in K_k$, $v_j(w)$ is an eigenvector of $G(w)$, and is orthogonal to $\{v_{j-k}(w), \ldots, v_{j-1}(w)\}$. It is also orthogonal to $v_i(w)$, $i < j - k$ as well, since $v_i(w)$ is an eigenvector that corresponds to the eigenvalue $\Lambda_i(w)$ which is different from the eigenvalue $\Lambda_j(w)$ of $v_j(w)$.

Incidentally, the proof of Theorem 2.3.6 shows that every FSI space can be written as a finite orthogonal sum of PSI spaces. This fact was established before in [BDR1] (cf. Theorem 3.5 there). It leads to the following interesting corollary.

**Corollary 2.3.8.** Given any FSI space $S$, there exists a finite subset $\Psi \subset S$ whose corresponding shift-invariant set $E_\Psi$ is a tight frame for $S$.

**Proof.** We write $S$ as a finite orthogonal sum of PSI spaces $\{S(\eta)\}_{\eta \in \Psi}$. By (d) of Theorem 2.2.7, each $S(\eta)$ contains a function $\psi_\eta$ whose shifts $E_{\psi_\eta}$ form a tight frame for $S(\eta)$, say, with frame bound 1. The totality $\{\psi_\eta\}_{\eta \in \Psi}$ is the required $\Psi$.

In general, there are many ways to write $S$ as an orthogonal sum, and, therefore, $S$ contains many tight frames. Though the norms of the individual generators $\psi \in \Psi$ depend in general on the specific $\Psi$ chosen, the sum $\sum_{\psi \in \Psi} \|\psi\|^2$ depends only on the space $S$, that is: it is the same for all tight frames $E_\Psi$ whose frame bound is 1, and whose corresponding $S(\psi)$, $\psi \in \Psi$ form an orthogonal decomposition of $S$.

### 2.4. Frames in quasi-regular FSI spaces

We had proved in the last subsection that every FSI space contains a shift-invariant tight frame. However, not every FSI space contains a shift-invariant stable basis. A partial solution to
that difficulty was offered in [BDR1] via the more general notion of quasi-stable generating sets. That notion was defined in (3.16) of [BDR1], and is closely related to the notion of frames. In fact, Definition 1.3.5 here allows us to rephrase Definition 3.16 of [BDR1] as follows:

**Definition 2.4.1.** Let $\Phi$ be a finite generating set for the FSI space $S$. We say that (the shifts $E_\phi$ of) $\Phi$ is (are) a quasi-stable generating set, if (i): $E_\phi$ is a frame for $S$; (ii):

$$C_{T_\phi} = \{ c = (c_\phi)_{\phi \in \Phi} \in \ell_2(E_\phi) : \text{supp } \hat{\phi}_c \subseteq \sigma S, \forall \phi \in \Phi \}.$$ 

Note that quasi-stability coincides with stability whenever $\sigma S = \mathbb{T}^d$, i.e., whenever $S$ is regular (indeed, if $S$ is regular and $\Phi$ is quasi-stable, then $C_{T_\phi} = \ell_2(E_\phi)$, and hence $\ker T_\phi = \{0\}$). Even with this weakening of the stability notion, [BDR1] shows that not every FSI space has a quasi-stable basis (we have proved, in Corollary 2.3.8, that every FSI space has a shift-invariant frame, and even a tight one, therefore, the existence of a quasi-stable basis really relies on the structure of $C_{T_\phi}$). Spaces that do have quasi-stable bases are termed in [BDR1] as quasi-regular. We discuss here several properties of frames in quasi-regular FSI spaces, which may not be valid in more general FSI spaces. One of these is an explicit representation for the orthogonal projector onto $S$: [BDR1] obtains such formulas for quasi-regular spaces by a Cramer-rule-like expression (see (1.9) there). On the other hand, we know from Proposition 1.3.8 that the orthogonal projector can also be represented by using a frame for $S$ and its dual frame, and this will lead us to an alternative representation of this projector.

Before we state our first result, we recall the definition of a quasi-basis from [BDR1]: The finite $\Phi$ is a quasi-basis for the FSI space $S$ if $\det G_\Phi$ is non-zero a.e. on $\sigma S$. We mention, [BDR1], that the existence of a quasi-basis for $S$ is equivalent to the quasi-regularity of $S$, and that every quasi-stable basis is also a quasi-basis but not vice versa. The cardinality of the quasi-basis is the length $\text{len} S$ of $S$ and is shown in [BDR1] to depend only on $S$.

**Proposition 2.4.2.** Let $\Phi$ be a finite quasi-basis for the (quasi-regular) FSI space $S$. Assume that $E_\Phi$ is a Bessel set. Then,

$$C_{T_\Phi} = \{ c = (c_\phi)_{\phi \in \Phi} \in \ell_2(E_\phi) : \text{supp } \hat{\phi}_c \subseteq \sigma S \}. \tag{2.4.3}$$

**Proof.** Denoting the right hand side of equation (2.4.3) by $C_\Phi$, we will show that (i): $C_{T_\Phi} \subseteq C_\Phi$, and (ii): $\ker T_\Phi \cap C_\Phi = \{0\}$. Since $C_{T_\Phi}$ is the orthogonal complement of $\ker T_\Phi$, (2.4.3) would then follow from (i) and (ii) combined.

The required (ii) was proved in [BDR1]: Corollary 3.11 there asserts that, since $\Phi$ is a quasi-basis, the map

$$\ell_2(E_\phi) \ni c \mapsto \hat{T_\Phi} c = \sum_{\phi \in \Phi} \hat{\phi}_c$$

is 1-1 on $C_\Phi$.

As for (i), given $f \in S$, supp $\hat{f}$ lies in the $2\pi$-periodic extension $(\sigma S)^c$ of $\sigma S$. Thus, if, for some $c = (c_\phi)_{\phi \in \Phi} \in \ell_2(E_\phi)$, each supp $\hat{\phi}_c$ is disjoint of $\sigma S$, we have

$$\hat{T_\Phi} c = \sum_{\phi \in \Phi} \hat{\phi}_c = 0.$$
This means that the space
\[ K_\Phi := \{ c \in \ell_2(E_\Phi) : \text{supp} \hat{c}_\phi \cap \sigma S \text{ is a null-set, } \forall \phi \in \Phi \} \]
lies in \( \ker \mathcal{T}_\Phi \). Since \( C_\Phi \) is clearly the orthogonal complement of \( K_\Phi \), we obtain (i) by applying orthogonal complements to the inclusion \( K_\Phi \subset \ker \mathcal{T}_\Phi \).

**Theorem 2.4.4.** Let \( \Phi \) be a finite generating set for the quasi-regular FSI space \( S \). Then \( \Phi \) is a quasi-stable generating set if and only if it is a quasi-basis and its corresponding shifts \( E_\Phi \) form a frame for \( S \).

**Proof.** If \( E_\Phi \) is quasi-stable, then, by definition, it is a frame, and it is also a quasi-basis by virtue of Proposition 3.18 of [BDR1].

Conversely, if \( \Phi \) is a quasi-basis and \( E_\Phi \) is a frame, then, for the quasi-stability of \( \Phi \), it remains to show that \( C_{\mathcal{T}_\Phi} \) has the required structure. This follows from Proposition 2.4.2 and the assumption that \( \Phi \) is a quasi-basis.

We mention that, given a quasi-regular FSI space \( S \), there exist shift-invariant frames \( E_\Phi \) for \( S \) which are not quasi-stable (hence do not form a quasi-basis). For example, the length of a PSI space is 1, and hence any quasi-basis for it is formed by the shifts of single function \( \phi \). At the same time, frames for PSI spaces that consist of the shifts of several functions exist, and, in fact, were discussed in detail in §2.1.

The proof of the second implication in the above theorem could also be done through eigenvalue functions. The argument is as follows. Since \( E_\Phi \) is a frame, Theorem 2.3.6 implies that the eigenvalue function \( \Lambda(w) \) \( (\lambda^+(w)) \) is essentially bounded above (away from zero) on \( \sigma S \). However, since \( G_\Phi(w) \) is invertible a.e. on \( \sigma S \) (since \( \Phi \) is a quasi-basis), it follows that \( \lambda(w) = \lambda^+(w) \), a.e. on \( \sigma S \), where \( \lambda(w) \) is the smallest eigenvalue function. Thus \( \Lambda(w) \) is essentially bounded above and \( \lambda(w) \) is bounded below on \( \sigma S \). By Corollary 3.30 of [BDR1], \( \Phi \) is a quasi-stable generating set.

In the rest of the subsection, we consider frame-dual frame representations of the orthogonal projector onto a quasi-regular FSI space \( S \). The idea is to use the fact that, given a general frame \( X \) for \( H \) and a dual frame \( RX \), the map \( TRX^* \) is always the identity on \( H \). Before we develop that direction further, we point out a relevant result. If \( X \) is a stable basis, then the condition \( TRX^* = I_H \) is not only necessary but also sufficient for \( RX \) to be the dual of \( X \). The result below shows that, in the shift-invariant setup, that sufficiency assertion extends to quasi-stable sets:

**Corollary 2.4.5.** Let \( E_\Phi \) be a quasi-stable basis for the FSI space \( S \), and let \( R \) be some map from \( \Phi \) into \( S(\Phi) \). If \( E_{R\Phi} \) is a Bessel set, then \( E_{R\Phi} \) is the dual frame of \( E_\Phi \) if (and only if) \( TR_{R\Phi}^* \) is the identity on \( S \), that is, if

\[
(2.4.6) \quad f = \sum_{\phi \in \Phi, \alpha \in \mathbb{Z}^d} \langle f, E^\alpha \phi \rangle E^\alpha R\phi, \quad \forall f \in S.
\]

**Proof.** After extending \( R \) from \( \Phi \) to \( E_\Phi \) by the rule \( R^\alpha \phi := E^\alpha R\phi \), we appeal to Proposition 1.3.7. That proposition validates the “only if” implication, and reduces the proof of the “if” implication to proving that \( C_{R^\Phi} = C_{\mathcal{T}_\Phi} \). Furthermore, Proposition 2.4.2 asserts that \( C_{\mathcal{T}_\Phi} \) is the same for all quasi-bases \( \Psi \) of \( S \).
Since \( \Phi \) is already known to be a quasi-basis (by virtue of its quasi-stability, cf. Theorem 2.4.4), it suffices to show that \( R\Phi \) is also a quasi-basis. The proof of this statement goes as follows. Since \( R\Phi \subset S \), we have \( S(R\Phi) \subset S \). This, together with (2.4.6), shows that \( E_{R\Phi} \) is fundamental in \( S \), and hence \( S(R\Phi) = S(\Phi) \). Since \( \Phi \) is a quasi-basis for \( S \), its cardinality is the length, \( \text{len} S \), of \( S \). Therefore, \( \#(R\Phi) \leq \# \Phi = \text{len} S \). However, as asserted by Theorem 3.12 of [BDR1], every generating set of a quasi-regular FSI space \( S \) that contains no more than \( \text{len} S \) elements must be a quasi-basis.  

**Theorem 2.4.7.** Assume that the shifts \( E_\Phi \) of the finite \( \Phi \) form a quasi-stable basis for the FSI space \( S \). Then the Fourier transforms of the generators \( R\Phi \) of the dual quasi-stable basis are given, on \( \sigma S \), by

\[
\overline{R\Phi} = \overline{G_{\Phi}^{-1}} \hat{\Phi}.
\]

with \( G_{\Phi}^{-1} \) the (pointwise) inverse of \( G_{\Phi} \).

**Proof.** Since \( R \) should invert \( T_\phi T_\phi^* \), we compute first \( T_\phi T_\phi^* \Phi \). Here, we use (2.1.2) (and the fact that \( T_\phi T_\phi^* = \sum_{\phi \in \Phi} T_\phi T_\phi^* \)) to conclude that

\[
(T_\phi T_\phi^* \Phi)^\wedge = (\sum_{\phi \in \Phi} [\hat{\phi}, \hat{\phi}^*])_{\phi \in \Phi} = G_{\Phi} \hat{\Phi}.
\]

Since \( G_{\Phi} \) is invertible on \( \sigma S \) (and is zero elsewhere), the claim follows.  

By Proposition 1.3.8, \( T_{R\Phi} T_{R\Phi}^* \) is the orthogonal projector \( \mathcal{P}_S \) of \( L_2(\mathbb{R}^d) \) on \( S \). The last result thus allows us to write

\[
\mathcal{P}_S f = \sum_{\phi, \phi' \in \Phi} [\hat{f}, \hat{\phi}] g_{\phi, \phi'} \hat{\phi'},
\]

with \((g_{\phi, \phi'})_{\phi, \phi' \in \Phi} = G_{\Phi}^{-1}\). Instead, we could have solved the equation \( G_{\Phi} \overline{R\Phi} = \hat{\Phi} \) by applying Cramer’s rule. That attempt would have resulted in the form for \( \mathcal{P}_S \) that was discussed in [BDR1].

3. Infinitely generated SI spaces

3.1. General

The study of FSI subspaces of \( L_2(\mathbb{R}^d) \) is pertinent to Approximation Theory, where one attempts to approximate from small, simple spaces of approximants. In other areas (such as wavelets) the main goal is to find an attractive basis for the entire \( L_2(\mathbb{R}^d) \) or to a “big” subspace of it. We therefore analyse in this section shift-invariant subspaces of \( L_2(\mathbb{R}^d) \) generated by a countable set of generators.
Our results on FSI spaces were stated in terms of the matrix spectrum of each of the “fiber” matrices $G(w), w \in \mathbb{T}^d$. We pause here momentarily in order to have a closer look at the potential practical value of the obtained characterizations. Assuming we hold in hand the Gramian matrix, the characterization of stability and of the Bessel property are of a more favorable nature than those of frames and tight frames: in many cases, the estimation of the largest eigenvalue $\Lambda(w)$ and the smallest eigenvalue $\lambda(w)$ of $G(w)$ can be done directly in terms of the entries of $G(w)$ (as we did in §1.6). However, estimating the smallest non-zero eigenvalue $\lambda^+(w)$, would, almost certainly, require the application of a costly iterative process. Consequently, the kind of characterization of FSI frames that was obtained in Theorem 2.3.6 seems to be practically less useful than its stability counterpart. This can also be viewed as follows: the invertibility of a certain operator is a more accessible property than its partial invertibility.

A partial solution to the above problem is obtained with the addition of the complementary dual Gramian analysis that will be developed. Indeed, as was already explained in the introduction, the Gramian analysis is engaged with the decomposition of the operator $T_\Phi^* T_\Phi$, while in the dual case the operator $T_\Phi^* T_\Phi^*$ is the object. In two respects, there is a significant difference between these two operators: the stability of a Bessel set $E_\Phi$ is equivalent to the invertibility of $T_\Phi^* T_\Phi$, but is not so nicely reflected by $T_\Phi T_\Phi^*$ (this latter operator should be partially invertible and onto $\ell_2(E_\Phi)$, two hard-to-verify properties). On the other hand, a fundamental frame for $L_2(\mathbb{R}^d)$ is characterized nicely through $T_\Phi T_\Phi^*$ (should be invertible), and is hard to be analysed via $T_\Phi^* T_\Phi$.

In summary, Gramian analysis is best suited for the study of stable bases, while dual Gramian analysis is particularly good for fundamental frames for $L_2(\mathbb{R}^d)$, hence, indeed, the two approaches complement each other.

In view of the above, one may wonder why we have not employed the dual Gramian analysis for the study of frames in FSI spaces. The answer for that is as follows: since an FSI space is always a proper subspace of $L_2(\mathbb{R}^d)$, a frame for it is never fundamental in $L_2(\mathbb{R}^d)$. For the analysis of frames which are not fundamental, both Gramian analysis and dual Gramian analysis require the (hard-to-verify) partial invertibility of their associated operator, hence the switch from the finite-order Gramian $G$ to infinite-order dual Gramian $\hat{G}$ provides no gain.

Throughout the section, we use the notation $\Sigma A$ for the spectrum of the operator $A$; namely, given a bounded linear endomorphism $A$ of a Hilbert space $H$, we denote

$$\Sigma A := \{ \lambda \in \mathbb{C} : \text{the inverse of } \lambda I - A \text{ is undefined or unbounded} \}.$$  

To make a clear distinction between this notion and the spectrum $\sigma S(\Phi)$ of $S(\Phi)$, we will always refer to the former as the the operator spectrum.

### 3.2. Gramian Analysis: SI spaces as the limit of FSI spaces

Two different approaches for the study of SI spaces are employed here. The first, that we discuss in the present subsection, attempts to extend the results from §2 on FSI spaces to general SI spaces, by viewing the latter as a certain limit of the former. That approach leads to the desired characterizations of the Bessel property and of the stability property, but is short of characterizing frames. Therefore, we will develop, (in §3.4) an alternative method, where we inspect directly the
operator spectrum of each of the “fibers” $G(w)$. This latter direction is more powerful, alas, much more involved, whence our decision to present both approaches.

The “going-to-the-limit” argument is almost self-suggestive, and is based on an elementary observation. Let $X$ be a countable subset of the Hilbert space $H$. Given any subset $Y \subset X$, let $H_Y$ be the closure in $H$ of the finite span of $Y$ (that is, $Y$ is fundamental in $H_Y$). As before, the operator $T_Y$ is defined on $\ell_0(Y)$, and, if bounded, is extended to the entire $\ell_2(Y)$ by continuity. Further, $\ell_2(Y)$ is isometrically embedded in $\ell_2(X)$ in the usual way.

For a set $X \subset H$, a chain

$$\ldots \subset X_{n-1} \subset X_n \subset X_{n+1} \subset \ldots$$

that satisfies $\cup_n X_n = X$ is called a \textit{filtration} of $X$.

\textbf{Theorem 3.2.1}. Let $X$ be a countable fundamental set of the Hilbert space $H$. Suppose that $\{X_n\}_n$ is a filtration of $X$, i.e., $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$, and $\cup_n X_n = X$. Denote $T := T_X$, $T_n := T_{X_n}$, $H_n := H_{X_n}$. Then:

(a) $X$ is a Bessel set if and only if the following condition holds “each $X_n$ is a Bessel set, and

$$\sup_n \|T_n\| < \infty$$. In such a case, $\|T\| = \sup_n \|T_n\| = \lim_{n \to \infty} \|T_n\|$.

(b) Assume $X$ is a Bessel set. Then, $X$ is a stable basis for $H$ if and only if the following condition holds “each $X_n$ is a stable basis for $H_n$, and $\sup_n \|T_n^{-1}\| < \infty$”. In such a case,

$$\|T^{-1}\| = \sup_n \|T_n^{-1}\| = \lim_{n \to \infty} \|T_n^{-1}\|.$$

(c) Assume $X$ is a Bessel set. Then, $X$ is a frame for $H$ if the following condition holds “for infinitely many $n$, $X_n$ is a frame for $H_n$, and $\liminf_n \|T_{n1}^{-1}\| < \infty$”. In such a case, $\|T_{1}^{-1}\| \leq \liminf_n \|T_{n1}^{-1}\|$.

\textbf{Proof}. The boundedness and invertibility of $T (T_n)$ is determined by its action on the finitely supported sequences $\ell_0(X) (\ell_0(X_n))$ in $\ell_2(X) (\ell_2(X_n))$. Assertions (a) and (b) thus follow from the fact that, since $\{X_n\}_n$ is a filter of $X$, $\ell_0(X)$ is the union of $\ell_0(X_n)$.

(c): Without loss, we may assume that each $X_n$ is a frame for $H_n$, and that $(\|T_{n1}^{-1}\| = \|T_{n1}^{-1}\|)_n$ converges (otherwise, we take a subsequence). Set $A := \lim \|T_{n1}^{-1}\|^{-1}$. Since $T$ is bounded, $A < \infty$. More importantly, by our assumptions here $A > 0$. Now, let $f \in H$. Given $\varepsilon > 0$, we can find, for all sufficiently large $k$, an element $f_k \in H_k$ so that $\|f - f_k\| \leq \max \{\|T\|, \|T_k^{-1}\|^{-1}\}$. Then,

$$\|T^* f\| \geq \|T^* f_k\| - \varepsilon \geq \|T_k^* f_k\| - \varepsilon.$$ Also,

$$\|T_k^* f_k\| \geq \|T_k^* f_k\|^{-1} \|f_k\| \geq \|T_k f_k\|^{-1} \|f\| - \varepsilon.$$ In summary, for every $f \in H$ and for all sufficiently large $k$,

$$\|T^* f\| \geq \|T_k^* f_k\|^{-1} \|f\| - 2\varepsilon.$$ By taking $k \to \infty$, we obtain that $\|T^* f\| \geq A \|f\| - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the desired result follows. \hfill \Box

Let $S$ be a shift-invariant space generated by the countable set $\Phi$. Let $(\Phi_n)_n$ be a filtration of $\Phi$ by finite sets. Then, $(E_n := E_{\Phi_n})_n$ is a filtration of $E_{\Phi}$ that employs FSI sets. Let $\Lambda_n$, $\lambda_n$ and $\lambda_n^*$ be the eigenvalue functions of $E_n$ (cf. the paragraph after Proposition 2.3.3). Combining Theorem 2.3.6 and Theorem 3.2.1, we obtain the following result.

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Corollary 3.2.2. With \( \Phi \subset L_2(\mathbb{R}^d) \) a countable set, with \( (\Phi_n)_n \) a filtration of \( \Phi \) that is made of finite sets, and with \( \Lambda_n, \lambda_n \) and \( \lambda_n^+ \) as above, we have

(a) \( E_\Phi \) is a Bessel set if and only if the function set \( \{ \Lambda_n \}_n \) is bounded in \( L_\infty(\mathbb{T}^d) \). Furthermore,
\[
\| \Phi \|^2 = \sup_n \| \Lambda_n \|_{L_\infty(\mathbb{T}^d)}.
\]

(b) Assume \( E_\Phi \) is a Bessel set. Then it is also a stable basis for \( S \) if and only if the function set \( \{ 1/\lambda_n \}_n \) is bounded in \( L_\infty(\mathbb{T}^d) \). Furthermore, \( \| T^{-1} \| = \sup_n \| 1/\lambda_n \|_{L_\infty(\mathbb{T}^d)} \). (Here, \( 1/0 := \infty \).)

(c) Assume \( E_\Phi \) is a Bessel set. Then it is also a frame if the following holds: “for each \( n \), the function \( 1/\lambda_n^+ \) is bounded on the spectrum \( \sigma_n \) of the FSI space \( S(\Phi_n) \), and \( \liminf_n \| 1/\lambda_n^+ \|_{L_\infty(\sigma_n)} < \infty \).”

The analysis of \( E_\Phi \) for a finite \( \Phi \) was done by a spectral-like decomposition of \( T_\Phi \) into the simpler fiber operators. For a countable \( \Phi \), we can still derive from (a) and (b) of the last corollary similar decomposition results.

We recall the functions \( \Lambda(w) \), \( \lambda(w) \) and \( \lambda^+(w) \) that were defined in the introduction. Note that for a finite \( \Phi \) these definitions coincide with the definitions of \( \Lambda(w) \), \( \lambda(w) \) and \( \lambda^+(w) \) as eigenvalue functions. Given now a filtration \( (\Phi_n)_n \) of \( \Phi \), Corollary 3.2.2 implies that \( \| T_\Phi \|^2 = \lim_n \| \Lambda_n \|_{L_\infty(\mathbb{T}^d)} \). Moreover, it is straightforward to show that, monotonically, \( \Lambda_n(w) \rightarrow \Lambda(w) \), and \( \lambda_n(w) \rightarrow \lambda(w) \) a.e. on \( \mathbb{T}^d \). This implies that \( \Lambda \) and \( \lambda \) are measurable, and, further, since the convergence \( \Lambda_n \rightarrow \Lambda \) and \( \lambda_n \rightarrow \lambda \) is monotone,
\[
\| \Lambda \|_{L_\infty(\mathbb{T}^d)} = \lim_{n \rightarrow \infty} \| \Lambda_n \|_{L_\infty(\mathbb{T}^d)},
\]
and
\[
\| 1/\lambda \|_{L_\infty(\mathbb{T}^d)} = \lim_{n \rightarrow \infty} \| 1/\lambda_n \|_{L_\infty(\mathbb{T}^d)}.
\]

Thus we obtain the following extension of (a) and (b) of Theorem 2.3.6:

Theorem 3.2.3. Let \( \Phi \) be a countable subset of \( L_2(\mathbb{R}^d) \) with Gramian matrix \( G \). Let \( \Lambda(w) := \| G(w) \| \) and \( \lambda(w) := \| G(w)^{-1} \|^{-1} \). Then:

(a) \( E_\Phi \) is a Bessel set if and only if \( \Lambda \) is essentially bounded. Moreover, we have \( \| T_\Phi \|^2 = \| \Lambda \|_{L_\infty(\mathbb{T}^d)} \).

(b) Suppose \( E_\Phi \) is a Bessel set. Then \( E_\Phi \) is a stable basis if and only if \( 1/\lambda \) is essentially bounded. Moreover, we have \( \| T_\Phi^{-1} \|^2 = \| 1/\lambda \|_{L_\infty(\mathbb{T}^d)} \).

Theorem 3.2.3 provides characterizations of the Bessel property and the stability property that, though were derived with the aid of the FSI results, are stated explicitly in terms of the fiber operators \( G(w), w \in \mathbb{T}^d \). Such a characterization is valid for frames, but, cannot be derived with the aid of the filtration argument. Therefore, we develop in §3.4 a direct approach that decompose \( G \) without the use of a filter. Since the proofs there are lengthy and technical, we postpone that development until after the dual Gramian analysis is presented.

3.3. Dual Gramian analysis

The starting point of the Gramian analysis is the fact that both \( G \) and its fibers \( (G(w))_w \) can be viewed as densely defined operator on \( L_2^2 \) and \( \ell_2(\Phi) \), respectively. An analogous statement about the dual Gramian is less obvious, and we need surmount here new obstacles.
The first (though, minor) difficulty that one should note is the well-definedness of the entries of the dual Gramian: while the Gramian entries \([\hat{\phi}, \hat{\psi}]\), \(\phi, \psi \in \Phi\) are in \(L_1(\mathbb{R}^d)\) hence well-defined a.e. regardless of the choice of the set \(\Phi\), the same cannot be said about the entries

\[
\sum_{\phi \in \Phi} \hat{\phi}(\cdot + \alpha)\hat{\phi}(\cdot + \beta), \quad \alpha, \beta \in 2\pi \mathbb{Z}^d
\]

of the dual Gramian \(\tilde{G}\). We start our discussion by settling that question.

Assume that \(E_\Phi\) is a Bessel set. Then, since \(\sum_{\phi \in \Phi} \|\mathcal{T}_\phi f\|^2 = \|\mathcal{T}_{\hat{\phi}} f\|^2 < \infty\), and since the Fourier transform is an isometry on \(L_2(\mathbb{R}^d)\), we conclude from (2.1.2) that

\[
\sum_{\phi \in \Phi} \|\hat{f}, \hat{\phi}\|_{L_2(\mathbb{R}^d)}^2 < \infty, \quad \forall f \in L_2(\mathbb{R}^d).
\]

Choosing now \(f\) as the inverse Fourier transform of the characteristic function of the cube \(\alpha + [-\pi, \pi]^d, \alpha \in 2\pi \mathbb{Z}^d\), we compute that \([\hat{f}, \hat{\phi}] = \hat{\phi}(\cdot - \alpha)|_c\), and therefore,

\[
\|\hat{f}, \hat{\phi}\|_{L_2(\mathbb{R}^d)}^2 = \|\hat{\phi}\|_{L_1(\mathbb{R}^d)}^2.
\]

Thus, we have proved that the sum

\[
\sum_{\phi \in \Phi} |\hat{\phi}(\cdot + \alpha)|^2
\]

is \(L_1(\mathbb{R}^d)\)-convergent, hence is also convergent pointwise a.e. Since that sum is the \((\alpha, \alpha)\)-entry of the dual Gramian, we conclude the following:

**Proposition 3.3.1.** Let \(\Phi\) be a countable subset of \(L_2(\mathbb{R}^d)\), and assume that \(E_\Phi\) is a Bessel set. Then, for each \(\alpha, \beta \in 2\pi \mathbb{Z}^d\), the \((\alpha, \beta)\)-entry

\[
\sum_{\phi \in \Phi} \hat{\phi}(\cdot + \alpha)\hat{\phi}(\cdot + \beta)
\]

of the dual Gramian matrix converges absolutely a.e. to an element of \(L_1(\mathbb{R}^d)\).

**Proof.** For \(\alpha = \beta\), the assertion was proved in the paragraph preceding the proposition. The extension to a general pair \((\alpha, \beta)\) follows from Schwartz’ inequality. ♠

Since the Bessel property of the set \(E_\Phi\) is the weakest property of that set of interest to us here, we may assume hereafter that, for all \(\alpha, \beta \in 2\pi \mathbb{Z}^d\), the sum that defines the \((\alpha, \beta)\)-entry

\[
\tilde{G}_{\alpha, \beta}
\]

of the dual Gramian converges absolutely a.e.

Another, more substantial, difficulty occurs upon attempting to prove that dual Gramian operator can be evaluated, i.e., that, under “reasonable assumptions”

\[
(\tilde{G}f)(w) = \tilde{G}(w)f_w, \quad \text{for a.e. } w \in \mathbb{R}^d.
\]
Here, as before \( f_{w} := f_{|w+2\pi \mathbb{Z}^{d}} \). Recall that the dual Gramian operator \( \tilde{G} \) is defined as \( \tilde{G} := J_{\Phi}J_{\Phi}^{*} \), i.e.,

\[
\tilde{G} : f \mapsto \sum_{\phi \in \Phi}[f, \hat{\phi}]\hat{\phi}.
\]

If \( E_{\Phi} \) is a Bessel set, the above sum must converge in \( L_{2}(\mathbb{R}^{d}) \), for every \( f \in L_{2}(\mathbb{R}^{d}) \). However, interpreting the above sum in the non-Bessel case is a non-obvious task. On the other hand, the connection between \( \tilde{G} \) and its evaluation \( \tilde{G}(w) \) is important even when \( E_{\Phi} \) is not Bessel, since, otherwise, we will not be able to use the fibers \( \{ \tilde{G}(w) \}_{w \in \mathbb{T}^{d}} \) for the characterization of the Bessel property. For this reason, we view, to this end, \( \tilde{G} \) as a quadratic form rather than as an operator, i.e., make use of the connection

\[
\langle \tilde{G}\hat{f}, \hat{f} \rangle = \sum_{\phi \in \Phi} \| (T_{\phi}^{*} f) \|^2 = \| \sum_{\phi \in \Phi} [\hat{f}, \hat{\phi}]^2 \|_{L_{2}(\mathbb{T}^{d})}.
\]

Assuming \( \hat{f} \) is compactly supported, we may use the a.e. finiteness of \( \sum_{\phi \in \Phi} \| \hat{\phi}(+\alpha)\hat{\phi}(+\beta) \| \) to sum by parts as follows:

\[
\sum_{\phi \in \Phi} \| \hat{\phi}(w) \|^2 = \sum_{\phi \in \Phi, \alpha, \beta \in 2\pi \mathbb{Z}^{d}} \hat{f}(w+\alpha)\overline{\hat{f}(w+\beta)}\hat{\phi}(w+\beta)\overline{\hat{\phi}(w+\alpha)}
\]

\[
= \sum_{\alpha, \beta} \hat{f}(w+\alpha)\overline{\hat{f}(w+\beta)} \sum_{\phi \in \Phi} \hat{\phi}(w+\beta)\overline{\hat{\phi}(w+\alpha)}
\]

\[
= (f_{lw})^{*} \tilde{G}(w)f_{lw}.
\]

Therefore, we conclude that

**Lemma 3.3.2.** Let \( \Phi \) be a countable subset of \( L_{2}(\mathbb{R}^{d}) \).

(a) If, for some \( \alpha, \beta \in 2\pi \mathbb{Z}^{d} \), the sum \( \sum_{\phi \in \Phi} \| \hat{\phi}(+\alpha)\hat{\phi}(+\beta) \| \) is infinite on a set of positive measure, then \( E_{\Phi} \) is not a Bessel set.

(b) If the above sum is finite a.e. for every \( \alpha, \beta \in 2\pi \mathbb{Z}^{d} \), then, for every band-limited \( f \),

\[
\| T_{\Phi} f \|^2 = (2\pi)^{-d} \int_{\mathbb{T}^{d}} (f_{lw})^{*} \tilde{G}(w)f_{lw} \, dw.
\]

The dual Gramian analysis can now be developed along lines parallel to the development of the Gramian analysis. For that, we set, for \( \alpha \in 2\pi \mathbb{Z}^{d} \), \( S_{\alpha} \) to be the subspace of \( L_{2}(\mathbb{R}^{d}) \) consisting of those functions whose Fourier transform is supported (up to a null-set) in \( \alpha + [-\pi..\pi]^{d} \). \( S_{\alpha} \) is a translation-invariant space. In fact, it is also a PSI space, and is generated by \( \chi_{\alpha} \), with \( \chi_{\alpha} \) the support function of \( \alpha + [-\pi..\pi]^{d} \) (cf. Result 2.2.9). We consider the restriction \( T_{\Phi, \alpha} \in \mathbb{Z}^{d} \) of \( T_{\phi}^{*} \) to the space \( S_{\alpha} \), and observe that, for \( \alpha \in \mathbb{T}^{d} \) and \( f \in S_{\alpha} \), the quadratic form \( f_{lw}^{*} \tilde{G}(w)f_{lw} \), \( w \in \mathbb{T}^{d} \), is reduced to \( \hat{f}(w+\alpha)\tilde{G}_{\alpha, \alpha}(w)\hat{f}(w+\alpha) = \tilde{G}_{\alpha, \alpha}(w)\| \hat{f}(w+\alpha) \|^2 \), and therefore

\[
\| T_{\Phi, \alpha} f \|^2 = (2\pi)^{-d} \| \tilde{G}_{\alpha, \alpha} \| \hat{f}(w+\alpha) \|^2 \|_{L_{2}(\mathbb{T}^{d})}.
\]

Since also \( \| f \|_{L_{2}(\mathbb{T}^{d})} = (2\pi)^{-d} \| \hat{f}(w+\alpha) \|^2 \|_{L_{1}(\mathbb{T}^{d})} \), the norm bounds on the restricted operator \( T_{\Phi, \alpha} \) and its inverse are the same as those of the map

\[
L_{1}(\mathbb{T}^{d}) \ni \tau \mapsto \tilde{G}_{\alpha, \alpha} \tau.
\]

Thus, in complete analogy with Theorem 2.2.7 (cf. the argument used in the proof of Theorem 2.2.14) we have the following.
Proposition 3.3.3. Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable (or finite), and assume that the sum $\sum_{\phi \in \Phi} |\phi|^2$ converges a.e. Then, for every $\alpha \in 2\pi\mathbb{Z}^d$:

(a) The restricted operator $T_{\Phi,\alpha}^*$ is bounded if and only if the function $\tilde{G}_{\alpha,\alpha}$ is essentially bounded. Furthermore,
\[
\|T_{\Phi,\alpha}^*\|^2 = \|\tilde{G}_{\alpha,\alpha}\|_{L_\infty(\mathbb{T}^d)}.
\]

(b) Assume $T_{\Phi,\alpha}^*$ is bounded. Then it is also invertible if and only if the function $1/\tilde{G}_{\alpha,\alpha}$ is essentially bounded. Further,
\[
\|T_{\Phi,\alpha}^{-1}\|^2 = \|1/\tilde{G}_{\alpha,\alpha}\|_{L_\infty(\mathbb{T}^d)}.
\]

(c) Assume $T_{\Phi,\alpha}^*$ is bounded. Then it is also partially invertible if and only if $1/\tilde{G}_{\alpha,\alpha}$ is essentially bounded on its support $\tilde{\sigma}_\alpha \subset \mathbb{T}^d$. Further,
\[
\|T_{\Phi,\alpha}^{-1}\|^2 = \|1/\tilde{G}_{\alpha,\alpha}\|_{L_\infty(\tilde{\sigma}_\alpha)}.
\]

The dual Gramian analogue of the FSI results (i.e., Theorem 2.3.6) is obtained by restricting $T_{\Phi}^*$ to a larger space of band-limited functions. Here, we take $Z$ to be any finite subset of $2\pi\mathbb{Z}^d$, and define $\Omega_Z := Z + [-\pi, \pi]^d$. We then consider the restriction $T_{\Phi, Z}^*$ of $T_{\Phi}^*$ to the space
\[
S_Z := \{ f \in L_2(\mathbb{R}^d) : \text{supp } \hat{f} \subset \Omega_Z \}.
\]

Given $g$ defined on $\Omega_Z$, and $w \in \mathbb{T}^d$, we denote by
\[
g_Z(w)
\]
the vector $(g(w + z) : z \in Z)$. Also,
\[
\tilde{G}_Z
\]
stands for the finite-order matrix obtained from the dual Gramian $\tilde{G}_\Phi$ by deleting all rows and columns not in $Z$. From Lemma 3.3.2,
\[
\|T_{\Phi} f\|^2 = (2\pi)^{-d} \|\hat{f}_Z \tilde{G}_Z \hat{f}_Z\|_{L_1(\mathbb{T}^d)}, \quad \forall f \in S_Z.
\]

Then, following the arguments in §2.3 (that is, establishing the analogous result of Proposition 2.3.3 and invoking then Lemma 2.3.5), we obtain the following analogue of Theorem 2.3.6:

Proposition 3.3.4. Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable and assume that $\sum_{\phi \in \Phi} |\phi|^2$ is finite a.e. Let $Z$ be a finite subset of $2\pi\mathbb{Z}^d$, and let $T_{\Phi, Z}^*$ be the restriction of $T_{\Phi}^*$ to $S_Z$. Let $\tilde{\Lambda}_Z$, $\tilde{\lambda}_Z$ and $\tilde{\lambda}_Z^+$ be the eigenvalue functions defined as $\Lambda$, $\lambda$ and $\lambda^+$ of §2.3, but with respect to the dual Gramian $\tilde{G}_Z$. Then:

(a) $T_{\Phi, Z}^*$ is bounded if and only if $\tilde{\Lambda}_Z$ is essentially bounded on $\mathbb{T}^d$. Furthermore, $\|T_{\Phi, Z}^*\|^2 = \|\tilde{\Lambda}_Z\|_{L_\infty(\mathbb{T}^d)}$.

(b) Assume $T_{\Phi, Z}^*$ is bounded. Then it is also invertible if and only if $1/\tilde{\lambda}_Z$ is essentially bounded on $\mathbb{T}^d$. Furthermore, $\|T_{\Phi, Z}^{-1}\|^2 = \|1/\tilde{\lambda}_Z\|_{L_\infty(\mathbb{T}^d)}$.

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(c) Assume $T_{\Phi, z}$ is bounded. Then it is also partially invertible if and only if $\tilde{\lambda}^+_{z}$ is essentially bounded on $\tilde{\sigma}_{z} := \{w \in \mathbb{T}^d : \tilde{G}_z (w) \neq 0\}$. Furthermore, $\|T_{\Phi, z}\|^{-1} = \|1/\tilde{\lambda}^+_{z}\|_{L_{\infty}(\tilde{\sigma}_{z})}$.

To extend Proposition 3.3.4 from spaces of the form $S_z$ to the entire $L_2(\mathbb{R}^d)$, we use some filtration

$$Z_0 \subset Z_1 \subset Z_2 \subset \ldots$$

of $2\pi \mathbb{Z}^d$. It induces a corresponding filtration of $\mathbb{R}^d$:

$$\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \ldots,$$

where $\Omega_j := Z_j + [-\pi..\pi]^d$. In this way we obtain the increasing space sequence

$$S_{Z_0} \subset S_{Z_1} \subset S_{Z_2} \subset \ldots$$

whose union $S$ is dense in $L_2(\mathbb{R}^d)$. Denoting by $T_n^*$ the restriction of $T_{\phi}^*$ to $S_{Z_n}$, we conclude that the boundedness and invertibility of $T_{\phi}^*$ are completely determined by its restriction to $S$ (which is the space of all band-limit functions). Therefore, we have the following analog of Theorem 3.2.3:

**Theorem 3.3.5.** Let $\Phi$ be a countable subset of $L_2(\mathbb{R}^d)$. Then:

(a) If the sum $\sum_{\phi \in \Phi} |\tilde{\phi}|^2$ diverges on some positive measure set, $E_\phi$ is not a Bessel set.

(b) Assume that $\sum_{\phi \in \Phi} |\tilde{\phi}|^2$ is finite a.e., and let $\tilde{G}$ be the dual Gramian of $E_\phi$. Further, let $\tilde{\lambda}$ and $\tilde{\lambda}$ be defined by

$$\tilde{\lambda}(w) := \|\tilde{G}(w)\|, \quad \tilde{\lambda}(w) := 1/\|\tilde{G}(w)^{-1}\|, \quad w \in \mathbb{T}^d.$$  

Then:

(b1) $E_\phi$ is Bessel set if and only if $\tilde{\lambda}$ is essentially bounded. Furthermore,

$$\|T_\phi\|^2 = \|\tilde{\lambda}\|_{L_{\infty}(\mathbb{T}^d)}.$$  

(b2) Assume that $E_\phi$ is a Bessel set. Then $E_\phi$ is a fundamental frame if and only if the following condition holds: “for a.e. $w$, $\tilde{G}(w)$ is boundedly invertible, and the hence well-defined function $1/\tilde{\lambda}$ is essentially bounded”. Furthermore, $\|T_\phi^*\|^{-2} = \|1/\tilde{\lambda}\|_{L_{\infty}(\mathbb{T}^d)}$.

Theorem 3.3.5 leads to an interesting conclusion concerning tight frames. Tight frames $E_\phi$ are characterized by the equality $\|T_\phi\| \|T_{\phi}^*\|^{-1} = 1$. The theorem shows that the latter condition is equivalent to the equality

$$\tilde{\lambda}(w) = \tilde{\lambda}(w) = \text{const, } \quad \text{for a.e. } w \in \mathbb{T}^d.$$  

The equality $\tilde{\lambda}(w) = \tilde{\lambda}(w)$ says that the operator spectrum of $\tilde{G}(w)$ consists of a single point, which can happen if and only if $\tilde{G}(w)$ is a scalar operator. This leads to the following:

**Corollary 3.3.6.** Let $\Phi$ be a countable subset of $L_2(\mathbb{R}^d)$. Then $E_\phi$ is a fundamental tight frame for $L_2(\mathbb{R}^d)$ if and only if there exists a constant $\text{const}$ such that, for every $\alpha, \alpha' \in 2\pi \mathbb{Z}^d$, and for almost every $w \in \mathbb{T}^d$,

$$\sum_{\phi \in \Phi} \phi(w + \alpha) \overline{\phi(w + \alpha')} = \text{const} \delta_{\alpha, \alpha'}.$$  

(3.3.7)
Proof. If the sum in (3.3.7) does not converge absolutely for some \( \alpha, \alpha' \) and on a set of positive measure, then, by Theorem 3.3.5, \( E_\Phi \) is not a Bessel set. Otherwise, the condition in (3.3.7) implies, Theorem 3.3.5, that \( E_\Phi \) is a Bessel set. Also, that condition implies that \( E_\Phi \) is fundamental: if not, there exists \( f \in L_2(\mathbb{R}^d) \) so that \( T_\Phi^* f = 0 \), hence \( \hat{G} \hat{f} = 0 \), implying thus that \( \hat{G}(w)\hat{f}_w = 0 \), a.e., in contradiction to the assumed structure of \( \hat{G}(w) \) in (3.3.7).

Therefore, when proving the required equivalence, we may assume, without loss, that \( E_\Phi \) is a fundamental Bessel set. The claim then follows from the arguments preceding the present corollary.

If \( X \) is a tight frame, then, up to a scalar multiple, it forms its own dual. The above result is thus a special case of a general relation between a shift-invariant fundamental frame and its dual (cf. Corollary 4.2).

### 3.4. Analysis of frames which are not fundamental in \( L_2(\mathbb{R}^d) \)

Theorems 3.2.3 and 3.3.5 provide us with the desired characterizations of the Bessel property (twice), the stability property, and the property of being a fundamental frame for \( L_2(\mathbb{R}^d) \). It fails to provide similar characterizations for frames of a shift-invariant proper subspace of \( L_2(\mathbb{R}^d) \) (unless that frame happens to be a stable basis). The present subsection is aimed at settling this remaining problem. After a brief introduction, we state the main theorem that will be proved here. The proof details then follow.

Let \( B \) be a bounded operator from a Hilbert space \( H \) into a Hilbert space \( H' \), and let \( A := B^*B \). Let \( \Sigma A \) be the operator spectrum of \( A \). We define

\[
\lambda^+(A) := \inf \{ \mu : \mu \in \Sigma A \setminus 0 \}.
\]

The operator \( A \) is partially invertible if and only if \( \lambda^+(A) > 0 \), and the norm of the partial inverse is \( 1/\lambda^+(A) \) (the “only if” implication is quite clear. The argument for the “if” statement can be found in the proof of the implication (b) \( \implies \) (a) of Theorem 3.4.1).

Given a Bessel set \( E_\Phi \) with Gramian \( G \) and dual Gramian \( \tilde{G} \), our two objectives are to connect (a): between the function

\[
\lambda^+(w) := \lambda^+(G(w)), \quad w \in \mathbb{T}^d,
\]

and the number \( \lambda^+(G) \); (b) between the function

\[
\tilde{\lambda}^+(w) := \lambda^+(\tilde{G}(w)), \quad w \in \mathbb{T}^d,
\]

and the number \( \lambda^+(\tilde{G}) \). Since \( \lambda^+(G) = \lambda^+(\tilde{G}) = \|T_{\Phi}^{-1}\|^{-1} \) (with \( \infty^{-1} := 0 \)), we will obtain in this way two characterizations of frames. In fact, we will prove the following:

**Theorem 3.4.1.** Let \( \Phi \) be a countable subset of \( L_2(\mathbb{R}^d) \), and assume that \( E_\Phi \) is a Bessel set. Let \( \sigma \Phi := \text{supp } G = \text{supp } \tilde{G} \subset \mathbb{T}^d \). Then the following conditions are equivalent:

(a) \( E_\Phi \) is a frame, and the norm of the partial inverse of \( T_{\Phi} \) is \( K < \infty \).

(b) The function \( \lambda^+ \) is bounded away from zero on \( \sigma \Phi \), and \( \|1/\lambda^+\|_{L^{\infty}(\sigma \Phi)} = K^2 \).


(c) The function $\tilde{\lambda}^+$ is bounded away from zero on $\sigma \Phi$, and $\|1/\tilde{\lambda}^+\|_{L_\infty(\sigma \Phi)} = K^2$.

The equivalence of (b) and (c) is quite straightforward. (Since $E_\Phi$ is Bessel, then, by Theorems 3.2.3 and 3.3.5, both $G(w)$ and $\hat{G}(w)$ are bounded for a.e. $w$. Since $G(w)$ is the product $J_\Phi^*(w)J_\Phi(w)$, and $\hat{G}(w)$ is the product of the same matrices in reversed order, $\Sigma(G(w))$ and $\Sigma(\hat{G}(w))$ can differ only by the single point $\{0\}$. Thus, $\lambda^+$ and $\tilde{\lambda}^+$ are equal pointwise.) We will prove here the equivalence of (a) and (b). The proof of the implication $(b) \implies (a)$ is based on the following lemma.

**Lemma 3.4.2.** Let $E_\Phi$ be a Bessel set, and let $\tau \in L_2^\Phi$, $G := G_\Phi$. Then,

(a) $\tau \in \ker G$ if and only if $\tau(w) \in \ker G(w)$ for almost every $w$.

(b) $\tau \in C_G := (\ker G)^\perp$ if and only if $\tau(w) \in C_w := (\ker G(w))^\perp$, for a.e. $w$.

**Proof.** The first assertion is obvious, since $(G\tau)(w)$ is $G(w)\tau(w)$. As for (b), assume first that $\tau(w) \in C_w$ for a.e. $w$. Then, for an arbitrary $\tau' \in \ker G$,

$$\langle \tau, \tau' \rangle_{L_2^\Phi} = \int_{\mathbb{T}^d} \langle \tau(w), \tau'(w) \rangle_{\ell_2(\Phi)} dw = 0,$$

since, by (a), $\tau'(w) \in \ker G(w) = C_w^\perp$, a.e. Therefore, $\tau \in (\ker G)^\perp = C_G$.

Conversely, assume that $\tau \in C_G$. If $\tau \in \text{ran } G$, then $\tau = G\tau_0$, for some $\tau_0$, hence, for a.e. $w$ (precisely, whenever $G(w)$ is bounded, and $\tau_0(w) \in \ell_2(\Phi)$), $\tau(w) = G(w)\tau_0(w) \in \text{ran } G(w) \subset C_w$. If $\tau \notin \text{ran } G$, it can still be approximated in $L_2^\Phi$ by a sequence $(\tau_n)_n \subset \text{ran } G$ (since $\text{ran } G$ is dense in $C_G$). By switching to a subsequence, if necessary, we may assume that, for almost every $w$, $(\tau_n(w))_n$ converges in $\ell_2(\Phi)$ to $\tau(w)$. Combining this with the argument in the beginning of the paragraph, we conclude that, for almost every $w$, $(\tau_n(w))_n$ is in $C_w$ and converges in the $\ell_2(\Phi)$-norm to $\tau(w)$. Since $C_w$ is certainly closed, we obtained that $\tau(w) \in C_w$, a.e. $\blacksquare$

**Proof of the implication (b) $\implies$ (a) in Theorem 3.4.1.**

We will prove that, assuming (b), $E_\Phi$ is a frame, and $\|T_\Phi^{-1}\| \leq 1/\lambda^+ \|_{L_\infty(\sigma \Phi)}$.

Assume that $1/\lambda^+$ is essentially bounded on $\sigma \Phi$, and let $\tau \in C_G \setminus 0$. By Lemma 3.4.2, $\tau(w) \in C_w$, a.e. on $\mathbb{T}^d$. We claim that, a.e., if $G(w) \neq 0$, it is partially invertible, i.e., bounded below on $C_w$. Indeed, the restriction $G(w)_\| \text{ of } G(w)$ to $C_w$ is (always) injective. Furthermore, since $\lambda^+(w) > 0$, the operator spectrum of $G(w)$ is disjoint from the non-empty interval $(0, \lambda^+(w))$. Therefore, the operator spectrum of $G(w)_\|$ is also disjoint from $(0, \lambda^+(w))$. Since $G(w)_\|$ is non-negative and injective, 0 cannot be an isolated point of its spectrum, hence it must be invertible. The argument also shows that $\|G(w)_\|^{-1}\| = 1/\lambda^+(w)$.

This means that, for a.e. $w$, if $\tau(w) \neq 0$, then

$$(3.4.3) \quad \|G(w)\tau(w)\|_{\ell_2(\Phi)} \geq \frac{\|\tau(w)\|_{\ell_2(\Phi)}}{\|G(w)_\|^{-1}} \geq \frac{\|\tau(w)\|_{\ell_2(\Phi)}}{1/\lambda^+ \|_{L_\infty(\sigma \Phi)}}.$$

For $\tau \in L_2^\Phi$,

$$\|\tau\|^2_{L_2^\Phi} = \int_{\mathbb{T}^d} \|\tau(w)\|^2_{\ell_2(\Phi)},$$

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hence also
\[ \|G\tau\|_{L^2_\Phi}^2 = \int_{\mathbb{T}^d} \|G(w)\tau(w)\|_{\ell^2_\Phi}^2, \]

hence (3.4.3) implies that
\[ \|G\tau\|_{L^2_\Phi} \geq \frac{\|\tau\|_{L^2_{\Phi}}}{{1/\lambda^+}\|\Omega\|_{L^\infty(\sigma\Phi)}}. \]

Therefore, \( G \) is partially invertible, and hence, Proposition 1.4.11, \( E_{\Phi} \) is a frame. Also, \( \|\mathcal{T}_{\phi}^{-1}\|^2 = \|G^{-1}\|^2 \leq \|1/\lambda^+\|_{L^\infty(\sigma\Phi)} \), with the inequality by the proof above, and the equality by Proposition 1.4.11.

\[ \star \]

**Proof of the implication (a) \implies (b) in Theorem 3.4.1.**
Since we will need, in the next section, a closely related result, we will prove herein the following more general statement:

**Theorem 3.4.4.** Let \( G \) be a non-negative self-adjoint bounded endomorphism of \( L^2_\Phi \). Let \( (G(w))_w \) be a collection of non-negative self-adjoint bounded endomorphisms of \( \ell^2_\Phi \), that satisfy, for every \( \tau \in L^2_\Phi \), and for a.e. \( w \in \mathbb{T}^d \), \( (G\tau)(w) = G(w)\tau(w) \). Let \( \Lambda(w) := \|G(w)\| \), and assume that \( \Lambda \in L^\infty(\mathbb{T}^d) \). Let \( \lambda^+(w) := \inf\{\mu \in \Sigma(G(w)) \setminus \{0\} \}. \) Let \( \Omega \) be the set \( \Omega := \{w \in \mathbb{T}^d : G(w) \neq 0 \} \).

If \( G \) is partially invertible, then \( 1/\lambda^+ \) is essentially bounded on \( \Omega \), and
\[ \|1/\lambda^+\|_{L^\infty(\Omega)} \leq \|G^{-1}\|. \]

The fact that Theorem 3.4.4 is a generalization of the required implication (a) \implies (b) is clear.

To this end, we prove Theorem 3.4.4.

In the proof, we use the following lemma, whose proof is postponed until after the proof of Theorem 3.4.4 is done.

**Lemma 3.4.5.** Under the conditions of Theorem 3.4.4, there exists a countable dense subset \( D \) of \( \ell^2_\Phi \), and a null-set \( Z \subset \Omega \), such that, for every \( c \in D \), for every \( w' \in \Omega \setminus Z \), and every \( \varepsilon > 0 \), the set
\[ K_{c,w',\varepsilon} := \{w \in \Omega : \|(G(w) - G(w'))c\|_{\ell^2_\Phi} < \varepsilon \|c\|_{\ell^2_\Phi} \} \]
has a positive measure.

**Proof of Theorem 3.4.4.** Let \( D \) and \( Z \) be the sets specified in the above lemma. Recall also the notations \( C_G := (\ker G)^\perp \), \( C_w := (\ker G(w))^\perp \).

Choose any \( w' \in \Omega \setminus Z \), and let \( \mu > 0 \) be any point in the operator spectrum \( \Sigma(G(w')) \). We will construct an element \( \tau \in C_G \), for which
\[ \|G\tau\|_{L^2_\Phi} \leq (1 + \delta)\mu\|\tau\|_{L^2_\Phi}, \]
with \( \delta \) positive and arbitrarily close to 0. This would yield that \( \|G^{-1}\| \geq 1/\mu \), implying thus that \( \lambda^+(w') > 0 \), and that
\[ \|G^{-1}\| \geq 1/\lambda^+(w'). \]
Since \( Z \) is a null-set, we will then conclude that
\[
\|G_1^{-1}\| \geq \|1 / \lambda^+\|_{L^\infty(\Omega)},
\]
which is the desired result.

The actual construction of \( \tau \) in (3.4.6) is as follows: we will find \( 0 \neq \tau \in L_2^\Phi \), supported in \( A \times \Phi \), where \( A \subset \Omega \) is some set of positive measure, such that (i): \( \tau(w) \in C_w \), for every \( w \in \mathbb{T}^d \), and (ii): \( \|G(w)\tau(w)\|_{\ell_2(\Phi)} \leq (1 + \delta)\mu\|\tau(w)\|_{\ell_2(\Phi)} \). Condition (i) would imply (as in Lemma 3.4.2) that \( \tau \in C_G \), while condition (ii) is needed for the conclusion that \( \|G\tau\|_{L_2^\Phi} \leq (1 + \delta)\mu\|\tau\|_{L_2^\Phi} \) (cf. the two displays after (3.4.3)).

In general, for the sake of (i) above, it might be hard to know whether a particular sequence lies in \( C_w \). The most efficient way is, probably, to select elements in ran \( G(w) \) (and use the fact that ran \( G(w) \) is dense in \( C_w \), by virtue of the self-adjointness of \( G(w) \)). Indeed, our element \( \tau \) will be defined as
\[
\tau(w) := \begin{cases} G(w)c, & w \in A, \\ 0, & \text{otherwise}, \end{cases}
\]
with \( c \) some fixed sequence in \( \ell_2(\Phi) \).

Here are the details: since \( \mu \in \Sigma(G(w')) \), \( G(w') - \mu I \) has no bounded inverse, and so we can find an element \( c \in \ell_2(\Phi) \), such that \( \|c\|_{\ell_2(\Phi)} = 1 / \mu \), and
\[
\|G(w')c - \mu c\|_{\ell_2(\Phi)} \leq \varepsilon,
\]
with \( \varepsilon > 0 \) arbitrarily small. It follows then that
\[
\|G(w')c\|_{\ell_2(\Phi)} \leq 1 + \varepsilon.
\]

Since \( G(w') \) is bounded and \( D \) is dense in \( \ell_2(\Phi) \), we may assume that span \( c \cap D \neq \emptyset \). Therefore, by Lemma 3.4.5, there exists a subset \( A \) of \( \Omega \) with positive measure, such that
\[
\|(G(w) - G(w'))c\|_{\ell_2(\Phi)} \leq \varepsilon / \mu, \quad \forall w \in A.
\]

We define \( \tau \in L_2^\Phi \) by
\[
\tau(w) := \begin{cases} G(w)c, & w \in A, \\ 0, & \text{otherwise}. \end{cases}
\]

Thus, condition (i) (i.e., that \( \tau(w) \in C_w \), all \( w \)) is satisfied. Also, the uniform boundedness of the operators \( \{G(w)\}_{w \in \mathbb{T}^d} \) easily implies that \( \tau \in L_2^\Phi \). Thus, to complete the proof, it remains to show that, for almost all \( w \in \mathbb{T}^d \),
\[
\|G(w)\tau(w)\|_{\ell_2(\Phi)} \leq (1 + \delta)\mu\|\tau(w)\|_{\ell_2(\Phi)}.
\]

This last claim is trivial for \( w \in \mathbb{T}^d \setminus A \), so we may assume that \( w \in A \). We first choose \( w = w' \).

For that specific choice, we get
\[
\|G(w')\tau(w')\|_{\ell_2(\Phi)} = \|G(w')G(w)c\|_{\ell_2(\Phi)} \leq \|\mu G(w')c\|_{\ell_2(\Phi)} + \|G(w')(G(w')c - \mu c)\|_{\ell_2(\Phi)}.
\]

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Denoting
\[ C := \| \Lambda \|_{L_{\infty}(\mathbb{R}^d)} < \infty, \]
we obtain from (3.4.9), (3.4.8), and (3.4.7) that
\[ \| G(w')\tau(w') \|_{\ell_2(\Phi)} \leq \mu(1 + \varepsilon) + C\varepsilon. \]
On the other hand, by (3.4.7),
\[ 1 - \| \tau(w') \|_{\ell_2(\Phi)} = \varepsilon \| \varepsilon \|_{\ell_2(\Phi)} - \| G(w')c \|_{\ell_2(\Phi)} \leq \varepsilon. \]
Altogether, we obtained for that case the inequality
\[ \| G(w')\tau(w') \|_{\ell_2(\Phi)} \leq \frac{\mu(1 + \varepsilon) + C\varepsilon}{1 - \varepsilon} \| \tau(w') \|_{\ell_2(\Phi)}. \]
By choosing \( \varepsilon \) sufficiently small (and adjusting \( A \) if necessary to that \( \varepsilon \)), we obtain that
\[ \| G(w')\tau(w') \|_{\ell_2(\Phi)} \leq (1 + \delta)\mu \| \tau(w') \|_{\ell_2(\Phi)}. \]
To extend that to a general \( w \in A \), we show that both \( \tau(w) - \tau(w') \) and \( G(w')\tau(w') - G(w)\tau(w) \) can be made arbitrarily small (in norm), and then invoke (3.4.11). First, since span \( c \cap D \neq \emptyset \),
\[ \| \tau(w) - \tau(w') \|_{\ell_2(\Phi)} = \| (G(w) - G(w'))c \|_{\ell_2(\Phi)} \leq \varepsilon/\mu. \]
Therefore, \( \| \tau(w) \|_{\ell_2(\Phi)} \geq \| \tau(w') \|_{\ell_2(\Phi)} - \varepsilon/\mu \geq 1 - \varepsilon - \varepsilon/\mu \), the second inequality by (3.4.10). This verifies that \( \tau(w) - \tau(w') \) is, indeed, small, and also means that, on \( A \), \( \tau(w) \) is being kept away from zero, a consequence that will be required shortly. Second, to estimate \( G(w)\tau(w) - G(w')\tau(w') \), we write
\[ G(w)^2 - G(w')^2 = G(w)(G(w) - G(w')) + (G(w) - G(w'))G(w'). \]
Now, since \( \| (G(w) - G(w'))c \|_{\ell_2(\Phi)} \leq \varepsilon/\mu \), we have that
\[ \| G(w)(G(w) - G(w'))c \|_{\ell_2(\Phi)} \leq C\varepsilon/\mu. \]
Also, due to (3.4.7) and the fact that \( \| (G(w) - G(w'))c \|_{\ell_2(\Phi)} \leq \varepsilon/\mu \),
\[ \| (G(w) - G(w'))G(w')c \|_{\ell_2(\Phi)} \leq \mu \| G(w) - G(w') \|_{\ell_2(\Phi)} + \| (G(w) - G(w'))c \|_{\ell_2(\Phi)} \leq \varepsilon + C\varepsilon. \]
So, we conclude from (3.4.13) that
\[ \| G(w)\tau(w) - G(w')\tau(w') \|_{\ell_2(\Phi)} = \| G(w)^2c - G(w')^2c \|_{\ell_2(\Phi)} \leq (C/\mu + 2C + 1)\varepsilon. \]
Therefore, by (3.4.11) and (3.4.12),
\[ \| G(w)\tau(w) \|_{\ell_2(\Phi)} \leq \| G(w')\tau(w') \|_{\ell_2(\Phi)} + (C/\mu + 2C + 1)\varepsilon \leq (1 + \delta)\mu \| \tau(w') \|_{\ell_2(\Phi)} + (C/\mu + 2C + 1)\varepsilon \leq (1 + \delta)\mu \| \tau(w) \|_{\ell_2(\Phi)} + (C/\mu + 2C + 2 + \delta)\varepsilon. \]
Since we have already proved that \( \| \tau(w) \|_{\ell_2(\Phi)} \) is kept away from zero, we can modify \( \varepsilon \) (hence \( A \)) to guarantee that, say,
\[ \| G(w)\tau(w) \|_{\ell_2(\Phi)} \leq (1 + 2\delta)\mu \| \tau(w) \|_{\ell_2(\Phi)}, \]
and the desired result then follows.

Finally we prove Lemma 3.4.5. For that we first recall the definition of measurable maps:
**Definition 3.4.14.** Let \( M \) be a measure space, and \( B \) a topological space. A map \( f : M \to B \) is **measurable** provided that \( f^{-1}(\Omega) \) is a measurable set in \( M \) for every open set \( \Omega \) in \( B \).

Clearly, if \( f : M \to B \) is measurable, then \( f^{-1}(U) \) is measurable for every Borel set \( U \subset B \).

**Proposition 3.4.15.** Let \( M \) be a positive measure space and \( B \) be a separable normed space. If the map \( f : M \to B \) is measurable, then, there exists a null-set \( Z \subset M \), such that, for every \( w' \in M \setminus Z \) and for arbitrary \( \varepsilon > 0 \), there is a positive-measure set \( A := A_{w',\varepsilon} \subset M \), such that for arbitrary \( w \in A \),

\[
\| f(w) - f(w') \|_B < \varepsilon.
\]

**Proof.** All norms in the proof below are B-norms.

Let \( X \) be a countable dense subset of \( B \), and let “<” be some well-ordering of \( X \). Given \( n \in \mathbb{N} \), let

\[
O_{x,n} := \{ u \in B : \| u - x \| < 1/n \}.
\]

Then \( (O_{x,n})_{x \in X} \) is an open covering of \( B \), and, defining

\[
U_{x,n} := O_{x,n} \setminus \bigcup_{y < x} O_{y,n}, \quad x \in X,
\]

we obtain a partition of \( B \) into Borel sets. That partition induces a partition

\[
(A_{x,n} := f^{-1}(U_{x,n}))_{x \in X}
\]

of \( M \) into measurable sets. We then define a map \( s_n : M \to B \) (as a matter of fact, \( \text{ran} \ s_n \subset X \)) as follows:

\[
s_n(w) = \sum_{x \in X} x \chi_{A_{x,n}}(w).
\]

Then, \( s_n \) converges to \( f \) uniformly. Indeed, we have that \( \| f(w) - s_n(w) \| < 1/n \) for all \( n \in \mathbb{N} \), \( w \in M \).

Let \( Z \) be a null-set that contains all those \( A_{x,n} \) \( (x \in X, n \in \mathbb{N}) \) whose measure is zero. Let \( w' \in M \setminus Z \). For arbitrary \( \varepsilon \), pick \( n \) with \( 2/n < \varepsilon \). Since \( w' \notin Z \), \( w' \) is in some positive-measure \( A_{x,n} \). For \( w \in A_{x,n} \),

\[
\| f(w') - f(w) \| \leq \| f(w') - s_n(w') \| + \| s_n(w') - s_n(w) \| + \| s_n(w) - f(w) \| < \varepsilon.
\]

**Proof of Lemma 3.4.5.** Let \( D \) be a dense countable subset of \( \ell_2(\Phi) \). Given \( c \in D \), let \( B_c \) be the space of all (bounded) linear operators from \( \text{span}\{c\} \) into \( \ell_2(\Phi) \).

Since we know that \( G(w), w \in \Omega \), is a bounded linear endomorphism of \( \ell_2(\Phi) \), then, certainly, \( G(w)|_{\text{span}\{c\}} \) is bounded for every \( w \in \Omega \). This defines a.e. the map

\[
f : \Omega \to B_c : w \mapsto G(w)|_{\text{span}\{c\}}.
\]

We need to prove that this map is measurable.
Given $L \in B_c$ and $w \in \Omega$, one observes that

$$\|G(w) - L\|_{B_c} = \frac{\|G(w)c - Lc\|_{\ell_2(\Phi)}}{\|c\|_{\ell_2(\Phi)}}.$$ 

Further, since $G$ is bounded, $Gc \in L^2_{\mathbb{R}^d}$, and in particular, its entries are measurable functions (for the sake of applying $G$ to $c$, $c$ should be interpreted as the element $\tau \in L^2_{\mathbb{R}^d}$ with constant entries $\tau_\phi = c_\phi$). Also, since $\|G(w)c - Lc\|_{\ell_2(\Phi)}$ is finite, the series that defines $\|G(w)c - Lc\|_{\ell_2(\Phi)}$ converges (unconditionally). Combining that with the previous observation, viz., that the entries of $Gc - Lc$ are measurable, we conclude that the map $w \mapsto \|G(w)c - Lc\|_{\ell_2(\Phi)}$ is measurable, hence so is our $f$.

An application of Proposition 3.4.15 with respect to the map $f$, yields the existence of a null-set $Z_\epsilon \subset \Omega$, such that for every $\epsilon > 0$ and every $w' \in \Omega \setminus Z_\epsilon$, the set

$$\{w : \|G(w) - G(w')\|_{B_c} < \epsilon\}$$

has a positive measure. Defining $Z := \cup_{\epsilon \in \mathbb{D}} Z_\epsilon$, we obtain that (a): $Z$ is a null-set, (b) the claim of the lemma holds for this $Z$.

4. Dual frames

Let $\Phi$ be a countable (or finite) subset of $L_2(\mathbb{R}^d)$, and assume that $E_{\Phi}$ is a frame. Let $R : \Phi \rightarrow L_2(\mathbb{R}^d)$ be some map, and assume that $E_{R\Phi}$ is a Bessel set. Let $J_{\Phi}$ and $J_{R\Phi}$ be the pre-Gramian of $\Phi$ and $R\Phi$ respectively. Our objective in this brief section is to study the property “$E_{R\Phi}$ is the dual frame of $E_{\Phi}$” via the fiber matrices $J_{\Phi}(w)$ and $J_{R\Phi}(w)$.

Our initial tool is Corollary 1.3.9. Part (b) of that Corollary says that, if $E_{R\Phi}$ is the dual of $E_{\Phi}$, then $T_{\Phi}^*T_{R\Phi}^*$ is an orthogonal projector. On the Fourier domain, this operator is represented by $J_{\Phi}J_{R\Phi}^*$ whose matrix representation is

$$J_{\Phi}J_{R\Phi}^* = \sum_{\phi \in \Phi} \hat{\phi}(\cdot + \alpha) \overline{\hat{\phi}(\cdot + \alpha')} \alpha, \alpha' \in \mathbb{Z}^{2d}.$$ 

The sum above that defines the entries of $J_{\Phi}J_{R\Phi}^*$ can be shown to converge absolutely for every $\alpha, \alpha' \in 2\pi \mathbb{Z}^{d}$, and for almost every $w \in \Pi^d$ (Schwartz’ inequality followed by an application of Proposition 3.3.1). Corollary 1.3.9 also implies that the operator $T_{\Phi}^*T_{R\Phi}^*$ is an orthogonal projector. Here, the Fourier transform analogue is $J_{\Phi}^*J_{R\Phi}$, whose matrix representation is

$$J_{\Phi}^*J_{R\Phi} = \langle [\hat{\phi'}], \hat{\phi} \rangle_{\phi, \hat{\phi} \in \Phi}.$$ 

The entries of this latter matrix are certainly well-defined (a.e.).
Lemma 4.1. With \( \Phi \) and \( \Phi R \) as above,
(a) \( T_\Phi^* T_\Phi^* \) is an orthogonal projector if and only if, for almost every \( w \in \mathbb{T}^d \), \( J_\Phi(w) J_\Phi^* R_\Phi(w) \) is an orthogonal projector (on \( \ell_2(2\pi \mathbb{Z}^d) \)).
(b) \( T_\Phi^* R_\Phi \) is an orthogonal projector if and only if, for almost every \( w \in \mathbb{T}^d \), \( J_\Phi(w) J_\Phi^* R_\Phi(w) \) is an orthogonal projector (on \( \ell_2(\Phi) = \ell_2(\Phi R) \)).

Proof. The arguments for proving (a) and (b) are essentially the same, hence we prove only (b).

Since the Fourier transformation is an isometry, we may replace in the proof the operator \( T_\Phi^* T_\Phi^* \) by its Fourier transform analogue \( J_\Phi^* R_\Phi \). Also, for the sake of notational simplicity, we set \( G := J_\Phi^* R_\Phi \), though, of course, this \( G \) is the Gramian of neither \( \Phi \) nor \( \Phi R \).

First, one checks that \( G \) is non-negative self-adjoint if and only if almost every \( G(w) \) is so.

Assume that \( G(w) \) is an orthogonal projector for a.e. \( w \). In particular, each \( G(w) \) is self-adjoint, hence, by the above, \( G \) is self-adjoint, too. To show that \( G \) is an orthogonal projector, we need to prove that \( G \tau = \tau \) for every \( \tau \in (\ker G)^\perp \). Let, therefore, \( \tau \in (\ker G)^\perp \). By a proof identical to that of Lemma 3.4.2, for a.e. \( w \in \mathbb{T}^d \), \( \tau(w) \in (\ker G(w))^\perp \). Since \( G(w) \) is assumed to be an orthogonal projector (a.e.), we conclude that \( G(w) \tau(w) = \tau(w) \) (a.e.), implying that \( G \tau = \tau \). This proves that \( G \) is an orthogonal projector, as needed.

Now assume that \( G \) is an orthogonal projector. We want to invoke here Theorem 3.4.4, hence need to verify its assumptions. The basic relation \( (G \tau)(w) = G(w) \tau(w) \) is straightforward. The fact that each \( G(w) \) is non-negative self-adjoint follows from the fact that \( G \) is assumed to be so. Finally, analogously to the derivation of (a) in Theorem 3.2.3, one proves the relation \( \|G\| = \|\Lambda\|_{L_\infty(\mathbb{T}^d)} \), with \( \Lambda(w) := \|G(w)\| \). Since \( ||G|| = 1 \) here, we conclude that, for a.e. \( w \in \mathbb{T}^d \), \( \Sigma(G(w)) \subset [0,1] \).

Now, we invoke Theorem 3.4.4. Since \( G \) is partially invertible (being an orthogonal projector), and \( \|G\|^{-1} = 1 \), that theorem tells us that \( \lambda^+(w) \geq 1 \), for almost all \( w \) that satisfy \( G(w) \neq 0 \). This implies that, a.e., \( \Sigma(G(w)) \in \{0\} \cup [1,\infty) \). Combining that with the result of the previous paragraph, we conclude that, a.e., \( \Sigma(G(w)) \subset \{0,1\} \). Each such \( G(w) \) is also known to be self-adjoint, hence must be an orthogonal projector.

In case \( E_\Phi \) is fundamental in \( L_2(\mathbb{R}^d) \), \( J_\Phi J_\Phi^* R_\Phi \) is the identity operator, and this immediately implies that almost every operator \( J_\Phi(w) J_\Phi^* R_\Phi(w) \) is the identity. Thus, we get the following:

Corollary 4.2. Let \( E_\Phi \) be a frame and let \( E_\Phi^* R_\Phi \) be its dual. Then:
(a) For every \( \alpha, \alpha' \in 2\pi \mathbb{Z}^d \), and for almost every \( w \in \mathbb{T}^d \),
\[
\sum_{\phi \in \Phi} \hat{\phi}(w + \alpha) \overline{\hat{R}\phi}(w + \alpha') = \sum_{\phi \in \Phi} \hat{\phi}(w + \alpha) \overline{\hat{\phi}}(w + \alpha').
\]
(b) If \( E_\Phi \) is fundamental in \( L_2(\mathbb{R}^d) \), then, for every \( \alpha, \alpha' \in 2\pi \mathbb{Z}^d \) and for almost every \( w \in \mathbb{T}^d \),
\[
\sum_{\phi \in \Phi} \hat{\phi}(w + \alpha) \overline{\hat{R}\phi}(w + \alpha') = \delta_{\alpha,\alpha'}.
\]

Proof. The first claim follows from the self-adjointness of the \( J_\Phi(w) J_\Phi^* R_\Phi(w) \)-matrices. The second claim follows from Lemma 4.1 and also directly from the remarks preceding the present corollary.
Corollary 1.3.9 provides us also with a sufficient condition for $E_{R\Phi}$ to be the dual frame of the frame $E_\Phi$. In the shift-invariant case, that corollary, combined with Lemma 4.1, leads to the following conclusion:

**Corollary 4.3.** Let $H$ be a closed subspace of $L_2(\mathbb{R}^d)$, and let $E_\Phi$ be a frame for $H$. Let $E_{R\Phi}$ be a Bessel set which is fundamental in $H$. Then $E_{R\Phi}$ is the dual of $E_\Phi$ if and only if for almost every $w \in \mathbb{T}^d$ each of the operators $J_\Phi^*(w)J_\Phi(w)$, $J_\Phi^*(w)J_{R\Phi}(w)$, $J_{R\Phi}(w)J_\Phi^*(w)$, and $J_\Phi(w)J_{R\Phi}^*(w)$ is an orthogonal projector.

We have stated Corollary 4.3 primarily for proving our following result. That result, though might look very special, will play a crucial role in the development of the duality principle of Weyl-Heisenberg frames in [RS1].

**Corollary 4.4.** Let $E_\Phi$ be a frame for $H \subset L_2(\mathbb{R}^d)$, with a dual $E_{R\Phi}$. Let $E_\Psi$ be a frame for $H' \subset L_2(\mathbb{R}^d)$, and let $\mathbb{R}': \Psi \rightarrow L_2(\mathbb{R}^d)$. Assume that, for almost every $w \in \mathbb{T}^d$,

$$J_\Phi(w) = J_\Psi^*(w), \quad J_{R\Phi}(w) = J_{R\Psi}^*(w).$$

(That is, for some indexing $\Phi = (\phi_\alpha)_{\alpha \in 2\pi\mathbb{Z}^d}$, and $\Psi = (\psi_\beta)_{\beta \in 2\pi\mathbb{Z}^d}$, $\tilde{\phi}_\alpha(w + \beta) = \overline{\psi}_\beta(w + \alpha)$, etc.) Then $E_{R\Psi}$ is the dual frame of $E_\Psi$.

**Proof.** Since $E_{R\Phi}$ is a frame, the equality $J_{R\Phi}(w) = J_{R\Psi}^*(w)$ easily implies (by Theorems 3.2.3, 3.3.5, and 3.4.1) that $E_{R\Psi}$ is a frame, as well.

Since $E_{R\Phi}$ is the dual frame of $E_\Phi$, then, by Corollary 4.3, for almost every $w \in \mathbb{T}^d$ each of the operators $J_{R\Phi}^*(w)J_\Phi(w)$, $J_\Phi^*(w)J_{R\Phi}(w)$, $J_{R\Phi}(w)J_\Phi^*(w)$, and $J_\Phi(w)J_{R\Phi}^*(w)$ is an orthogonal projector. By virtue of (4.5), we get that for almost every $w \in \mathbb{T}^d$ each of the operators $J_{R\Phi}^*(w)J_\Phi(w)$, $J_\Phi^*(w)J_{R\Phi}(w)$, $J_{R\Phi}(w)J_\Phi^*(w)$, and $J_\Phi(w)J_{R\Phi}^*(w)$ is an orthogonal projector. Therefore, Corollary 4.3 would imply that $E_{R\Psi}$ is a frame dual to $E_\Psi$ as soon as we show that $E_{R\Psi}$ is a fundamental set of $H'$.

Let $H''$ be the closure of the algebraic span of $E_{R\Psi}$. If $H'' \neq H'$, then, since $E_\Psi$ is fundamental in $H'$, there exists, say, some $f \in L_2(\mathbb{R}^d)$ such that $T_\Psi f = 0$, but $T_{R\Psi}^* f \neq 0$. (Otherwise, there exists $f$ such that $T_\Psi^* f \neq 0$, but $T_{R\Psi} f = 0$, and the argument below can be adapted to this case, as well). By Lemma 1.4.8, this implies that, while

$$J_\Psi^*(w)\hat{f}_w = 0, \quad \text{a.e. } w,$$

$$J_{R\Psi}^*(w)\hat{f}_w \neq 0, \quad \text{on a set of positive measure}.$$ 

On the other hand, since $E_{R\Phi}$ is the dual frame of $E_\Phi$, Proposition 1.3.7 implies that $\ker T_\Phi = \ker T_{R\Phi}$, and hence that, for a.e. $w$, $\ker J_\Phi^*(w) = \ker J_{R\Phi}(w) = \ker J_{R\Phi}^*(w) = \ker J_{R\Psi}^*(w)$, and we have reached a contradiction.

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References


