DISTRIBUTIONAL SOLUTIONS OF NONHOMOGENEOUS DISCRETE AND CONTINUOUS REFINEMENT EQUATIONS

RONG-QING JIA, QINGTANG JIANG AND ZUOWEI SHEN

ABSTRACT. Discrete and continuous refinement equations have been widely studied in the literature for the last few years, due to their applications to the areas of wavelet analysis and geometric modeling. However, there is no 'universal' theorem that deals with the problem about the existence of compactly supported distributional solutions for both discrete and continuous refinement equations simultaneously. In this paper, we provide a uniform treatment for both equations. In particular, a complete characterization of the existence of distributional solutions of nonhomogeneous discrete and continuous refinement equations is given, which covers all cases of interest.

1. Introduction and Notation

Let M be a dilation matrix, that is, an $s \times s$ real matrix whose eigenvalues lie outside the closed unit disk. We are interested in the following nonhomogeneous refinement equation:

$$\phi(x) = g(x) + \int_{\mathbb{R}^s} d\mu(y)\phi(Mx - y), \quad x \in \mathbb{R}^s, \tag{1.1}$$

where $\phi = (\phi_1, \ldots, \phi_r)^T$ is the unknown, $g = (g_1, \ldots, g_r)^T$ is a given $r \times 1$ vector of compactly supported distributions on $\mathbb{R}^s$, and μ is an $r \times r$ matrix of finite complex Borel measures on $\mathbb{R}^s$ with compact supports. Let $\mu = (\mu_{lj})_{1 \le l,j \le r}$. Then, (1.1) can be written in the component form:

$$\phi_l(x) = g_l(x) + \sum_{j=1}^{r} \int_{\mathbb{R}^s} \phi_j(Mx - y)\, d\mu_{lj}(y), \quad l = 1, \ldots, r.$$

When each μ_{lj} is a discrete Borel measure, (1.1) becomes a discrete refinement equation; when each μ_{lj} is absolutely continuous with respect to the Lebesgue measure, (1.1)

1991 *Mathematics Subject Classification.* 41A58, 42C15, 41A17, 42C99.

Key words and phrases. nonhomogeneous, discrete and continuous refinement equations.

is a continuous refinement equation. If $g = 0$, then (1.1) becomes a homogeneous refinement equation:

$$\phi(x) = \int_{\mathbb{R}^s} d\mu(y)\phi(Mx - y). \tag{1.2}$$

Refinement equations are fundamental to wavelet theory and subdivision. In the context of wavelet theory, the key step to the construction of wavelets is to construct suitable refinable functions. In the context of subdivision, the limiting surface of a subdivision process is a linear combination of integer translates of the refinable function corresponding to the subdivision scheme.

For the scalar case (i.e. $r = 1$), a homogeneous discrete refinement equation can be written as

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(Mx - \alpha), \quad x \in \mathbb{R}^s, \tag{1.3}$$

where the refinement mask a is finitely supported. When the dilation matrix M is two times the $s \times s$ identity matrix I_s, existence and uniqueness of the solutions of (1.3) were studied in [3] and [7]. In particular, for the univariate case $s = 1$, it was proved in [7] that (1.3) has a nontrivial L_1–solution with compact support only if $\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = 2^n$ for some positive integer n.

For the vector case (i.e. $r > 1$), the coefficients $a(\alpha)$, $\alpha \in \mathbb{Z}^s$ in (1.3) become $r \times r$ complex matrices. The existence of compactly supported distributional solutions is characterized by spectral properties of the matrix $\Delta := \sum_\alpha a(\alpha)/|\det(M)|$. The spectral radius of Δ is denoted by $\rho(\Delta)$.

When $s = 1$ and $M = (2)$, the existence (and uniqueness) of compactly supported distributional solutions of the vector refinement equation were first investigated in [14]. One of the main results of [14] states as follows: Suppose that there is a single eigenvalue λ of Δ with $|\lambda| = \rho(\Delta) < 2$. then the vector refinement equation (1.3) has k independent compactly supported distributional solutions, where k is the multiplicity of the eigenvalue 1 of Δ. This result was improved by [5] by showing that it is still valid under a weak assumption that $\rho(\Delta) < 2$. A complete characterization of the existence of the compactly supported distributional solutions (for the case $M = 2I_s$) was given in [17]. It states that the vector refinement equation (1.3) has a nontrivial compactly supported distributional solution if and only if there exists a nonnegative integer n such that 2^n is an eigenvalue for Δ (for the case that $r = 2$ and $s = 1$ also see [24]).

Nonhomogeneous discrete refinement equations were investigated in [9] and [22]. For the case $s = 1, M = (2)$ and $r = 1$, necessary and sufficient conditions for existence and uniqueness of nontrivial compactly supported distributional solutions were given independently by [9] and [22].

Continuous homogeneous refinement equations were studied by many authors (see [4], [6], [8], [12], [15], [16], [18], [19] and [21]). The interested readers should consult the aforementioned references for details.

Although a lot of work has been done in this subject, there is no "universal" theorem that covers all cases. In this paper, we give a uniform treatment of the existence and uniqueness of distributional solutions of both discrete and continuous nonhomogeneous refinement equations in the most general setting, *i.e.*, for the case of an arbitrary dilation matrix, any number of functions and any number of variables. The main idea is to use an iteration scheme in the Fourier domain with *real* variables. This approach enables us to unify the treatment for both discrete and continuous refinement equations.

While revising this paper, we became aware of recent paper of [10] and [23] related to our work. Both papers deal with some special setting of this paper (e.g. the case $M = 2I_s$).

Section 2 is devoted to a complete characterization of the existence of compactly supported distributional solutions of (1.1) in terms of g and μ. In Section 3, several examples are given to illustrate our theory.

We now turn to the basics needed in this paper.

For a given vector $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$, the norm (or length) of the vector ξ is defined as

$$|\xi| := |\xi_1| + \cdots + |\xi_r|, \quad \xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r. \tag{1.4}$$

Similarly, for a $r \times r$ complex matrices $A = (a_{ij})_{1 \le i,j \le r} \in \mathbb{C}^{r \times r}$, its norm is defined to be the maximum of the norm of its column vectors, i.e.

$$\|A\| := \max\Big\{ \sum_{i=1}^{r} |a_{ij}| : j = 1, 2, \dots, r\Big\}.$$

In general, for a linear space F, F^r is denoted as the linear space

$$\Big\{ (f_1, \dots, f_r)^T : f_1, \dots, f_r \in F \Big\}.$$

When F is a Banach space equipped with the norm $\|\cdot\|$, the space F^r is also a Banach space with the norm given by

$$\|f\| := \sum_{j=1}^{r} \|f_j\|, \qquad f = (f_1, \ldots, f_r)^T \in F^r.$$

The space $\mathbb{R}^s$ is the s-dimensional Euclidean space equipped with the norm in **??**. The set of all positive integers is denoted by $\mathbb{N}$; and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is the set of all nonnegative integers.

A nonnegative integer $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s$ is also used as a multi-index. Its length is the norm of α given in **??**. For two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $\beta = (\beta_1, \ldots, \beta_s)$, $\beta \le \alpha$ whenever $\beta_j \le \alpha_j$ for $j = 1, \ldots, s$.

For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$, set $x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$. We also use x^α to denote the function whose value at any x is x^α. The space P_n is the set of all polynomials of (total) degree at most n. For $j = 1, \ldots, s$, D_j denotes the partial derivative with respect to the jth coordinate and D^α is the differential operator $D_1^{\alpha_1} \cdots D_s^{\alpha_s}$. More general, for a given polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha$, $x \in \mathbb{R}^s$, the corresponding differential operator is

$$p(D) := \sum_\alpha c_\alpha D^\alpha.$$

Finally, for a given nonnegative integer α, the factorial of α is defined as $\alpha! := \alpha_1! \cdots \alpha_s!$.

Next, we list some basic notations of tempered distribution used in this paper. Let φ be a C^∞ function on $\mathbb{R}^s$. The semi-norm $\|\cdot\|_{(m,\alpha)}$ of φ for a nonnegative integer m and a multi-index α is defined as

$$\|\varphi\|_{(m,\alpha)} := \sup_{x \in \mathbb{R}^s} \{(1 + |x|)^m |D^\alpha \varphi(x)|\}.$$

A function $\varphi \in C^\infty(\mathbb{R}^s)$ is said to be rapidly decreasing if $\|\varphi\|_{(m,\alpha)} < \infty$ for all $m \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^s$. On the other hand, a continuous function f on $\mathbb{R}^s$ is said to be slowly increasing if there exists a polynomial p in s variables such that

$$|f(x)| \le |p(x)| \qquad \forall\, x \in \mathbb{R}^s.$$

Let $\mathcal{S}(\mathbb{R}^s)$ be the Schwartz space which is the space of all rapidly decreasing functions on $\mathbb{R}^s$ equipped with the metric

$$d(f,g) := \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{1}{2^m} \frac{\|f-g\|_{(m,\alpha)}}{1 + \|f-g\|_{(m,\alpha)}}, \qquad f, g \in \mathcal{S}(\mathbb{R}^s).$$

A linear continuous functional on $\mathcal{S}(\mathbb{R}^s)$ is called a tempered distribution. The space $\mathcal{S}'(\mathbb{R}^s)$ is the linear space of all tempered distributions on $\mathbb{R}^s$. For example, the Dirac function δ given by

$$\langle \delta, \varphi \rangle := \varphi(0) \qquad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^s)$$

is a tempered distribution. More general, a slow increasing continuous function $f \in \mathbb{R}^s$ induces a tempered distribution by

$$\langle f, \varphi \rangle := \int_{\mathbb{R}^s} f(x)\varphi(x)dx \qquad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^s).$$

Let f be a tempered distribution on $\mathbb{R}^s$. We say that f vanishes on open set $G \in \mathbb{R}^s$, if $\langle f, \varphi \rangle = 0$ for every $\varphi \in \mathcal{S}(\mathbb{R}^s)$ supported in G, Let W be the union of all open subsets G of $\mathbb{R}^s$ in which f vanishes. The complement of W is the support of f and denoted by $\operatorname{supp} f$. If $\operatorname{supp} f$ is a compact subset of $\mathbb{R}^s$, then we say that f is compactly supported.

The Fourier transform of a function φ in $\mathcal{S}(\mathbb{R}^s)$ is defined by

$$\widehat{\varphi}(\omega) := \int_{\mathbb{R}^s} \varphi(x) e^{-ix\cdot\omega} dx, \qquad \omega \in \mathbb{R}^s,$$

where i stands for the imaginary unit, and $x \cdot \omega := x_1\omega_1 + \cdots + x_s\omega_s$ for $x = (x_1, \ldots, x_s)$ and $\omega = (\omega_1, \ldots, \omega_s)$.

The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^s)$ is the tempered distribution $\widehat{f}$ defined by

$$\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle \qquad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^s).$$

For example, the Fourier transform of the Dirac function δ is the constant 1. Let p be a polynomial and $f \in \mathcal{S}'(\mathbb{R}^s)$, the Fourier transform of $p(-iD)f$ is $p\widehat{f}$. In particular, the Fourier transform of $p(-iD)\delta$ is p.

The Fourier transform of a compactly supported distribution is an analytic function. Recall that a function f on $\mathbb{R}^s$ is said to be analytic if f can be expanded into a power series

$$f(x) = \sum\nolimits_{\alpha \in \mathbb{N}_0^s} c_\alpha x^\alpha,$$

which converges for every $x \in \mathbb{R}^s$. The coefficients c_α are given by $c_\alpha = D^\alpha f(0)/\alpha!$. We use $\mathcal{A}(\mathbb{R}^s)$ to denote the linear space of all analytic functions on $\mathbb{R}^s$.

We will also use the following identity

$$\langle f, \varphi \rangle = (2\pi)^{-s/2} \langle \widehat{f}, \widehat{\varphi} \rangle \qquad \forall\, f \in \mathcal{S}'(\mathbb{R}^s), \varphi \in \mathcal{S}(\mathbb{R}^s).$$

A vector of distributions $\phi = (\phi_1, \ldots, \phi_r)^T \in (\mathcal{S}'(\mathbb{R}^s))^r$ is called a solution of (1.1) if

$$\langle \phi, \varphi \rangle = \langle g, \varphi \rangle + \Big\langle \phi, \int_{\mathbb{R}^s} \overline{(d\mu(y))^T} \varphi(M^{-1}(\cdot + y))/|\det(M)| \Big\rangle$$

holds for all $\varphi = (\varphi_1, \ldots, \varphi_r)^T \in (\mathcal{S}(\mathbb{R}^s))^r$.

2. Existence of Solutions

The problem of the existence of distributional solutions of discrete and continuous refinement equations can be handled simultaneously in the Fourier domain. For this, recall that the Fourier transform of μ_{lj} is given by

$$\widehat{\mu}_{lj}(\omega) = \int_{\mathbb{R}^s} e^{-i\omega\cdot y}\, d\mu_{lj}(y), \quad \omega \in \mathbb{R}^s.$$

Thus, refinement equation (1.1) can be written as

$$\widehat{\phi}(\omega) = \widehat{g}(\omega) + H(N\omega)\widehat{\phi}(N\omega), \quad \omega \in \mathbb{R}^s, \tag{2.1}$$

where $N := (M^{-1})^T$ and

$$H(\omega) := (1/|\det(M)|)\,\widehat{\mu}(\omega) = (1/|\det(M)|)(\widehat{\mu}_{lj}(\omega))_{1\le l,j\le r}, \quad \omega \in \mathbb{R}^s. \tag{2.2}$$

For a nonnegative integer n, we denote by P_n^r the linear space of all $r \times 1$ vectors of polynomials of degree at most n. For $f \in (\mathcal{A}(\mathbb{R}^s))^r$, define

$$f^{[n]}(\omega) := \sum\nolimits_{|\alpha|\le n} D^\alpha f(0)\omega^\alpha/\alpha!, \qquad \omega \in \mathbb{R}^s.$$

Clearly, $f^{[n]}$ belongs to P_n^r. Let L_n be the linear operator defined on P_n^r by

$$L_n p := \Big(H(N\cdot)\,p(N\cdot)\Big)^{[n]}, \quad p \in P_n^r.$$

The linear operator L_n can be viewed as follows: Let $\sum_{\alpha\in\mathbb{N}_0^s} v_\alpha \omega^\alpha$, $\omega \in \mathbb{R}^s$ be the Taylor expansion of $H(N\omega)p(N\omega)$. Then, $L_n p(\omega) = \sum_{|\alpha|\le n} v_\alpha \omega^\alpha$.

Suppose $\widehat{\phi}$ satisfies (2.1). Then for any $n \in \mathbb{N}_0$,

$$\widehat{\phi}^{[n]} = \widehat{g}^{[n]} + \Big(H(N\cdot)\,\widehat{\phi}(N\cdot)\Big)^{[n]} = \widehat{g}^{[n]} + L_n\widehat{\phi}^{[n]}.$$

Hence, $p := \widehat{\phi}^{[n]} \in P_n^r$ is a solution of the following linear equation:

$$p - L_n p = \widehat{g}^{[n]}. \tag{2.3}$$

Next, we show that if (2.3) has a solution $p \in P_n^r$ for a sufficiently large integer n, then (2.1) has a compactly supported distributional solution ϕ such that $\widehat{\phi}^{[n]} = p$. For this, we first note that if f is a compactly supported *continuous function*, then using Taylor's formula (see, *e.g.*, [20], Theorem 7.7) we have

$$\widehat{f}(\omega) = \sum_{|\alpha|\le n} \frac{D^\alpha \widehat{f}(0)}{\alpha!}\omega^\alpha + \sum_{|\alpha|=n+1} \frac{D^\alpha \widehat{f}(\xi)}{\alpha!}\omega^\alpha,$$

where ξ is a point on the straight line segment from 0 to ω. Note that

$$D^\alpha \widehat{f}(\xi) = \int_{\mathbb{R}^s} (-ix)^\alpha f(x) e^{-ix\cdot\xi}\, dx.$$

Since f is a compactly supported continuous function, the set $K := \operatorname{supp} f$ is a compact set of $\mathbb{R}^s$. Hence,

$$\left|D^\alpha \widehat{f}(\xi)\right| \le \int_K |x|^{|\alpha|} |f(x)|\, dx.$$

Therefore, there exists a constant C_n such that

$$\left|\widehat{f}(\omega) - \sum\nolimits_{|\alpha|\le n} D^\alpha \widehat{f}(0)\omega^\alpha/\alpha!\right| \le C_n |\omega|^{n+1} \qquad \forall\, \omega \in \mathbb{R}^s. \tag{2.4}$$

The following lemma extends the above estimate to compactly supported *distributions*. The key to our extension is the well-known fact that a *compactly supported* distribution is of finite order (see [2], Theorem 2.22).

Lemma 2.1. *Suppose f is a compactly supported distribution on $\mathbb{R}^s$. Then for a given nonnegative integer n, there exists a polynomial q_n in s variables such that*

$$\left|\widehat{f}(\omega) - \sum\nolimits_{|\alpha|\le n} D^\alpha \widehat{f}(0)\omega^\alpha/\alpha!\right| \le |\omega|^{n+1} q_n(\omega) \qquad \forall\, \omega \in \mathbb{R}^s. \tag{2.5}$$

Proof. Since f is compactly supported, there exists a positive integer m and compactly supported continuous functions f_β ($|\beta| \le m$) such that $f = \sum_{|\beta|\le m} D^\beta f_\beta$. Hence,

$$\widehat{f}(\omega) = \sum\nolimits_{|\beta|\le m} (i\omega)^\beta \widehat{f}_\beta(\omega), \qquad \omega \in \mathbb{R}^s.$$

Set $c_{\alpha,\beta} := D^\alpha \widehat{f}_\beta(0)/\alpha!$ for $\alpha \in \mathbb{N}_0^s$ and $|\beta| \le m$. Write $\widehat{f}$ as the sum of h_1 and h_2, where

$$h_1(\omega) := \sum\nolimits_{|\beta|\le m} (i\omega)^\beta \sum\nolimits_{|\alpha|\le n} c_{\alpha,\beta}\omega^\alpha, \qquad \omega \in \mathbb{R}^s,$$

and

$$h_2(\omega) := \sum\nolimits_{|\beta|\le m} (i\omega)^\beta \Big(\widehat{f}_\beta(\omega) - \sum\nolimits_{|\alpha|\le n} c_{\alpha,\beta}\omega^\alpha\Big), \qquad \omega \in \mathbb{R}^s.$$

It follows from (2.4) that there exists a polynomial u such that

$$|h_2(\omega)| \le |\omega|^{n+1} u(\omega) \qquad \forall\, \omega \in \mathbb{R}^s.$$

But h_1 is a polynomial, so there exists a polynomial v such that

$$\Big|h_1(\omega) - \sum_{|\alpha|\le n} D^\alpha h_1(0)\omega^\alpha/\alpha!\Big| \le |\omega|^{n+1} v(\omega) \qquad \forall\, \omega \in \mathbb{R}^s.$$

Since $D^\alpha \widehat{f}(0) = D^\alpha h_1(0)$ for all $|\alpha| \le n$, we conclude that (2.5) holds with $q_n = u + v$. □

To state the next theorem, we define

$$c_0 := \sup_{\omega\in\mathbb{R}^s} \|H(\omega)\|. \tag{2.6}$$

Since each measure μ_{lj} $(l, j = 1, \ldots, r)$ is finite, by (2.2), $\|H(\omega)\|$ is bounded on $\mathbb{R}^s$. Hence, $c_0 < \infty$. We also recall that $\rho(N)$ is the spectral radius of the matrix N.

Theorem 2.1. *Suppose* (2.3) *has a solution $p \in P_n^r$ for some nonnegative integer n satisfying $\rho(N)^{n+1} < 1/c_0$. Then* (2.1) *has a compactly supported distributional solution ϕ such that $\widehat{\phi}^{[n]} = p$.*

Proof. In this proof, the number n is fixed. The proof is based on the following iteration scheme. It starts with the $r\times 1$ vector $\phi_0 := p(-iD)\delta$, with each the jth entry $p_j(-iD)\delta$, where p_j is the jth entry of the vector p and δ is the Dirac function. The jth entry of ϕ_0 is supported at the origin and $\widehat{\phi}_0 = p$. For $k = 1, 2, \ldots$, the $r \times 1$ vectors ϕ_k are defined recursively by

$$\widehat{\phi}_k(\omega) := \widehat{g}(\omega) + H(N\omega)\widehat{\phi}_{k-1}(N\omega). \tag{2.7}$$

In particular, $\widehat{\phi}_1^{[n]} = \widehat{g}^{[n]} + L_n p = p$. By (2.7) we have

$$\begin{aligned} \widehat{\phi}_{k+1}(\omega) - \widehat{\phi}_k(\omega) &= H(N\omega)\Big(\widehat{\phi}_k(N\omega) - \widehat{\phi}_{k-1}(N\omega)\Big) \\ &= \Big(\prod_{j=1}^k H(N^j\omega)\Big)\Big(\widehat{\phi}_1(N^k\omega) - \widehat{\phi}_0(N^k\omega)\Big). \end{aligned}$$

Since $\|H(\omega)\| \le c_0$ for all $\omega \in \mathbb{R}^s$, we have

$$|\widehat{\phi}_{k+1}(\omega) - \widehat{\phi}_k(\omega)| \le c_0^k |\widehat{\phi}_1(N^k\omega) - \widehat{\phi}_0(N^k\omega)| \qquad \forall\, \omega \in \mathbb{R}^s \text{ and } k \in \mathbb{N}_0. \tag{2.8}$$

In particular, Since $\widehat{\phi}_1^{[n]} - \widehat{\phi}_0^{[n]} = 0$. By Lemma 2.1, there exists a polynomial q (depending on n) such that

$$|\widehat{\phi}_1(N^k\omega) - \widehat{\phi}_0(N^k\omega)| \le |N^k\omega|^{n+1} q(N^k\omega) \qquad \forall\, \omega \in \mathbb{R}^s \text{ and } k \in \mathbb{N}_0. \tag{2.9}$$

Since $\rho(N)^{n+1} < 1/c_0$ and $\rho(N) < 1$, there is $\varepsilon > 0$ such that $t := (\rho(N)+\varepsilon)^{n+1}c_0 < 1$ and $\rho(N)+\varepsilon < 1$. Hence,

$$|N^k\omega| \le C(\rho(N)+\varepsilon)^k|\omega| \qquad \forall\, \omega \in \mathbb{R}^s \text{ and } k \in \mathbb{N}_0, \tag{2.10}$$

where the constant C depends on ε. This implies that there exists a polynomial Q with $q(N^k\omega) \le Q(\omega)$ for all $k \in \mathbb{N}_0$ and all $\omega \in \mathbb{R}^s$. Combining (2.8), (2.9), and (2.10), we obtain

$$|\widehat{\phi}_{k+1}(\omega) - \widehat{\phi}_k(\omega)| \le t^k|C\omega|^{n+1}Q(\omega) \qquad \forall\, \omega \in \mathbb{R}^s \text{ and } k \in \mathbb{N}_0, \tag{2.11}$$

which means that for each $\omega \in \mathbb{R}^s$, $(\widehat{\phi}_k(\omega))_{k\in\mathbb{N}}$ is a Cauchy sequence. Hence,

$$f(\omega) := \lim_{k\to\infty} \widehat{\phi}_k(\omega), \quad \omega \in \mathbb{R}^s$$

is well defined. Moreover, $(\widehat{\phi}_k)_{k\in\mathbb{N}}$ converges to f uniformly on an arbitrary compact subset of $\mathbb{R}^s$. So f is an $r \times 1$ vector of continuous functions on $\mathbb{R}^s$. Furthermore, we deduce from (2.11) that

(2.12)
$$|\widehat{\phi}_k(\omega) - p(\omega)| \le \sum\nolimits_{j=0}^{k-1}\left|\widehat{\phi}_{j+1}(\omega) - \widehat{\phi}_j(\omega)\right| \le (1-t)^{-1}|C\omega|^{n+1}Q(\omega), \quad \omega \in \mathbb{R}^s.$$

Consequently,

$$|f(\omega) - p(\omega)| \le (1-t)^{-1}|C\omega|^{n+1}Q(\omega), \quad \omega \in \mathbb{R}^s.$$

Hence, f is an $r \times 1$ vector of slowly increasing continuous functions with $f^{[n]} = p$. Therefore, there is a unique $\phi \in (\mathcal{S}'(\mathbb{R}^s))^r$ such that $f = \widehat{\phi}$.

It remains to prove that ϕ is compactly supported. Let K be a compact subset of $\mathbb{R}^s$ such that

$$\{0\} \cup \operatorname{supp} \mu \cup \Big(M(\operatorname{supp} g)\Big) \subseteq K.$$

Let

$$\Omega := \sum\nolimits_{n=1}^{\infty} M^{-n}K.$$

Recall that $\phi_0 = p(-iD)\delta$. By our choice of K,

$$\operatorname{supp} \phi_0 = \{0\} \subseteq K.$$

It can be easily proved that $\operatorname{supp}\phi_k \subseteq \Omega$ for all $k \in \mathbb{N}_0$ inductively (see [13]). Suppose φ belongs to $(\mathcal{S}(\mathbb{R}^s))^r$ and $\operatorname{supp}\varphi \subset \mathbb{R}^s \setminus \Omega$. Since $\widehat{\varphi}$ is rapidly decreasing and since (2.12) is valid, there exists a constant C such that

$$\left|\widehat{\phi}_k(\omega)^T\widehat{\varphi}(\omega)\right| \le C(1+|\omega|)^{-s-1} \qquad \forall\,\omega \in \mathbb{R}^s \text{ and } k \in \mathbb{N}_0.$$

Thus, the Lebesgue dominated convergence theorem leads to

$$\lim_{k\to\infty}\int_{\mathbb{R}^s}\widehat{\phi}_k(\omega)^T\widehat{\varphi}(\omega)\,d\omega = \int_{\mathbb{R}^s} f(\omega)^T\widehat{\varphi}(\omega)\,d\omega.$$

In other words, $\lim_{k\to\infty}\langle\widehat{\phi}_k,\widehat{\varphi}\rangle = \langle\widehat{\phi},\widehat{\varphi}\rangle$. Therefore, we obtain

$$\langle\phi,\varphi\rangle = (2\pi)^{-s/2}\langle\widehat{\phi},\widehat{\varphi}\rangle = (2\pi)^{-s/2}\lim_{k\to\infty}\langle\widehat{\phi}_k,\widehat{\varphi}\rangle = \lim_{k\to\infty}\langle\phi_k,\varphi\rangle = 0.$$

Hence, $\langle\phi,\varphi\rangle = 0$ for all $\varphi \in (\mathcal{S}(\mathbb{R}^s))^r$ supported in $\mathbb{R}^s \setminus \Omega$, which implies that ϕ is supported in Ω. □

Theorem 2.1 reduces the problem of the existence of solutions of (1.1) to that of (2.3).

In order to study equation (2.3), we shall use the notation introduced in [1]. For $|\beta| = k$, write

$$(M^T\omega)^\beta = \sum\nolimits_{|\alpha|=k} m_{\alpha,\beta}\omega^\alpha, \quad \omega \in \mathbb{R}^s.$$

The coefficients $m_{\alpha,\beta}$ ($|\alpha| = k, |\beta| = k$) are uniquely determined by the matrix M and the number k. The matrix $(m_{\alpha,\beta})_{|\alpha|=k,|\beta|=k}$ will be denoted by M_k. For $k \in \mathbb{N}_0$, let J_k be the set $\{\alpha \in \mathbb{N}_0^s : |\alpha| = k\}$. The cardinality of J_k is $d_k := \binom{k+s-1}{s-1}$. The ordering $\prec$ on J_k is defined as follows: For $\alpha = (\alpha_1, \dots, \alpha_s) \in J_k$ and $\beta = (\beta_1, \dots, \beta_s) \in J_k$, $\alpha \prec \beta$ whenever there exists some j, $1 \le j \le s$, such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for $i = j+1, \dots, s$.

Replacing ω by $M^T\omega$ in (2.1), we have

$$\hat{\phi}(M^T\omega) = \hat{g}(M^T\omega) + H(\omega)\hat{\phi}(\omega), \quad \omega \in \mathbb{R}^s. \tag{2.13}$$

Write $\hat{\phi}(\omega) = \sum_{\beta\in\mathbb{N}_0^s} v_\beta\omega^\beta$, $H(\omega) = \sum_{\beta\in\mathbb{N}_0^s} H_\beta\omega^\beta$, and $\hat{g}(\omega) = \sum_{\beta\in\mathbb{N}_0^s} g_\beta\omega^\beta$, $\omega \in \mathbb{R}^s$. Substituting the above expressions into (2.13) and comparing the coefficients of both sides, we obtain

$$\sum_{|\beta|=k} m_{\alpha,\beta}v_\beta - \sum_{0\le\gamma\le\alpha} H_{\alpha-\gamma}v_\gamma = h_\alpha, \qquad |\alpha| = k, \tag{2.14}$$

where

$$h_\alpha := \sum\nolimits_{|\beta|=k} m_{\alpha,\beta} g_\beta, \qquad |\alpha| = k. \tag{2.15}$$

Denote by $v_{[k]}$ the $(rd_k)\times 1$ column vector defined by $v_{[k]} := (v_\beta)_{|\beta|=k}$. The column vector $v_{[k]}$ is ordered from the top to the bottom as follows: For α, β with $|\alpha| = |\beta| = k$, if $\alpha \prec \beta$, then the segment v_α is put at the top of the segment v_β. The $(rd_k)\times 1$ column vector $h_{[k]} := (h_\beta)_{|\beta|=k}$ is defined similarly.

The notation $B\otimes C$ stands for $(b_{ij}C)$, the (right) Kronecker product of two matrices $B = (b_{ij})$ and C. With this, equation (2.14) can be rewritten as

$$(M_k \otimes I_r)\, v_{[k]} - \sum\nolimits_{j=0}^{k} (H_{\alpha-\gamma})_{|\alpha|=k,|\gamma|=j} v_{[j]} = h_{[k]}, \tag{2.16}$$

where $H_{\alpha-\gamma}$ is understood to be 0 if $\gamma \le \alpha$ does not hold. When $|\alpha| = k$ and $|\gamma| = k$, we have $(H_{\alpha-\gamma})_{|\alpha|=k,|\gamma|=k} = I_{d_k} \otimes H(0)$. It follows from (2.16) that

$$T_k \begin{bmatrix} v_{[0]} \\ v_{[1]} \\ \vdots \\ v_{[k]} \end{bmatrix} = \begin{bmatrix} h_{[0]} \\ h_{[1]} \\ \vdots \\ h_{[k]} \end{bmatrix}, \tag{2.17}$$

where the matrix T_k is given by

$$T_k := \begin{bmatrix} I_r & 0 & 0 & \cdots & 0 \\ 0 & M_1\otimes I_r & 0 & \cdots & 0 \\ 0 & 0 & M_2\otimes I_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_k\otimes I_r \end{bmatrix} - \begin{bmatrix} H(0) & 0 & 0 & \cdots & 0 \\ (H_\alpha)_{|\alpha|=1} & I_{d_1}\otimes H(0) & 0 & \cdots & 0 \\ (H_\alpha)_{|\alpha|=2} & (H_{\alpha-\gamma})_{|\alpha|=2,|\gamma|=1} & I_{d_2}\otimes H(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (H_\alpha)_{|\alpha|=k} & (H_{\alpha-\gamma})_{|\alpha|=k,|\gamma|=1} & (H_{\alpha-\gamma})_{|\alpha|=k,|\gamma|=2} & \cdots & I_{d_k}\otimes H(0) \end{bmatrix}. \tag{2.18}$$

Therefore, ϕ satisfies (2.1) if and only if, for each $k \in \mathbb{N}_0$, $v_{[0]}, v_{[1]}, \ldots, v_{[k]}$ satisfy (2.17).

Let $\lambda_1, \ldots, \lambda_s$ be the eigenvalues of M. As usual, for $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}_0^s$, $\lambda^\beta := \lambda_1^{\beta_1} \cdots \lambda_s^{\beta_s}$.

Lemma 2.2. *Let k be a nonnegative integer. Suppose λ^β is not an eigenvalue of $H(0)$ for any $\beta \in \mathbb{N}_0^s$ with $|\beta| = k$. Then the matrix*

$$M_k \otimes I_r - I_{d_k} \otimes H(0)$$

is non-singular.

Proof. Suppose B is an $s \times s$ matrix. For $\alpha \in J_k$ we write

$$(B\omega)^\alpha = \sum_{|\beta|=k} b_{\alpha,\beta}^{[k]} \omega^\beta, \quad \omega \in \mathbb{R}^s,$$

where $b_{\alpha,\beta}^{[k]}$ are complex numbers. Let $B^{[k]}$ denote the matrix $(b_{\alpha,\beta}^{[k]})_{|\alpha|=k,|\beta|=k}$. Suppose C is also an $s \times s$ matrix. It is easily seen that

$$(BC)^{[k]} = B^{[k]} C^{[k]}.$$

Since the eigenvalues of M are $\lambda_1, \ldots, \lambda_s$, there exists an invertible $s \times s$ matrix U such that the matrix $\Lambda := U^{-1} M^T U$ is a lower triangular matrix with $\lambda_1, \ldots, \lambda_s$ being the entries in its main diagonal. We also note that $M_k = ((M^T)^{[k]})^T$ by the definition of M_k.

In order to establish the lemma, it suffices to show that the matrix $(M^T)^{[k]} \otimes I_r - I_{d_k} \otimes H(0)^T$ is non-singular. For that, we observe that

$$\left[(U^{-1})^{[k]} \otimes I_r\right] \left[(M^T)^{[k]} \otimes I_r - I_{d_k} \otimes H(0)^T\right] \left[U^{[k]} \otimes I_r\right] = \Lambda^{[k]} \otimes I_r - I_{d_k} \otimes H(0)^T.$$

Clearly, $\Lambda^{[k]}$ is a lower triangular matrix with λ^β ($|\beta| = k$) being the entries in its main diagonal. Thus, $\Lambda^{[k]} \otimes I_r - I_{d_k} \otimes H(0)^T$ is a lower triangular block matrix with diagonal blocks $\lambda^\beta I_r - H(0)^T$. Since λ^β is not an eigenvalue of $H(0)$ for any β with $|\beta| = k$, we conclude that the matrix $\Lambda^{[k]} \otimes I_r - I_{d_k} \otimes H(0)^T$ is non-singular. □

We are in a position to establish the main result of this paper. In what follows, for $k \in \mathbb{N}_0$, T_k is the matrix given in (2.18), and $h_{[k]}$ is the vector $(h_\alpha)_{|\alpha|=k}$ with h_α given in (2.15). Finally, $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of M.

Theorem 2.2. *Suppose $H(0)$ has no eigenvalues of the form λ^β, $\beta \in \mathbb{N}_0^s$, then equation* (1.1) *has a unique compactly supported distributional solution. Suppose $H(0)$ has eigenvalues of the form λ^β for some $\beta \in \mathbb{N}_0^s$. Let $n_0 := \max\{|\beta| : \lambda^\beta$ is an eigenvalue of $H(0)\}$.*

Then (1.1) *has a compactly supported distributional solution* ϕ *if and only if the linear equation*

$$T_{n_0}\begin{bmatrix} v_{[0]} \\ v_{[1]} \\ \vdots \\ v_{[n_0]} \end{bmatrix} = \begin{bmatrix} h_{[0]} \\ h_{[1]} \\ \vdots \\ h_{[n_0]} \end{bmatrix}, \tag{2.19}$$

has a solution. Furthermore, let $v_{[0]}, v_{[1]}, \ldots, v_{[n_0]}$ *be a solution of the above linear equation and* $v_{[j]} = (v_\alpha)_{|\alpha|=j}$ $(j = 0, \ldots, n_0)$*. Then, there is a unique compactly supported distributional solution* ϕ *of* (1.1) *satisfying* $\widehat{\phi}^{[n_0]}(\omega) = \sum_{|\alpha|\le n_0} v_\alpha \omega^\alpha$, $\omega \in \mathbb{R}^s$*.*

Proof. Let n be a nonnegative integer satisfying $\rho(N)^{n+1} < 1/c_0$, where c_0 is given by (2.6). Suppose that $H(0)$ has no eigenvalues of the form λ^β, $\beta \in \mathbb{N}_0^s$. Then, $M_j \otimes I_r - I_{d_j} \otimes H(0)$ is non-singular for every $j \in \mathbb{N}_0$ by Lemma 2.2. Hence, there is a unique solution $v_{[0]}, v_{[1]}, \ldots, v_{[n]}$ that satisfies the linear equation (2.17) for $k = n$. Hence, (1.1) has a unique compactly supported distributional solution by Theorem 2.1.

Next, suppose $H(0)$ has eigenvalues of the form λ^β for some $\beta \in \mathbb{N}_0^s$. Let $v_{[0]}, v_{[1]}, \ldots, v_{[n_0]}$ be a solution of the linear equation (2.19). By Lemma 2.2, $M_k \otimes I_r - I_{d_k} \otimes H(0)$ is non-singular for $k > n_0$. Hence, we can find $v_{[n_0+1]}, \ldots, v_{[n]}$ from $v_{[0]}, \ldots, v_{[n_0]}$ by using (2.17) for $k = n_0+1, \ldots, n$. This implies that (2.3) has a solution $p = \sum_{|\beta|\le n} p_\beta \omega^\beta \in P_n^r$ with $(p_\beta)_{|\beta|=k} = v_{[k]}$, $0 \le k \le n$. By Theorem 2.1, (1.1) has a compactly supported distributional solution ϕ such that $\widehat{\phi}^{[n]} = p$. Consequently, $\widehat{\phi}^{[n_0]}(\omega) = \sum_{|\alpha|\le n_0} v_\alpha \omega^\alpha$, $\omega \in \mathbb{R}^s$, where v_α $(|\alpha| \le n_0)$ are determined by $(v_\alpha)_{|\alpha|=j} = v_{[j]}$ for $j = 0, \ldots, n_0$.

Finally, we establish the uniqueness of the solution. Let ϕ and ψ be two compactly supported distributional solutions of (1.1) with $\hat{\phi}^{[n_0]} = \hat{\psi}^{[n_0]}$. Write $\hat{\phi}(\omega) = \sum_{\alpha\in\mathbb{N}_0^s} v_\alpha \omega^\alpha$ and $\hat{\psi}(\omega) = \sum_{\alpha\in\mathbb{N}_0^s} u_\alpha \omega^\alpha$, $\omega \in \mathbb{R}^s$. For $k \in \mathbb{N}_0$, let $v_{[k]} := (v_\alpha)_{|\alpha|=k}$ and $u_{[k]} := (u_\alpha)_{|\alpha|=k}$. We claim that $u_{[k]} = v_{[k]}$ for all $k \in \mathbb{N}_0$. This is shown by induction on k. It is clear that $u_{[k]} = v_{[k]}$ for $k = 0, \ldots, n_0$. Assume that $u_{[j]} = v_{[j]}$ for $j = 0, \ldots, k-1$. Consider $k > n_0$, it follows from (2.17) that

$$(M_k \otimes I_r)\, u_{[k]} - \sum_{j=0}^k (H_{\alpha-\gamma})_{|\alpha|=k,|\gamma|=j} u_{[j]} = (M_k \otimes I_r)\, v_{[k]} - \sum_{j=0}^k (H_{\alpha-\gamma})_{|\alpha|=k,|\gamma|=j} v_{[j]}.$$

Since $u_{[j]} = v_{[j]}$ for $j = 0, \ldots, k-1$, we have that

$$(M_k \otimes I_r - I_{d_k} \otimes H(0))\, u_{[k]} = (M_k \otimes I_r - I_{d_k} \otimes H(0))\, v_{[k]}.$$

But the matrix $M_k \otimes I_r - I_{d_k} \otimes H(0)$ is non-singular for $k > n_0$. Therefore, $u_{[k]} = v_{[k]}$. This shows $\phi = \psi$, as desired. □

When $H(0)$ has no eigenvalues of the form λ^β, $\beta \in \mathbb{N}_0^s$, the homogeneous equation (1.2) only has the trivial solution. The following corollary generalizes the result of [17] to an arbitrary dilation matrix.

Corollary 2.1. *Homogeneous refinement equation* (1.2) *has a nontrivial compactly supported distributional solution if and only if* $H(0)$ *has an eigenvalue of the form* λ^β, $\beta \in \mathbb{N}_0^s$. *Furthermore, the number of linearly independent compactly supported solutions of* (1.2) *is the same as the dimension of the space* $\ker(T_{n_0})$.

Suppose that ϕ and ψ are two compactly supported distributional solutions of (1.1). Then $\phi - \psi$ is a solution of the corresponding homogeneous equation (1.2). Thus, we have the following corollary.

Corollary 2.2. *Suppose* $H(0)$ *has eigenvalues of the form* λ^β *for some* $\beta \in \mathbb{N}_0^s$. *Let* S *be the set of all compactly supported distributional solutions of* (1.1). *If* (1.1) *has at least one solution, then* S *is a linear manifold whose dimension is the same as that of* $\ker(T_{n_0})$, *where* $n_0 := \max\{|\beta| : \lambda^\beta$ *is an eigenvalue of* $H(0)\}$.

3. Examples

In this section we give several examples to illustrate our theory.

Example 1. The following interesting example was first studied in [11]. Let $r = 2$ and $s = 1$. Consider the discrete refinement equation

$$\phi = \sum_{j=0}^{3} a(j)\phi(2\cdot - j), \tag{3.1}$$

where

$$a(0) = \begin{bmatrix} h_1 & 1 \\ h_2 & h_3 \end{bmatrix}, \qquad a(1) = \begin{bmatrix} h_1 & 0 \\ h_4 & 1 \end{bmatrix},$$

$$a(2) = \begin{bmatrix} 0 & 0 \\ h_4 & h_3 \end{bmatrix}, \qquad a(3) = \begin{bmatrix} 0 & 0 \\ h_2 & 0 \end{bmatrix},$$

and h_1, h_2, h_3, h_4 are given by

$$h_1 = -\frac{t^2 - 4t - 3}{2(t+2)}, \qquad h_2 = -\frac{3(t^2-1)(t^2-3t-1)}{4(t+2)^2},$$

$$h_3 = \frac{3t^2 + t - 1}{2(t+2)}, \qquad h_4 = -\frac{3(t^2-1)(t^2-t+3)}{4(t+2)^2}.$$

It was proved in [11] that the refinement equation has a unique continuous nontrivial solution $\phi = (\phi_1, \phi_2)^T$ for $|t| < 1$. In particular, when $t = -0.2$, the shifts of ϕ_1 and ϕ_2 are orthogonal, and corresponding orthogonal double wavelets were constructed there.

Consider the case $|t| > 1$. We note that $H(\omega) = \sum_{j=0}^{3} a(j)e^{-ij\omega}/2$, $\omega \in \mathbb{R}$. The matrix $H(0)$ has two eigenvalues 1 and t. Therefore, (3.1) has compactly supported distributional solutions only if $t = 2^n$ for some positive integer (see [14]). The case $t = 2$ was discussed in [24], and it was shown there that (3.1) has *two* linearly independent solutions.

Here, we consider the case $t = 4$. Write $H(\omega) = H_0 + H_1\omega + H_2\omega^2 + \cdots$, $\omega \in \mathbb{R}$, where $H_0, H_1, H_2, \ldots$ are 2×2 matrices. For the case $t = 4$, $n_0 = 2$, the corresponding matrix T_2 is given by

$$T_2 = -\begin{bmatrix} H(0) - I_2 & 0 & 0 \\ H_1 & H(0) - 2I_2 & 0 \\ H_2 & H_1 & H(0) - 4I_2 \end{bmatrix}.$$

A simple computation yields $\dim(\ker(T_2)) = 1$ for $t = 4$. By Corollary 2.1 we conclude that (3.1) has *one* linearly independent compactly supported distributional solution. Moreover, if ϕ is a nontrivial solution of (3.1), then we must have $\hat{\phi}(0) = 0$. This is in sharp contrast to the case $t = 2$.

Example 2. Let $r = 2$, $M = 2I_s$ and $g = 0$. Suppose

$$(3.2) \qquad H(\omega) = \begin{bmatrix} h_{11}(\omega) & h_{12}(\omega) \\ h_{21}(\omega) & h_{22}(\omega) \end{bmatrix}, \quad \omega \in \mathbb{R}^s \quad \text{and} \quad H(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

In this case, $n_0 = 1$, and

$$T_1 = -\begin{bmatrix} H(0) - I_2 & & & & \\ D_1H(0) & H(0) - 2I_2 & & & \\ D_2H(0) & 0 & H(0) - 2I_2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ D_sH(0) & 0 & 0 & \cdots & H(0) - 2I_2 \end{bmatrix}.$$

If $D_jh_{21}(0) = 0$ for all $j = 1, \ldots, s$, then $\dim(\ker(T_1)) = s + 1$. So (1.2) has exactly $s + 1$ linearly independent solutions, by Corollary 2.1. Otherwise, (1.2) has exactly s

linearly independent solutions. Moreover, the homogeneous refinement equation (1.2) has a compactly supported distributional solution ϕ such that $\hat{\phi}(0) \neq 0$ if and only if $D_j h_{21}(0) = 0$ for all $j = 1, \ldots, s$. For discrete refinement equations, this recovers the result of Theorem 4 in [24].

The next two examples are devoted to nonhomogeneous refinement equations.

Example 3. Let $r = 2$, $s = 1$, $M = (2)$, and $g = (g_1, g_2)^T$. Suppose the conditions in (3.2) are satisfied. In this case, $n_0 = 1$, and T_1 is the 4×4 matrix given by

$$T_1 = -\begin{bmatrix} H(0) - I_2 & 0 \\ H'(0) & H(0) - 2I_2 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h'_{11}(0) & h'_{12}(0) & -1 & 0 \\ h'_{21}(0) & h'_{22}(0) & 0 & 0 \end{bmatrix}.$$

By Theorem 2.2, (1.1) has a compactly supported distributional solution if and only if the linear equation

$$T_1 v = [\hat{g}_1(0), \hat{g}_2(0), 2\hat{g}'_1(0), 2\hat{g}'_2(0)]^T$$

has a solution v in $\mathbb{C}^4$. Let S be the set of all compactly supported distributional solutions of (1.1). There are two possible cases: $h'_{21}(0) \neq 0$ and $h'_{21}(0) = 0$. In the former case, (1.1) has a solution if and only if $\hat{g}_1(0) = 0$, and $\dim(S) = 1$, by Corollary 2.2. In the latter case, *i.e.*, $h'_{21}(0) = 0$, (1.1) has a compactly supported distributional solution if and only if $\hat{g}_1(0) = 0$ and $\hat{g}_2(0) h'_{22}(0) = 2\hat{g}'_2(0)$. If this is the case, then $\dim(S) = 2$.

Example 4. Let $r = 2$, $s = 2$, $M = 2I_2$, and $g = (g_1, g_2)^T$. Suppose the conditions in (3.2) are satisfied. In this case, $n_0 = 1$, and T_1 is the 6×6 matrix given by

$$T_1 = -\begin{bmatrix} 0 & 0 & & & & \\ 0 & 1 & & & & \\ D_1 h_{11}(0) & D_1 h_{12}(0) & -1 & 0 & & \\ D_1 h_{21}(0) & D_1 h_{22}(0) & 0 & 0 & & \\ D_2 h_{11}(0) & D_2 h_{12}(0) & 0 & 0 & -1 & 0 \\ D_2 h_{21}(0) & D_2 h_{22}(0) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 2.2, (1.1) has a compactly supported distributional solution if and only if the linear equation

$$T_1 v = [\hat{g}_1(0), \hat{g}_2(0), 2D_1\hat{g}_1(0), 2D_1\hat{g}_2(0), 2D_2\hat{g}_1(0), 2D_2\hat{g}_2(0)]^T$$

has a solution v in $\mathbb{C}^6$. Let S be the set of all compactly supported distributional solutions of (1.1). There are two possible cases.

Case 1: Suppose $D_1 h_{21}(0) = 0$ and $D_2 h_{21}(0) = 0$. By Theorem 2.2, (1.1) has a compactly supported distributional solution if and only if

$$\hat{g}_1(0) = 0, \quad 2D_1\hat{g}_2(0) = \hat{g}_2(0)D_1 h_{22}(0), \quad \text{and} \quad 2D_2\hat{g}_2(0) = \hat{g}_2(0)D_2 h_{22}(0).$$

In this case, Corollary 2.2 confirms that $\dim(S) = 3$, since the dimension of $\ker(T_1)$ is 3.

Case 2: Suppose $D_1 h_{21}(0) \neq 0$ or $D_2 h_{21}(0) \neq 0$. In this case (1.1) has a compactly supported distributional solution if and only if $\hat{g}_1(0) = 0$ and

$$D_1 h_{21}(0)\Big(2D_2\hat{g}_2(0) - \hat{g}_2(0)D_2 h_{22}(0)\Big) = D_2 h_{21}(0)\Big(2D_1\hat{g}_2(0) - \hat{g}_2(0)D_1 h_{22}(0)\Big).$$

In this case, $\dim(S) = 2$ by the fact that the dimension of $\ker(T_1)$ is 2.

Acknowledgments. The research is supported in part by NSERC Canada under Grant OGP 121336 and the Wavelets Strategic Research Programme, National University of Singapore, under a grant from the National Science and Technology Board and the Ministry of Education, Singapore.

References

1. C. Cabrelli, C. Heil and U. Molter, *Accuracy of lattice translates of several multidimensional refinable functions*, J. Approx. Th. **95** (1998), 5–52.
2. J. Barros-Neto, An Introduction to the Theory of Distributions, Marcel Dekker, New York, 1973.
3. A. S. Cavaretta, W. Dahmen and C. A. Micchelli, Stationary Subdivision, Memoirs of Amer. Math. Soc., Volume 93, 1991.
4. C. K. Chui and X. Shi, *Continuous two-scale equations and dyadic wavelets,* Adv. in Comp. Math. **2** (1994), 185–213.
5. A. Cohen, I. Daubechies and G. Plonka, *Regularity of refinable function vectors*, J. Fourier Anal. and Appl. **3** (1997), 295–324.
6. W. Dahmen and C. A. Micchelli, *Continuous refinement equations and subdivision,* Adv. in Comp. Math. **1** (1993), 1–37.

7. I. Daubechies and J. Lagarias, *Two-scale difference equations: I. Existence and global regularity of solutions*, SIAM J. Math. Anal. **22** (1991), 1388–1410.
8. G. Derfel, N. Dyn and D. Levin, *Generalized refinement equations and subdivision processes*, J. Approx. Th. **80** (1995), 272–297.
9. T. B. Dinsenbacher and D. P. Hardin, *Nonhomogeneous refinement equations*, in "Wavelets, Multiwavelets, and their Applications", A. Aldroubi and E. Lin (eds.), pp. 117–127, Amer. Math. Soc., Providence, 1998.
10. T. B. Dinsenbacher and D. P. Hardin, *Multivariate nonhomogeneous refinement equations*, preprint, 1998.
11. G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Construction of orthogonal wavelets using fractal interpolation functions*, SIAM J. Math. Anal. **27** (1996), 1158–1192.
12. N. Dyn and A. Ron, *Multiresolution analysis by infinitely differentiable compactly supported functions,* Appl. and Comp. Harmon. Anal. **2** (1995), 15–20.
13. B. Han and R. Q. Jia, *Multivariate refinement equations and convergence of subdivision schemes*, SIAM J. Math. Anal. **29** (1998), 1177–1199.
14. C. Heil and D. Colella, *Matrix refinement equations: Existence and uniqueness*, J. Fourier Anal. and Appl. **2** (1996), 75–94.
15. R. Q. Jia, S. L. Lee and A. Sharma, *Spectral properties of continuous refinement operators*, Proc. Amer. Math. Soc. **126** (1998), 729–737.
16. Q. T. Jiang and S. L. Lee, *Spectral properties of matrix continuous refinement operators*, Adv. in Comp. Math. **7** (1997), 383–399.
17. Q. T. Jiang and Z. W. Shen, *On existence and weak stability of matrix refinable functions*, Constr. Approx. **15** (1999), 337–353.
18. K. Kabaya and M. Iri, *Sum of uniformly distributed random variables and a family of nonanalytic C^∞-functions,* Japan J. Appl. Math. **4** (1987), 1–22.
19. K. Kabaya and M. Iri, *On operators defining a family of nonanalytic C^∞-functions,* Japan J. Appl. Math. **5** (1988), 333–365.
20. M. H. Protter and C. B. Morrey, A First Course in Real Analysis, Second Edition, Springer-Verlag, New York, 1977.
21. V. A. Rvachev, *Compactly supported solutions of functional-differential equations and their applications,* Russian Mathematical Survey, **45:1** (1990), 87–120 .
22. G. Strang and D. X. Zhou, *Inhomogeneous refinement equations*, J. Fourier Anal. and Appl. **4** (1998), 733–747.
23. Q. Y. Sun, *Nonhomogeneous refinement equations: Existence, regularity and biorthogonality*, preprint, 1998.
24. D. X. Zhou, *Existence of multiple refinable distributions*, Michigan Math. J. **44** (1997), 317–329.

Department of Mathematical Sciences, University of Alberta, Edmonton, Canada T6G 2G1. E-mail: jiaxihu.math.ualberta.ca, jiangmath.ualberta.ca

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260. E-mail: matzuowsleonis.nus.edu.sg