Riesz bases in subspaces of $L_2(\mathbb{R}_+)$

Tim N. T. Goodman, Charles A. Micchelli and Zuowei Shen

Abstract

In recent investigation [8] concerning the asymptotic behavior of Gram Schmidt orthonormalization procedure applied to the nonnegative integer shifts of a given function, the problem of determining whether or not such functions form a Riesz system in $L_2(\mathbb{R}_+)$ arose. In this note, we provide a sufficient condition to determine whether the nonnegative translates form a Riesz system on $L_2(\mathbb{R}_+)$. This result is applied to identify a large class of functions for which very general translates enjoy the Riesz basis property in $L_2(\mathbb{R}_+)$.

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1. Introduction

For closed linear subspaces of $L_2(\mathbb{R})$ which are spanned by the integer shifts of a family of functions useful criteria are known to insure that these functions form a Riesz basis in a subspace of $L_2(\mathbb{R})$ (cf. [2] or [6]). In fact, the Riesz basis property is equivalent to the invertibility of a Hermitian matrix (of order equal to the cardinality of family) whose elements consist of $2\pi-$periodic functions. Underlying this criterion is the fact that the relevant Gram matrix on $L_2(\mathbb{R})$ is Toeplitz and so its invertibility is equivalent to the invertibility of its symbol. This observation is very useful in any general study of Riesz basis of shift invariant subspaces which typically arises in wavelet analysis as well as in any investigation of concrete such spaces, for example, cardinal spline spaces (c.f. Schoenberg [1]).

In a recent investigation [8] concerning the asymptotic behavior of Gram-Schmidt orthonormalization procedure applied to the nonnegative integer shifts of a given function, the problem of determining whether or not such functions form a Riesz basis in $L_2(\mathbb{R}_+)$ arose. In this case, the relevant Gram matrix is *not* Toeplitz and therefore a general criterion to insure the Riesz basis property is not available. Therefore, in [8] numerical experiments were made to establish that the nonnegative integer shifts of the Gaussian function formed a Riesz basis of a subspace in $L_2(\mathbb{R}_+)$. Although these computations indicated they indeed form a Riesz basis, no proof was available. It is the intent of this paper to provide a result that not only establishes this fact, but identifies a large class of functions for which very general translates also enjoy the Riesz basis property in $L_2(\mathbb{R}_+)$.

2. Riesz basis in $L_2(\Omega)$

We begin our analysis with a lemma of some independent interest. In fact this lemma has already found another application in wavelet construction, [10].

Lemma 2.1. *Let H_1 and H_2 be closed subspaces of a Hilbert space H. For an arbitrary $x \in H_1$ and $y \in H_2$, with $\|x\| = \|y\| = 1$, and $\alpha, \beta \in \mathbb{C}$, we have*

$$\|\alpha x + \beta y\| \geq \left((1 - C_{H_1,H_2})(|\alpha|^2 + |\beta|^2)\right)^{1/2}$$

where

$$C_{H_1,H_2} := \sup\{\langle u, v\rangle : u \in H_1, v \in H_2, \|u\| = \|v\| = 1\}.$$

Proof. Since the minimum eigenvalue of the 2×2 Gram matrix generated by x and y is $1 - |\langle x, y\rangle|$ which is greater than or equal to $1 - C_{H_1,H_2}$, we have

$$\|\alpha x + \beta y\| \geq \left((1 - |\langle x, y\rangle|)(|\alpha|^2 + |\beta|^2)\right)^{1/2} \geq \left((1 - C_{H_1,H_2})(|\alpha|^2 + |\beta|^2)\right)^{1/2}.$$

□

We use this result to demonstrate that we can add to any Riesz basis a Riesz basis consisting of finite number of functions and still preserve the Riesz basis property.

Recall that a set X of a closed linear subspace H_1 of Hilbert space H is called a Riesz basis for H_1 provided that there exist $0 < c \le C < \infty$, such that for an arbitrary $a := \{a_x : x \in X\} \in \ell_2(X)$, the inequality

$$c\|a\|_{\ell_2(X)} \le \|\sum_{x\in X} a_x x\| \le C\|a\|_{\ell_2(X)}$$

holds. In this case, we also say that X forms a Riesz system on H.

Example. Let $\phi \in L_2(\mathbb{R}^s)$. The system of functions $\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ form a Riesz system of $L_2(\mathbb{R}^s)$ if and only if there exist constants $0 < c \le C < \infty$, such that

$$c \le \sum_{\alpha\in 2\pi\mathbb{Z}^s} |\widehat{\phi}(\omega + 2\pi\alpha)|^2 \le C$$

holds for a.e. $\omega \in \mathbb{R}^s$ (cf. [2] or [6]).

Lemma 2.2. *Let H_1 and H_2 be closed subspaces of a Hilbert space H. Let X and Y be a Riesz basis of H_1 and H_2, respectively. Suppose that $H_1 \cap H_2 = \{0\}$ and that X is a finite set. Then $X \cup Y$ is a Riesz basis of $H_1 \bigoplus H_2$.*

Proof. By Lemma 2.1, $X \cup Y$ is a Riesz basis of $H_1 \bigoplus H_2$ provided that $C_{H_1,H_2} < 1$. We show that this is indeed the case by contradiction. To this end, assume that $C_{H_1,H_2} = 1$. Let $\{v_k : k \in \mathbb{Z}_+\} \subset H_1$ and $\{u_k : k \in \mathbb{Z}_+\} \subset H_2$, be sequences in H_1 and H_2, respectively with $\|v_k\| = \|u_k\| = 1$, such that

$$\lim_{k\to\infty} \langle v_k, u_k\rangle = 1.$$

Certainly, we conclude that

$$\lim_{k\to\infty} \|v_k - u_k\|^2 = 0. \tag{2.3}$$

Since H_1 is a finite dimension space, there is a subsequence $\{v_{k_j} : j \in \mathbb{Z}_+\}$ of $\{v_j : k \in \mathbb{Z}_+\}$ which converges to $v \in H_1$. This fact together with (2.3) implies that the subsequence $\{u_{k_j} : j \in \mathbb{Z}_+\}$ also converges to some $u \in H_2$. Again by (2.3), the element $v - u \in H_1 \bigoplus H_2$ is necessary zero. Hence, we conclude that $v = u = 0$, which contradicts the fact that $\langle u, v\rangle = 1$. $\square$

The above result does not hold if we drop the assumption that X is finite as shown in the next example.

Example. Let $H = \ell_2(\mathbb{Z}_+)$ and for $j \in \mathbb{Z}_+$ define e_j in H by $(e_j)_k = \delta_{j,k}$, $k \in \mathbb{Z}_+$. Let $X = \{u_j : j \in \mathbb{Z}_+\}$ and $Y = \{v_j : j \in \mathbb{Z}_+\}$, where $u_j = e_{2j-1}$, $v_j = e_{2j-1} + \frac{1}{j}e_{2j}$. Clearly X is a Riesz basis of its closed linear span H_1. For $a \in H$,

$$\|\sum_{j\in\mathbb{Z}_+} a_j v_j\|_2^2 = \sum_{j\in\mathbb{Z}_+} a_j^2(1 + \frac{1}{j^2})$$

and so Y is a Riesz basis for its closed linear span H_2.

Now, take x in $H_1 \cap H_2$. Since $x \in H_2$ we have $x = \sum_{j \in \mathbb{Z}_+} a_j(e_{2j-1} + \frac{1}{j} e_{2j})$ for some a in H. Moreover, when $x \in H_1$, for all $j \in \mathbb{Z}_+$, we have that $0 = x_{2j} = \frac{a_j}{j}$, and consequently $H_1 \cap H_2 = \{0\}$. With respect to the basis $X \cup Y$, the ℓ_2 norm of the coefficients of $v_j - u_j$ is $\sqrt{2}$ for all j in $\mathbb{Z}_+$ but

$$\lim_{j \to \infty} ||v_j - u_j||_2 = \lim_{j \to \infty} ||\frac{1}{j} e_{2j}||_2 = 0.$$

Therefore, we conclude that $X \cup Y$ is not a Riesz basis. □

Theorem 2.4. *For given measurable sets Ω and Ω' with $\Omega' \subset \Omega \subset \mathbb{R}^s$, let $\{\phi_j : j \in J\}$, where J is a subset of $\mathbb{Z}_+$, be a set of functions in $L_2(\Omega)$ which satisfies the following conditions:*

(i) The functions $\{\phi_j : j \in J\}$ form a Riesz basis of the smallest closed subspace of $L_2(\Omega)$ containing $\{\phi_j : j \in J\}$.

(ii) For any sequence $a := \{a_j : j \in J\} \in \ell_2(J)$, the equality

$$\sum_{j \in J} a_j \phi_j(x) = 0, \quad a.e. \quad x \in \Omega'$$

implies that $a_j = 0$, $j \in J$.

(iii) For $j \in J$, the sequence

$$\int_{\Omega \setminus \Omega'} |\phi_j(x)|^2 dx$$

is summable.

Then the functions $X = \{\phi_{j|_{\Omega'}} : j \in J\}$ form a Riesz basis of V_0, the smallest closed subspace of $L_2(\Omega')$ containing X.

Proof. First, we show that there exist $M \in J$ and constants $0 < c \leq C < \infty$, such that for arbitrary sequences $a \in \ell_2(J_M)$, where $J_M := \{j \in J : j \geq M\}$,

$$c\|a\|^2_{\ell_2(J_M)} \leq \int_{\Omega'} |\sum_{j \in J_M} a_j \phi_j(x)|^2 dx \leq C\|a\|^2_{\ell_2(J_M)}. \tag{2.5}$$

Since the functions $\{\phi_j : j \in J\}$ form a Riesz basis for the smallest closed linear space containing the functions ϕ_j, $j \in J$ in $L_2(\Omega)$, there are constants $0 < c' \leq C' < \infty$, such that for arbitrary sequences $b := \{b_j : j \in J_M\} \in \ell_2(J_M)$,

$$c'\|b\|^2_{\ell_2(J_M)} \leq \int_{\Omega} |\sum_{j \in J_M} b_j \phi_j(x)|^2 dx \leq C'\|b\|^2_{\ell(J_M)}.$$

Thus, the upper bound in (2.5) is clear for an arbitrary M. For the lower bound in (2.5), we use property (iii) and choose $M > 0$ so that

$$\int_{\Omega \setminus \Omega'} \sum_{j \in J_M} |\phi_j(x)|^2 dx \leq c'/2.$$

Hence,

$$\int_{\Omega\backslash\Omega'} |\sum_{j\in J_M} a_j\phi_j(x)|^2dx \le (c'/2)\|a\|^2_{\ell_2(J_M)}$$

and we conclude that

$$\int_{\Omega'} |\sum_{j\in J_M} a_j\phi_j(x)|^2dx \ge c'\|a\|^2_{\ell_2(J_M)} - (c'/2)\|a\|^2_{\ell_2(J_M)} = (c'/2)\|a\|^2_{\ell_2(J_M)},$$

thereby proving the lower bound of (2.5).

Denote by V_M the $L_2(\Omega')$ closure of all linear combinations of functions $\{\phi_{i|'_\Omega} : i \in J_M\}$. Then (2.5) shows that functions $\{\phi_i : i \in J_M\}$ form a Riesz basis of V_M. We also note from (ii) that functions $\{\phi_{i|'_\Omega} : i \in J\backslash J_M\}$ form a Riesz basis of the space $W := \text{span}\{\phi_{i|'_\Omega} : i < M, i \in J\}$. We invoke condition (ii) and Lemma 2.2 to conclude that functions $\{\phi_{j|_{\Omega'}} : j \in J\}$ form a Riesz basis of V_0. □

Applying this result to the case that ϕ_j is generated by shifts of a given set of a finite number of functions with $\Omega = \mathbb{R}$ and $\Omega' = \mathbb{R}_+$, we obtain the following result.

Corollary 2.6. *Let Ψ be a set of finite number of functions in $L_2(\mathbb{R})$ which satisfies the following conditions:*

(i) The nonnegative integer shifts of Ψ form a Riesz basis of the closure of the linear combinations of all nonnegative integer shifts of Ψ in $L_2(\mathbb{R})$.

(ii) For any sequence $\{a_\phi(j) : j \in \mathbb{Z}_+\} \in \ell_2(\mathbb{Z}_+)$, $\phi \in \Psi$, the equality

$$\sum_{j\in\mathbb{Z}_+\phi\in\Psi} a_\phi(j)\phi(x-j) = 0, \quad a.e. \quad x \in \mathbb{R}_+$$

implies that $a_\phi = 0$.

(iii) For $j \in \mathbb{Z}_+$, and $\phi \in \Psi$, the sequence

$$\int_{-\infty}^0 |\phi(x-j)|^2dx$$

is summable.

Then the functions $X = \{\phi(\cdot - j) : j \in \mathbb{Z}_+\}$, $\phi \in \Psi$ form a Riesz basis of V_0, the smallest closed subspace of $L_2(\mathbb{R}_+)$ containing X.

Remark. Condition (i) in Corollary 2.6 can be replaced by the following stronger condition which is much easier to check via its Fourier transform (cf. [2] and [6]):

(i') The integer shifts of Ψ form a Riesz basis of the closure of the linear combinations of all integer shifts of Ψ in $L_2(\mathbb{R})$.

Theorem 2.7. *Let ϕ be a continuous function on $\mathbb{R}$ and $\{x_j : j \in \mathbb{Z}_+\}$ be a sequence of points in $\mathbb{R}_+$ which satisfy the following conditions:*

(i) There exist positive constants $A > 0$, $\delta > 0$, such that

$$|\phi(x)| \leq \frac{A}{(1+|x|)^{1+\delta}}, \quad x \in \mathbb{R}. \tag{2.8}$$

(ii) The Fourier transform of ϕ is positive, i.e.

$$\widehat{\phi}(\omega) > 0, \quad \omega \in \mathbb{R}.$$

(iii) There exists a positive number q, such that $x_j - x_k \geq q$, $j > k$.
Then the functions $X = \{\phi(\cdot - x_j) : j \in \mathbb{Z}_+\}$ form a Riesz basis of V_0, the smallest closed subspace $L_2(\mathbb{R}_+)$ containing $\{\phi(\cdot - x_j) : j \in \mathbb{Z}_+\}$.

Proof. Let $\phi_j = \phi(\cdot - x_j)$ for $j \in \mathbb{Z}_+$.

The first step in the proof is to establish Condition (i) of Theorem 2.4, i.e. $\{\phi_j : j \in \mathbb{Z}_+\}$ is a Riesz basis on $L_2(\mathbb{R})$. To this end, we note that the Gram matrix is given by

$$\int_{\mathbb{R}} \phi_j(x)\overline{\phi_k(x)}dx = \Phi(x_j - x_k), \quad j,k \in \mathbb{Z}_+, \tag{2.9}$$

where Φ is the autocorrelation of ϕ. The lower bound of the system $\{\phi_j : j \in \mathbb{Z}_+\}$ then follows from Theorem 3.1 of [9] (see also the elegant results in [2,4,5]), because the Fourier transform of Φ is strictly positive on $\mathbb{R}$. Indeed, let $\rho = \min\{\widehat{\phi}(\omega) : |\omega| \leq 14/q\}$. (Obviously, since $\widehat{\phi} > 0$ is continuous, $\rho > 0$.) Then, Theorem 3.1 of [9] (specialized to the univariate case) says for any positive integer N the least eigenvalue of the positive definite matrix

$$(\Phi(x_j - x_k))_{j,k \leq N}$$

is at least $\rho^2(2q)^{-1}$. Since this bound is independent of N, the lower bound does follow from the continuity of the operator Φ, where Φ is considered as a linear operator on $\ell_2(\mathbb{Z}_+)$, which we now establish.

For the upper Riesz bound for the system $\{\phi_j : j \in \mathbb{Z}_+\}$, we note that the Riesz upper bound is the square root of the operator bound of Φ. For the Hermitian operator Φ, its operator norm is bounded by the supremum of the $\ell_1(\mathbb{Z}_+)$–norm of the rows, which is

$$\sup_{i \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} |\Phi_{i,k}|.$$

(In fact, it is a special case of the well-known Schur lemma, which holds for nonsymmetric matrices (cf. [7]).) Take any $j \in \mathbb{Z}_+$ and choose $x_j \leq x \leq x_{j+1}$. For $k \geq j+1$, $x_k - x \geq x_k - x_{j+1} \geq q(k-j-1)$ and for $k \leq j$, $x - x_k \geq x_j - x_k \geq q(j-k)$. Thus we have that

$$\begin{aligned}
\sum_{k \in \mathbb{Z}_+} |\phi(x - x_k)| &\leq \sum_{k \in \mathbb{Z}_+} \frac{A}{(1+|x-x_k|)^{1+\delta}} \\
&\leq 2 \sum_{k \in \mathbb{Z}_+} \frac{A}{(1+kq)^{1+\delta}} \\
&= C \\
&< \infty.
\end{aligned}$$

A similar proof leads to the above inequality when $x \leq x_j$ for all $j \in \mathbb{Z}_+$. Therefore, we conclude that

$$\begin{aligned}\sum_{k\in\mathbb{Z}_+} |\Phi_{i,k}| &\leq \int_{\mathbb{R}} |\phi(x-x_i)| \sum_{k\in\mathbb{Z}_+} |\phi(x-x_k)|dx \\ &\leq C \int_{\mathbb{R}} |\phi(x-x_i)|dx \\ &< \infty.\end{aligned}$$

For Condition (ii) of Theorem 2.4, we first observe that whenever $a = \{a_j : j \in \mathbb{Z}_+\}$ is in $\ell_2(\mathbb{Z}_+)$, the function

$$f = \sum_{j\in\mathbb{Z}_+} a_j \phi_j$$

is continuous, which follows directly by the hypotheses on the decay of ϕ. Hence we can establish (ii) of Theorem 2.4 by again applying Theorem 3.1 of [9] to the matrix

$$(\phi(x_j - x_k))_{j,k,\in\mathbb{Z}_+}$$

on $\ell_2(\mathbb{Z}_+)$.

Finally, for Condition (iii) of Theorem 2.4, we use (i) to obtain

$$\int_{-\infty}^{0} |\phi(x-x_j)|^2 dx \leq \frac{A^2}{(x_0 + jq)^{1+2\delta}}, \quad j > 0.$$

□

Remark: The above result applies to the choice $x_j = j$, $j \in \mathbb{Z}_+$ and any one of the functions

$$\phi(x) = e^{-\sigma x^2}, \quad x \in \mathbb{R},\ \sigma > 0,$$

$$\phi(x) = e^{-\sigma|x|}, \quad x \in \mathbb{R},\ \sigma > 0,$$

and

$$\phi(x) = \frac{1}{\sigma^2 + x^2}, \quad x \in \mathbb{R}, \sigma \neq 0,$$

which were considered in [8].

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