STABILITY AND ORTHONORMALITY OF MULTIVARIATE REFINABLE FUNCTIONS

W. LAWTON, S. L. LEE AND ZUOWEI SHEN

Abstract. This paper characterizes the stability and orthonormality of the shifts of a multidimensional (M,c) refinable function ϕ in terms of the eigenvalues and eigenvectors of the transition operator $W_{c_{au}}$ defined by the autocorrelation c_{au} of its refinement mask c, where M is an arbitrary dilation matrix. Another consequence is that if the shifts of ϕ form a Riesz basis, then $W_{c_{au}}$ has a unique eigenvector of eigenvalue 1, and all its other eigenvalues lie inside the unit circle. The general theory is applied to two-dimensional nonseparable (M,c) refinable functions whose masks are constructed from Daubechies' conjugate quadrature filters.

1. Introduction

In this paper we present a complete characterization of the stability and orthonormality of the shifts of a refinable function in terms of the refinement mask by analysing the simplicity of eigenvalue 1 of the transition operator.

Denote by $\ell^1(\mathbf{Z^d})$ and $\ell^2(\mathbf{Z^d})$ the spaces of absolutely summable and modulus square summable complex-valued sequences defined on $\mathbf{Z^d}$, respectively. Let $M \in \mathbf{Z^{d\times d}}$ be a $d\times d$ integer matrix with eigenvalues of modulus > 1 and with $|\det M| = m > 1$. Let $c \in \ell^1(\mathbf{Z^d})$ and $\phi : \mathbf{R^d} \to \mathbf{C}$ be a complex-valued function. The equation

$$\phi(x) = \sum_{q\in\mathbf{Z^d}} m\ c(q)\phi(Mx-q) \tag{1.1}$$

is called a *refinement equation.* The matrix M is called a *dilation matrix.* The sequence c is called a *refinement mask*, and the function ϕ is called a (M,c) *refinable function* or (M,c) *scaling function.* We assume $\int \phi(x)dx = 1$.

Date:

1991 *Mathematics Subject Classification.* 41A15, 41A30, 42C05, 42C15.

Key words and phrases. refinement equations, interpolatory refinable functions, dilation matrix, subdivision operators, transition operators.

Denote by $L^1(\mathbf{R^d})$ and $L^2(\mathbf{R^d})$ the spaces of Lebesque integrable and modulus square integrable functions defined on $\mathbf{R^d}$, respectively. The class of all tempered distributions on $\mathbf{R^d}$ will be denoted by $\mathcal{S}'$. The dilation operator M associated with the dilation matrix M is defined for all functions ϕ by $M\phi(x) := \phi(Mx),\ x \in \mathbf{R^d}$. This can be extended to all distributions $\phi \in \mathcal{S}'$ by defining

$$\langle M\phi, f\rangle := \frac{1}{m}\langle \phi, M^{-1}f\rangle \text{ for all } f \in \mathcal{S}\ ,$$

where $\mathcal{S}$ denotes the class of all infinitely differentiable functions with rapid decay at infinity. Similarly the shift operator $T^p\phi(x) := \phi(x-p),\ p \in \mathbf{Z^d}$, for functions may be extended to distributions by

$$\langle T^p\phi, f\rangle := \langle \phi, T^{-p}f\rangle \text{ for all } f \in \mathcal{S}\ .$$

The refinement equation (1.1) may now be extended to include distributions $\phi \in \mathcal{S}'$ by writing

$$\phi = \sum_{q\in\mathbf{Z^d}} m\ c(q) M T^q \phi\ . \tag{1.2}$$

A distribution ϕ which satisfies (1.2) is called a (M, c) *refinable distribution.*

The Fourier transform of a sequence $a \in \ell^1(\mathbf{Z^d})$ will be denoted by $\widehat{a}$ and is defined by

$$\widehat{a}(u) := \sum_{p\in\mathbf{Z^2}} a(p)e^{-ipu}\ .$$

where $i \equiv \sqrt{-1}$. Note that $\widehat{a}(u)$ is a complex-valued 2π-periodic continuous function on $\mathbf{R^d}$ and thus is defined on the d-dimensional torus $\mathbf{T^d}$. For a finitely supported sequence $(a_j)_{j\in\mathbf{Z^d}}$ with support in $[0, N-1]^d$, we define N as its *length* .

For any continuous function f defined on $\mathbf{R^d}$ we shall denote by $f_|$ the sequence $(f(p))_{p\in\mathbf{Z^d}}$, which is the restriction f to $\mathbf{Z^d}$.

The Fourier transform of a function $f \in L^1(\mathbf{R^d})$ is

$$\widehat{f}(u) := \int_{\mathbf{R^d}} f(x)e^{-ixu}dx\ .$$

This maps $\mathcal{S}$ onto itself, and extends to all tempered distributions $\mathcal{S}'$ by duality.

We shall assume throughout this paper that c is a finitely supported sequence satisfying

$$\sum_{p \in \mathbf{Z^d}} c(p) = 1 \ . \tag{1.3}$$

Then there exists a compactly supported (M, c) refinable distribution ϕ, unique up to a constant multiple, such that its Fourier transform admits the infinite product representation

$$\widehat{\phi}(u) = \widehat{\phi}(0) \prod_{j=1}^{\infty} \widehat{c}\left((M^T)^{-j} u\right), \quad u \in \mathbf{R^d} \tag{1.4}$$

(see [11]). Hereafter, we assume that $\widehat{\phi}(0) = 1$.

A (M, c) refinable function $\phi \in L^2(\mathbf{R^d})$ is *stable* if $\{\phi(x-p)\}_{p \in \mathbf{Z^d}}$ is a Riesz basis of its closed linear span. It is *orthonormal* if $\{\phi(x-p)\}_{p \in \mathbf{Z^d}}$ is an orthonormal basis of its closed linear span.

For a (M, c) refinable function $\phi \in L^2(\mathbf{R})$, define

$$\phi_{au}(x) := \int_{\mathbf{R^d}} \phi(x-t)\overline{\phi(-t)}\ dt, \quad x \in \mathbf{R^d}, \tag{1.5}$$

and

$$c_{au}(p) := \sum_{q \in \mathbf{Z^d}} c(p-q)\overline{c(-q)}, \quad p \in \mathbf{Z^d} \ . \tag{1.6}$$

Then ϕ_{au} is a continuous (M, c_{au}) refinable function. The function ϕ_{au} is called the *autocorrelation* of ϕ and the sequence c_{au} is called the *autocorrelation* of c.

A necessary condition for a (M, c) refinable function ϕ to be orthonormal is that the refinement mask c satisfies the conditions

$$m\ c_{au}(Mp) = \delta(p), \quad p \in \mathbf{Z^d} \ , \tag{1.7}$$

and

$$\sum_{q \in \mathbf{Z^d}} c(q) = 1 \ , \tag{1.8}$$

where $\delta(p) = 1$ for $p = 0$ and zero otherwise. A sequence c which satisfies (1.7) and (1.8) is called a *conjugate quadrature filter* with respect to the dilation matrix M (M-CQF). Note that (1.8) implies

$$\sum_{q \in \mathbf{Z^d}} c_{au}(q) = 1 \ . \tag{1.9}$$

For a dilation matrix M and any finitely supported refinement mask c, we define the (M, c) *subdivision operator* $S_c : \ell^2(\mathbf{Z^d}) \to \ell^{\mathbf{2}}(\mathbf{Z^d})$, by

$$(S_c b)(p) := \sum_{q \in \mathbf{Z^d}} m \ c(p - Mq) b(q), \quad b \in \ell^1(\mathbf{Z^d}) \ . \tag{1.10}$$

For the case $M = 2I$, this operator has been extensively studied in [1]. The adjoint of $S_{\tilde{c}}$, where $\tilde{c}(p) := \overline{c(-p)}$, $p \in \mathbf{Z^d}$, is the (M, c) *transition operator* which shall be denoted by W_c. Thus, the operator $W_c : \ell^2(\mathbf{Z^d}) \to \ell^{\mathbf{2}}(\mathbf{Z^d})$ is defined by

$$(W_c b)(p) = \sum_{q \in \mathbf{Z^d}} m \ c(Mp - q) b(q), \quad b \in \ell^2(\mathbf{Z^d}) \, . \tag{1.11}$$

We remark that the transition operator $W_{c_{au}}$, corresponding to the autocorrelation c_{au} of c, is called the *wavelet-Galerkin operator* in [18].

Note that if c is conjugate symmetric, i.e. $c = \tilde{c}$, then $S_c = S_{\tilde{c}}$, and W_c is the adjoint of the subdivision operator S_c. For our purpose we shall restrict the transition operator to the space $\ell^1(\mathbf{Z^d})$. If ϕ is a (M, c) refinable continuous function, where c is finitely supported, then ϕ is compactly supported, and the sequence $\phi_|$ is an eigenvector of W_c in $\ell^1(\mathbf{Z^d})$ of eigenvalue 1. If c is a M-CQF, then $\phi_{c_{au}|} = \delta$, which is an eigenvector of $W_{c_{au}}$ of eigenvalue 1.

The definition (1.11) shows that if the refinement mask c is supported in $[0, N-1]^d$, and $M = 2I$, then $W_{c_{au}}$ maps sequences supported on $[-N+1, N-1]^d$ into sequences supported on $[-N+1, N-1]^d$. For a general dilation matrix M, a more detailed discussion in §4 leads to the fact that $W_{c_{au}}$ has a finite dimensional invariant subspace consisting of sequences on a finite set. The operator $W_{c_{au}}$ (respectively W_c) restricted to any of its finite dimensional invariant subspace will be called a *restricted transition operator.*

The eigenvalues of $W_{c_{au}}$ hold the key to many important properties of the refinable function, for instance, stability, regularity and the convergence of the cascade algorithm (see [17], [5], [9], [8], [22]). The first indication of this role appeared in [17] and [18] where it was shown that for $d = 1$, and $M = (2)$, i.e. in one dimension with dyadic scaling, if ϕ is a $(2, c)$ refinable function, then ϕ is orthonormal if and only if 1 is a simple eigenvalue of the transition operator $W_{c_{au}}$.

The object of this note is to investigate further the relationship between stability and orthonormality of a (M, c) refinable function on one hand and the eigenvalues of the corresponding transition operator $W_{c_{au}}$ on the other, in the multivariate setting with an arbitrary dilation matrix. In particular we give a multidimensional extension of the results in [17] on the characterization of the orthonormality of the refinable function. We further show that a (M, c) refinable function ϕ is stable if and only if the transition operator $W_{c_{au}}$ has a unique eigenvector of eigenvalue 1, whose Fourier transform does not vanish on the torus. This is given in Theorem 2.2, where Theorem 2.1 plays a key role in the proof. Another consequence of Theorem 2.1 is the fact that if the shifts of a (M, c) refinable function ϕ form a Riesz basis, then the sequence $\phi_{c_{au}|}$ is the unique eigenvector of $W_{c_{au}}$ corresponding to the eigenvalue 1, and all the other eigenvalues of $W_{c_{au}}$ lie inside the unit circle. Section 3 deals with M-CQF's. In particular, it is shown that for a M-CQF the corresponding (M, c) refinable function belongs to $L^2(\mathbf{R^d})$, and further characterizations of orthonormality are also given. Restricted transition operators are studied in more detail in §4. It is shown that checking for stability and orthonormality is reduced to checking whether 1 is a simple eigenvalue of a finite order matrix, which is generated from the refinement mask of ϕ. The general theory is applied to the construction of nonseparable conjugate quadrature filters (M-CQF's) and the corresponding refinable functions from the one-dimensional CQF's of Daubechies.

Another approach in the characterization of stability and orthonormality of a refinable function ϕ with finitely supported refinement mask c makes use of the zero set of the Fourier transform of c. A detailed discussion for the univariate case can be found in [2] and [12]. In one dimension, both approaches characterize the stability and orthonormality of a refinable function ϕ in terms of its refinement mask c, using the equivalent characterization of the Fourier transform of ϕ. We prefer the eigenvalue approach for the following reasons. First, as was pointed out in [7], for a specified finitely supported mask, it is easier to check for stability and orthonormality of the corresponding refinable function using the eigenvalue characterization. In this case the problem of checking for stability and orthonormality is reduced to the simple routine of checking whether 1 is a simple eigenvalue of a finite order matrix . Second, the analysis of the zero set of the Fourier transform of the refinement mask relies on the fact that a univariate polynomial has a finite number of zeros. This no longer holds for multivariate polynomials. However, it is possible to extend the corresponding univariate results to higher dimensions, by imposng the condition that a certain multivariate polynomial has a finite number of zeros, as suggested by [11].

It is of particular interest to construct compactly supported wavelets from a compactly supported refinable function and its mask. In the univariate case, with dyadic scaling (M=(2)), the construction is simple. For a general integer dilation $M = (m)$, an algorithmic approach in the construction of compactly supported wavelets from a given refinable function ϕ and its refinement mask is given in [16]. The problem of wavelet construction from a (M, c) refinable function ϕ and its refinement mask is much more challenging in higher dimensions. However, in dimensions no greater than 3 and $M = 2I$, a method is given in [20] and [21], under a mild condition on the refinement mask c. In this case, compactly supported wavelets can be constructed based solely on c.

2. Stability and orthonormality of refinable functions

Let $c : \mathbf{Z}^\mathbf{d} \to \mathbf{C}$ be a finitely supported sequence satisfying (1.3) and M be a dilation matrix. This section studies the relationship between stability and orthonormality of the (M, c) refinable function ϕ on one hand and the spectral properties of the corresponding transition operator W_c, on the other. Recall that the sequence $\phi_|$ is an eigenvector of W_c of eigenvalue 1. We shall first establish a result relating the spectrum of W_c and the nonvanishing of the Fourier transform of $\phi_|$, and then deduce results on stability and orthonormality of ϕ.

Lemma 2.1. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$. Let ϕ be a continuous (M, c) refinable function and $b \in \ell^1(\mathbf{Z^d})$. Then for any integer $N \geq 1$ and for any $r \in \mathbf{Z^d}$,*

$$\sum_{p \in \mathbf{Z^d}} b(p)\phi\left(r - M^{-N}p\right) = \left(W_c^N(b * \phi_|)\right)(r) \ . \tag{2.1}$$

Proof. The proof is by induction on N. For $N = 1$, applying the refinement equation (1.1) gives

$$\begin{aligned} \sum_{p \in \mathbf{Z^d}} b(p)\phi\left(r - M^{-1}p\right) &= \sum_{p,q \in \mathbf{Z^d}} mb(p)c(q)\phi_| \left(Mr - p - q\right) \\ &= \sum_{q \in \mathbf{Z^d}} mc(q) b * \phi_|(Mr - q) \\ &= \left(W_c(b * \phi_|)\right)(r) \ . \end{aligned}$$

If (2.1) holds for N, then

$$\begin{aligned} \left(W_c^{N+1}(b * \phi_|)\right)(r) &= \sum_{q \in \mathbf{Z^d}} mc(Mr - q) \left(W_c^N(b * \phi_|)\right) \\ &= \sum_{q \in \mathbf{Z^d}} mc(Mr - q) \sum_{p \in \mathbf{Z^d}} b(p)\phi\left(q - M^{-N}p\right) \ . \end{aligned} \tag{2.2}$$

Interchanging the order of summation on the sum in (2.2) followed by a change of index, it can be written as

$$\sum_{p \in \mathbf{Z^d}} b(p) \sum_{k \in \mathbf{Z^d}} mc(k)\phi\left(M(r - M^{-N-1}p) - k\right) = \sum_{p \in \mathbf{Z^d}} b(p)\phi\left(r - M^{-N-1}p\right) \ .$$

The result now follows by induction. □

Corollary 2.1. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$ and ϕ is a continuous (M, c) refinable function. Then $\phi_|$ is the unique eigenvector of W_c in $\ell^1(\mathbf{Z^d}) * \phi_|$ of eigenvalue 1*

Proof. Suppose that $b * \phi_|,\ b \in \ell^1(\mathbf{Z^d})$, is another eigenvector of W_c of eigenvalue 1. Then (2.1) gives

$$\sum_{p\in\mathbf{Z^d}} b(p)\phi\left(r - M^{-N}p\right) = \left(W_c^N(b*\phi_|)\right)(r) = b*\phi_|(r)\ ,$$

for all integers $N \geq 0$ and $r \in \mathbf{Z^d}$. Letting $N \to \infty$, we have

$$\left(\sum_{p\in\mathbf{Z^d}} b(p)\right)\phi(r) = b*\phi_|(r)\ ,$$

which is equivalent to

$$\left(\sum_{p\in\mathbf{Z^d}} b(p)\right)\widehat{\phi}_|(u) = \widehat{b}(u)\widehat{\phi}_|(u), \quad u \in \mathbf{R^d}\ .$$

Since $\widehat{\phi}_|$ does not vanish on a set of positive measure, it follows that $\widehat{b}(u)$ is a constant. Equivalently, $b = \alpha\ \delta$, for some $\alpha \in \mathbf{C}$. Hence, $b*\phi_| = \alpha\phi_|$. □

Remark 1. *In general, if* $\widehat{\phi}_|$ *can be factored as*

$$\widehat{\phi}_|(u) = \widehat{h}(u)\widehat{g}(u)\ ,$$

where $\widehat{g}(u) \neq 0$ *for all* $u \in \mathbf{R^d}$, *and* $g \in \ell^1(\mathbf{Z})$, *then it follows from Corollary 2.1 and Wiener's Theorem, that* $\phi_|$ *is the unique eigenvector of* W_c *in* $h * \ell^1(\mathbf{Z})$ *of eigenvalue* 1.

Theorem 2.1. *Suppose that* c *is a finitely supported sequence satisfying* $\widehat{c}(0) = 1$ *and* ϕ *is a continuous* (M, c) *refinable function. If*

$$\widehat{\phi}_|(u) := \sum_{p\in\mathbf{Z^d}} \phi(p)e^{-ipu} \neq 0, \quad u \in \mathbf{R^d}\ , \tag{2.3}$$

then $\phi_|$ *is the unique eigenvector of* W_c *in* $\ell^1(\mathbf{Z^d})$ *of eigenvalue* 1, *and all the other eigenvalues of* W_c *lie inside the unit circle.*

Further, 1 *is a simple eigenvalue of any restricted* W_c.

Proof. Suppose (2.3) holds, and let

$$1/\widehat{\phi}_|(u) =: \sum_{p\in\mathbf{Z^d}} w(p)e^{-ipu}, \quad u \in \mathbf{R^d}\ .$$

Since $1/\widehat{\phi_|}(u)$ is smooth, it follows that $w \in \ell^1(\mathbf{Z^d})$. Hence $\ell^1(\mathbf{Z^d}) * \phi_| = \ell^\mathbf{1}(\mathbf{Z^d})$, and Corollary 2.1 implies that $\phi_|$ is the unique eigenvector of W_c in $\ell^1(\mathbf{Z})$ of eigenvalue 1.

Now, let $\lambda \neq 1$ be an eigenvalue of W_c and let $v \in \ell^1(\mathbf{Z^d})$ be the corresponding eigenvector. Equation (2.1) gives

$$\lambda^N v(r) = (W_c^N v)(r) = (W_c^N v * w * \phi_|)(r) = \sum_{p \in \mathbf{Z}} (v * w)(p) \phi\left(r - M^{-N} p\right) \ . \tag{2.4}$$

The limit, as $N \to \infty$, of the sum on the right of (2.4) exists and is equal to

$$\left(\sum_p v * w(p) \right) \phi(r), \quad r \in \mathbf{Z^d} \ .$$

Therefore, if $\lambda \neq 1$ then necessarily $|\lambda| < 1$. Further, $\sum_p v * w(p) = 0$.

If 1 is not a simple eigenvalue of a restricted transition operator W_c, then it must be a degenerate eigenvalue with only one eigenvector, say b. In this case, there exists a vector b_1 such that $W_c b_1 = b_1 + b$, which implies that $W_c^N b_1 = b_1 + Nb$, for all integers $N \geq 1$. Again, (2.4) gives

$$b_1(r) + Nb(r) = (W_c^N b_1)(r) = \sum_{p \in \mathbf{Z}} (b_1 * w)(p) \phi\left(r - M^{-N} p\right) \ ,$$

for all $N \geq 1$ which is impossible. □

A function $\phi \in L^2(\mathbf{R^d})$ is stable if $\{\phi(\cdot - p)\}_{p \in \mathbf{Z^d}}$ is a Riesz basis of its closed linear span. Recall that $\{\phi(\cdot - p)\}_{p \in \mathbf{Z}}$ is a Riesz basis if and only if there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \leq \sum_{q \in \mathbf{Z^d}} |\hat{\phi}(u + 2\pi q)|^2 \leq C_2, \text{ for almost all } u \in \mathbf{R^d}. \tag{2.5}$$

If $\phi_{au|} \in \ell^1(\mathbf{Z^d})$, the Poisson summation formula leads to the characterization that ϕ is stable if and only if

$$C_1 \leq \sum_p \phi_{au}(p) e^{-ipu} \leq C_2, \quad u \in \mathbf{R^d} \ . \tag{2.6}$$

Corollary 2.2. *Suppose that* c *is a finitely supported sequence satisfying* $\widehat{c}(0) = 1$ *and that* ϕ *is a stable* (M, c) *refinable function . Then* $\phi_{au|}$ *is the unique eigenvector in* $\ell^1(\mathbf{Z^d})$ *of* $W_{c_{au}}$ *corresponding to the eigenvalue* 1, *and all the other eigenvalues of* $W_{c_{au}}$ *lie inside the unit circle.*

The above corollary follows directly from Theorem 2.1 and equation (2.6). In one dimension with dyadic scaling, Cohen and Daubechies [3] have proved that for a stable $(2, c)$ refinable function ϕ, the eigenvalues of the corresponding transition operator $W_{c_{au}}$ restricted to its invariant subspace of finite sequences with zero mean, lie inside the unit circle. Their result has been extended to higher dimension with dilation matrix $M = 2I$ by Long and Chen [14]. We note that in one dimension, the result of Cohen and Daubechies has also been improved upon by Hervé [10].

The following theorem gives a characterization of stability of a (M, c) refinable function. A similar result in one-dimensional with dilation $M = (2)$ was obtained in [5].

Theorem 2.2. *Suppose that* c *is a finitely supported sequence satisfying* $\widehat{c}(0) = 1$. *An* (M, c) *refinable function* ϕ *is stable if and only if* $W_{c_{au}}$ *has a unique eigenvector of eigenvalue* 1 *whose Fourier transform does not vanish.*

Further, 1 *is a simple eigenvalue of any restricted* $W_{c_{au}}$.

Proof. If ϕ is stable, then condition (2.3) of Proposition 2.1 is satisfied for ϕ_{au}. Hence, $W_{c_{au}}$ has a unique eigenvector $\phi_{au|}$ of eigenvalue 1 which has a nonvanishing Fourier transform.

Conversely, since $\phi_{au|} \in \ell^1(\mathbf{Z^d})$ is an eigenvector of $W_{c_{au}}$ with eigenvalue 1, and since such an eigenvector is unique and has nonvanishing Fourier transform, it follows from (2.6) that the (M, c) refinable function ϕ is stable. □

A (M, c) refinable function $\phi \in L^2(\mathbf{R^d})$ is *interpolatory* if ϕ is continuous and satisfies

$$\phi(p) = \delta(p), \quad p \in \mathbf{Z^d} \ . \tag{2.7}$$

Theorem 2.3. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$. A necessary and sufficient condition for a continuous (M, c) refinable function ϕ to be interpolatory is that the sequence δ is the unique eigenvector of W_c of eigenvalue 1.*

Further, 1 is a simple eigenvalue of any restricted $W_{c_{au}}$.

Proof. Since ϕ is (M, c) refinable, the corresponding sequence $\phi_|$ is an eigenvector of W_c in $\ell^1(\mathbf{Z^d})$ with eigenvalue 1. If δ is the unique eigenvector of eigenvalue 1 then

$$\phi(p) = \delta(p), \quad p \in \mathbf{Z^d} \ ,$$

i.e. ϕ is interpolatory.

Conversely, if ϕ is interpolatory, then obviously $\phi_| = \delta$ is an eigenvector of W_c in $\ell^1(\mathbf{Z^d})$ with eigenvalue 1. Since

$$\sum_{p \in \mathbf{Z^d}} \phi(p) e^{-ipu} = 1, \qquad \text{for all } u \in \mathbf{R^d} \ ,$$

does not vanish, by Theorem 2.1, δ is the unique eigenvector of eigenvalue 1. □

Clearly $\phi \in L^2(\mathbf{R^d})$ is an orthonormal (M, c) refinable function if and only if ϕ_{au} is an interpolatory (M, c_{au}) refinable function.

Corollary 2.3. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$. An (M, c) refinable function ϕ is orthonormal if and only if the sequence δ is the unique eigenvector of $W_{c_{au}}$ of eigenvalue 1*

Further, 1 is a simple eigenvalue of any restricted $W_{c_{au}}$.

Combining this corollary with Theorem 2.2, we have the following proposition.

Proposition 2.1. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$ and that $\phi \in L^2(\mathbf{R^d})$ is (M, c) refinable. Then the following are equivalent:*

(i) *ϕ is orthonormal ,*

(ii) *c is an M-CQF and ϕ is stable,*

(iii) *c is M-CQF and $\widehat{\phi_{au|}}(u) \neq 0, \ u \in \mathbf{R^d}$.*

If c is a M-CQF, Corollary 2.3 says that the simplicity of the eigenvalue 1 of any restricted transition operator $W_{c_{au}}$ is equivalent to the orthonormality of the refinable function ϕ. On the other hand, Theorem 2.2 states that the stability of the refinable function ϕ is equivalent to the simplicity of the eigenvalue 1 of any restricted transition operator $W_{c_{au}}$ and the nonvanishing of the Fourier transform of the corresponding eigenvector. It will be shown in an example below that the simplicity of the eigenvalue 1 of a restricted $W_{c_{au}}$ does not imply the existence of an eigenvector with nonvanishing Fourier transform. This shows that the conditions in Theorem 2.2 are not superfluous. In fact, it will be interesting to know whether the simplicity of the eigenvalue 1 of a restricted transition operator together with the additional condition,

$$\sum_k |\widehat{c}(u + (M^T)^{-1} 2\pi\gamma_k)|^2 > 0, \tag{2.8}$$

would imply the nonvanishing of the Fourier transform of the corresponding eigenvector.

The following examples show that the simplicity of eigenvalue 1 together with (2.8) do not imply the nonvanishing of the Fourier transform of the corresponding eigenvector. In particular, they show that the converse of Theorem 2.1 is false even under the assumption (2.8).

Example 2.1

Let c be the sequence

$$\{\ldots, 0, 1/4, 1/2, 1/4, 1/4, 1/2, 1/4, 0, \ldots\},$$

where $c(0) = 1/4$, $c(1) = 1/2, \ldots$. It is straightforward to check that the sequence $a = \{\ldots, 0, 1, 2, 2, 2, 1, 0, \ldots\}$, where $a(0) = 1$, $a(1) = 2, \ldots$, is the unique eigenvector for W_c of eigenvalue 1, and that the Fourier transform of a is $(1+e^{iu})(1+e^{iu}+e^{i2u})$, which vanishes at -1.

We shall show that there is a compactly supported continuous $(2, c)$ refinable function ϕ, such that

$$\phi(n) = a(n), \quad n \in \mathbf{Z}\ , \tag{2.9}$$

and the Fourier transform $\widehat{c}(u)$ of the mask c satisfies

$$|\widehat{c}(u)|^2 + |\widehat{c}(u+\pi)|^2 > 0 \text{ for all } u \in \mathbf{R} \ . \tag{2.10}$$

The Fourier transform of c can be written as

$$\widehat{c}(u) = \widehat{b}(u)(1 - e^{iu} + e^{i2u}) = \widehat{b}(u)(e^{iu} + \omega)(e^{iu} + \omega^2) \ , \tag{2.11}$$

where

$$b(u) = \frac{1}{4}(1 + e^{iu})^3$$

is the Fourier transform of the mask

$$b = \{\ldots, 0, 1/4, 3/4, 3/4, 1/4, 0, \ldots\}$$

with $b(0) = 1/4, \ b(1) = 3/4, \ldots$, and $\omega \neq 1$ is a cube root of unity. All the roots of $\widehat{c}$ lie on the unit circle and they are precisely $-1, -1, -1, -\omega, -\omega^2$. Since no root is the negative of another root, $|\widehat{c}(u)|^2 + |\widehat{c}(u+\pi)|^2 > 0$ for all real u. Thus, the mask c satisfies (2.10).

Note that the sequence b is exactly the mask for the $(2, b)$ refinable quadratic B-spline g obtained by convolving the characteristic function $\chi_{[0,1)}$ with itself three times. Let

$$\phi(x) = g(x) + g(x-1) + g(x-2), \quad x \in \mathbf{R} \ . \tag{2.12}$$

Then the Fourier transform of ϕ is

$$\widehat{\phi}(u) = \widehat{g}(u)(e^{iu} - \omega)(e^{iu} - \omega^2) = \widehat{g}(u)(1 + e^{iu} + e^{i2u}) \ . \tag{2.13}$$

Since the set of roots of $(1-z)^3(1+z+z^2)$ is closed under the mapping $z \to z^2$, $\phi(x)$ is $(2, a)$ refinable by Theorem 2.1 of [15]. The function ϕ is not stable, since ω and ω^2 are zeros of $1 + z + z^2$. With a suitable normalization of g, we have

$$\phi_| = \{\ldots, 0, 1, 2, 2, 2, 1, 0, \ldots\} \ ,$$

which is (2.9) .

Example 2.2

Let c and ϕ be as in Example 1. Then

$$c_{au} = \{\ldots, 0, 1, 4, 6, 6, 9, 12, 9, 6, 6, 4, 1, 0, \ldots\} .$$

The sequence $\phi_{au|}$ is an eigenvector of $W_{c_{au}}$ of eigenvalue 1. The 11×11 linear system, satisfied by the eigenvectors for $W_{c_{au}}$ corresponding to the eigenvalue 1 has rank 10 (here $W_{c_{au}}$ is restricted to sequences supported on $[-5, 5]$, which form an invariant subspace containing all finitely supported eigenvectors). Hence $\phi_|$ is the unique eigenvector of $W_{c_{au}}$ of eigenvalue 1.

In summary, this example gives a $(2, c)$ refinable function which is not stable, but $W_{c_{au}}$ has a simple eigenvalue 1 and the mask c satisfies

$$|\widehat{c}(u)|^2 + |\widehat{c}(u+\pi)|^2 > 0, \text{ for all } u .$$

3. Multivariate Conjugate Quadrature Filters

Let M be an integer dilation matrix , and $c : \mathbf{Z^d} \to \mathbf{C}$ be a finitely supported sequence satisfying (1.3). Hence there is a unique compactly supported (M, c) refinable distribution ϕ, normalized so that $\widehat{\phi}(0) = 1$. Assuming that c is a M-CQF, we are interested to know when $\phi \in L^2(\mathbf{R^d})$, and to obtain further characterizations of orthonormality.

We first consider the cascade algorithm for the computation of the compactly supported (M, c) refinable distribution. Let $\phi^{(0)}$ be the indicator function of any fundamental region for $\mathbf{Z^d} \subset \mathbf{R^d}$ (thus $\widehat{\phi^{(0)}}$ is continuous at 0 and $\widehat{\phi^{(0)}}(0) = 1$). Starting from $\phi^{(0)}$, define a sequence of functions $\phi^{(n)}$ by

$$\phi^{(n)}(x) := \sum_{p \in Z^d} m\ c(p)\phi^{(n-1)}(Mx - p), \quad n = 1, 2, \ldots . \tag{3.1}$$

Then

$$\widehat{\phi^{(n)}}(u) = \widehat{\phi^{(0)}}\left((M^T)^{-n}u\right) \prod_{j=1}^{n} \left(\widehat{c}\left((M^T)^{-j}u\right)\right), \quad u \in \mathbf{R^d} . \tag{3.2}$$

The sequence $\widehat{\phi^{(n)}} \to \widehat{\phi}$ uniformly on compact subsets as $n \to \infty$, where

$$\widehat{\phi}(u) = \prod_{j=1}^{\infty} \widehat{c}\left((M^T)^{-j}u\right), \quad u \in \mathbf{R^d} , \tag{3.3}$$

since c satisfies (1.3). Further, $\widehat{\phi}$ is continuous at the origin and $\widehat{\phi}(0) = 1$.

It is clear that $\phi^{(n)} \in L^2(\mathbf{R^d})$ and is compactly supported. Next, we prove that $\|\phi^{(n)}\| = 1$ for all $n = 0, 1, \ldots$. Note that if c is a M-CQF, then (1.7) and (1.11) imply that δ is an eigenvector of $W_{c_{au}}$ of eigenvalue 1. Since $\phi^{(0)}_{au\,|} = \delta$ and $\phi^{(n)}_{au\,|} = W_{c_{au}} \phi^{(n-1)}_{au\,|}$, we have $\phi^{(n)}_{au\,|} = \delta$. Hence $\|\phi^{(n)}\| = 1$, for all $n = 0, 1, \ldots$.

Proposition 3.1. *If c is an M-CQF, then $\phi^{(n)}$ defined by* (3.1) *converges weakly to the* (M, c) *refinable function $\phi \in L^2(\mathbf{R^d})$.*

If, in addition, $\|\phi\| = 1$, then $\phi^{(n)}$ converges strongly in $L^2(\mathbf{R^d})$ to ϕ.

Proof. First, note that if c is a M-CQF, then $\|\phi^{(n)}\| = 1$ for all $n \geq 0$. Therefore, $\{\phi^{(n)}\}$ has a subsequence which converges weakly to $\varphi \in L^2(\mathbf{R^d})$. Since weak convergence is stronger than convergence in distribution, we have $\varphi = \phi$, and hence $\phi \in L^2(\mathbf{R^d})$.

Next, we show that the sequence $\phi^{(n)}$ itself converges weakly to ϕ. If $\phi^{(n)}$ does not converge to ϕ weakly, then there exists a subsequence $\phi^{(n_i)}$ which converges weakly to a function in $L^2(\mathbf{R^d})$ other than ϕ. This contradicts the fact that $\phi^{(n_i)}$ converges to ϕ in distribution.

In addition, if $\|\phi\| = 1$, then $\|\phi^{(n)}\| \to \|\phi\|$. With this, weak convergence of $\phi^{(n)} \to \phi$ implies strong convergence. □

Remark 2. *For a general finitely supported mask c, a similar proof shows that the corresponding (M, c) refinable distribution ϕ belongs to $L^2(\mathbf{R^d})$, if the ℓ^2 operator norm $\|W_{c_{au}}\| \leq 1$.*

Since ϕ is compactly supported, if $\phi \in L^2(\mathbf{R^d})$, then $\phi \in L^1(\mathbf{R^d})$. Hence $\widehat{\phi}(u) \to 0$ as $|u| \to \infty$. If ϕ is refinable, then for any $p \in 2\pi\mathbf{Z^d}/\{\mathbf{0}\}$,

$$\widehat{\phi}((M^T)^n p) = \widehat{\phi}(p) \prod_{j=1}^{n} \widehat{c}(((M^T)^{n-j})p) = \widehat{\phi}(p) \ .$$

Letting $n \to \infty$, implies $\widehat{\phi}(p) = 0$ for $p \in 2\pi\mathbf{Z^d}/\{\mathbf{0}\}$. This means that ϕ satisfies the Strang-Fix condition of order 1. By the Poisson summation formula, this condition is equivalent to the shifts of ϕ forming a partition of unity, i.e.

$$\sum_{p\in Z^d} \phi(x-p) = 1, \quad x \in \mathbf{R^d} \ . \tag{3.4}$$

Proposition 3.2. *Let M be a dilation matrix, c a M-CQF and ϕ the unique (M,c) refinable function normalized so that $\widehat{\phi}(0) = 1$. The following are equivalent:*

(i) *δ is the unique eigenvector of $W_{c_{au}}$ of eigenvalue* 1.

(ii) *the shifts of ϕ are orthonormal,*

(iii) *the shifts of ϕ are orthogonal .*

Proof. The equivalence of (i) and (ii) is given in Corollary 2.3 . We need only to show that (iii) implies (ii). If (iii) holds, multiplying both sides of (3.4) by ϕ and integrating term by term, the orthogonality of the shifts of ϕ and the fact that $\int \phi(x) = 1$, gives $||\phi||^2 = 1$. □

As a Consequence of Propositions 3.1 and 3.2, the cascade algorithm converges strongly if the corresponding (M,c) refinable function ϕ is orthonormal, a result which coincides with the well known fact that the stability of a (M,c) refinable L^2 function implies strong convergence of the cascade algorithm.

4. Restricted Transition Operators

We now discuss how to restrict the transition operator to a finite dimensional subspace. For a dilation matrix M and a finitely supported refinement mask c, a subset $D \subset \mathbf{Z^d}$ is called an *invariant support set* for the transition operator W_c if the following are satisfied:

(i) D is finite,

(ii) for all sequences b with support in D, the support of $W_c b$ is also in D, and

(iii) the support of every finitely supported eigenvector of W_c corresponding to a nonzero eigenvalue is contained in D.

Such a finite invariant support set D for W_c always exists. To construct D, choose a vector norm $||\cdot||$ on R^d and a number $0 < \alpha < 1$ such that for all $x \in R^d$,

$$||M^{-1}x|| \leq (1-\alpha)||x|| \ .$$

This is possible because the spectral radius $\rho(M^{-1}) < 1$. Now choose

$$r \geq r_{\text{min}} := \frac{(1-\alpha)}{\alpha} \max_{c(p)\neq 0} ||p||$$

and define

$$D_r = \{p \in \mathbf{Z^d} \ : \ ||\mathbf{p}|| \leq \mathbf{r}\} \ .$$

Clearly D_r is an invariant support set for W_c. Further, if a sequence b is supported in D_s with $s > r$, then $W_c b$ is supported in D_t where $t = \alpha r + (1-\alpha)s$. Therefore D_r contains the support of every compactly supported eigenvector of W_c. Further, any compactly supported eigenvector of W_c is also an eigenvector of the restricted operator $W_c|_{\ell(D_r)}$, where $\ell(D_r)$ is the space of all sequences supported on D_r. One may construct the disk D_r to be arbitrary close to the minimal size by using a vector norm so that the corresponding operator norm $||M^{-1}||$ is sufficiently closed to $\rho(M^{-1})$, and by choosing $r = r_{\text{min}}$.

For the case $M = 2I$ with refinement mask c supported in $[0, N-1]^d$, the support of ϕ is in $[0, N-1]^d$ and that of ϕ_{au} is contained in $[-N+1, N-1]^d$. A mimimal invariant supported set for the corresponding transition operator $W_{c_{au}}$ is $[-N+1, N-1]^d$.

For a dilation matrix M and a finitely supported mask c, let Ω be an invariant support set of $W_{c_{au}}$, and $\ell(\Omega)$ be the space of all sequences supported on Ω. Then the transition operator $W_{c_{au}}$ restricted to $\ell(\Omega)$ is represented by the matrix

$$A := (m \ c_{au}(Mp - q))_{p,q\in\Omega} \ , \tag{4.1}$$

and $\phi_{au|}$ is an eigenvector of A of eigenvalue 1.

Theorem 4.1. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$, and $\phi \in L^2(\mathbf{R^d})$ is the compactly supported (M, c) refinable function. Then ϕ is stable if and only if*

(i) *there is an eigenvector corresponding to the eigenvalue 1 of matrix A defined by (4.1) whose Fourier transform does not vanish, and*

(ii) 1 *is a simple eigenvalue of A.*

Proof. Conditions (i) and (ii) together with the fact that $\phi_{au|}$ is an eigenvector of A of eigenvalue 1 imply that $\phi_{au|}$ has nonvanishing Fourier transform. Hence conditions (i) and (ii) imply that ϕ is stable.

On the other hand, if ϕ is stable, then $\phi_{au|}$ is an eigenvector of A of eigenvalue 1 whose Fourier transform does not vanish, hence condition (i) holds. To show condition (ii), assume 1 is not a simple eigenvalue of A. Then there exists an eigenvector a of A of eigenvalue 1, and a is not a scalar multiple of the eigenvector $\phi_{au|}$. Since the transition operator $W_{c_{au}}$ maps $\ell(\Omega)$ into $\ell(\Omega)$, the vector a is an eigenvector in $\ell^1(\mathbf{Z})$ of the transition operator $W_{c_{au}}$ of eigenvalue 1. This contradicts Theorem 2.2. □

A similar argument using Theorem 2.3 and Corollary 2.3 respectively, leads to the following results.

Proposition 4.1. *Let c be a finitely supported sequence satisfying $\widehat{c}(0) = 1$, ϕ the (M, c) refinable function, and D an invariant support set of W_c, and suppose that ϕ is continuous. Then ϕ is interpolatory if and only if the sequence δ is a unique eigenvector of the matrix*

$$C := (m\ c(Mp - q))_{p,q \in D}$$

of simple eigenvalue 1.

Proposition 4.2. *Suppose that c is a finitely supported sequence satisfying $\widehat{c}(0) = 1$ and that ϕ is the (M, c) refinable function in $L^2(\mathbf{R}^\mathbf{d})$. Then ϕ is orthonormal if and only if the sequence δ is a unique eigenvector of the matrix A defined by* (4.1) *of simple eigenvalue 1.*

This proposition shows that the problem of checking whether ϕ has orthonormal shifts simply amounts to checking whether 1 is a simple eigenvalue of the matrix A. Similarly, checking whether ϕ is stable reduces to checking whether 1 is a simple eigenvalue of the matrix A and whether the Fourier transform of the corresponding eigenvector vanishes on the torus.

In the case $d = 2, 3$ and $M = 2I$, if ϕ has orthonormal shifts and the refinement mask c satisfies

$$\overline{\widehat{c}(u)} = e^{ip_0 \cdot y}\widehat{c}(u),$$

for some $p_0 \in \mathbf{Z^d}$, then it was shown in [20] and [21] that compactly supported orthonormal wavelets can be easily constructed from c and ϕ. Interested readers should consult [20] and [21] for details.

5. Construction of admissible refinement masks

This section constructs three 2×2 dilation matrices M with $|\det(M)| = 2$, an infinite family of two-dimensional finitely supported masks c, and shows the corresponding (M, c) refinable functions ϕ are orthonormal. The refinable functions ϕ are constructed so that the set of points satisfying the condition $\{u : \widehat{\phi}(u) \neq 0\}$ contains a connected open set containing a fundamental domain for $2\pi\mathbf{Z^2} \subset \mathbf{R^2}$. This implies that $\widehat{(\phi_{au|})}(u) > 0$, hence by Corollary 2.3 , ϕ has orthonormal shifts .

Up to a similarity transformation and multiplication by matrices which represent reflection about the origin and reflection about the x_1 axis, there are only three distinct 2×2 integer dilation matrices whose determinant equals 2 or -2. They are

$$M_1 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \text{ having eigenvalues } \pm i\sqrt{2},$$

$$M_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \text{ having eigenvalues } 1 \pm i,$$

and

$$M_3 = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}, \text{ having eigenvalues } \tfrac{(1 \pm i\sqrt{7})}{2}.$$

We shall now construct refinement masks $c : \mathbf{Z}^2 \to \mathbf{C}$ which generate nonseparable orthonormal refinable functions and wavelets for dilation matrices $M_1,\ M_2,\ M_3$.

For a given one-dimensional sequence b, we define the *induced two-dimensional mask* $c : \mathbf{Z^2} \to \mathbf{C}$ by

$$c\begin{pmatrix} m \\ n \end{pmatrix} = b(m)\delta(n), \quad (m, n)^T \in \mathbf{Z^2}\ . \tag{5.1}$$

Lemma 5.1. *Let* $Z_{\widehat{b}}$ *denote the set of real zeros of* $\widehat{b}$. *Then the zero set* $Z_{\widehat{c}}$ *of the Fourier transform of the induced mask* c *is given by*

$$Z_{\widehat{c}} = \left\{ (u_1, u_2)^T : u_1 \in Z_{\widehat{b}},\ u_2 \in R \right\} \tag{5.2}$$

Proof. From (5.1),

$$\widehat{c}((u_1, u_2)^T) = \widehat{b}(u_1), \quad (u_1, u_2)^T \in \mathbf{R^2} \ ,$$

which gives (5.2) . □

Lemma 5.2. *Let* b *and* c *be as above. Let* M *be a* 2×2 *dilation matrix, and let* ϕ *be the unique* (M, c) *refinable distribution. Then the zero set* $Z_{\widehat{\phi}}$ *of the Fourier transform of* ϕ *satisfies*

$$Z_{\widehat{\phi}} = \bigcup_{j \geq 1} (M^T)^j Z_{\widehat{c}} \ . \tag{5.3}$$

Proof. The assertion follows from the infinite product representation of $\widehat{\phi}$. □

Let $\mathcal{L}_1,\ \mathcal{L}_2$ be the lines $x_1 = \pi$ and $x_1 = -\pi$ in $\mathbf{R^2}$ respectively. For any 2×2 dilation matrix M and the corresponding (M, c) refinable function ϕ, the set K is defined to be the closure of the largest connected subset of $\mathbf{R^2}$ containing the origin and consisting of points where the Fourier transform $\widehat{\phi}$ is nonzero.

Lemma 5.3. *Let* b *be a one-dimensional mask and let* c *be the induced two-dimensional mask. Suppose that the zero set of* $\hat{b}$ *satisfies*

$$Z_{\widehat{b}} = \{(2n + 1)\pi \ : \ n \in \mathbf{Z}\} \ . \tag{5.4}$$

Then K *is bounded by a subset of lines* $(M^T)^k \mathcal{L}_j,\ k \geq 1,\ j = 1, 2.$

Proof. The assertion follows from Lemmas 5.1 and 5.2 . □

Remark 3. *The refinement masks* b, *used by Daubechies in* [6] *to construct orthonormal refinable functions of one variable, are CQF's and the zero set of* $\widehat{b}$ *satisfies condition* (5.4) .

For the three dilation matrices $M_n,\ n = 1, 2, 3$, the set K can be computed explicitly if the zero set of $\hat{b}$ satisfies condition (5.4) . In each case the set K is a polygon whose vertices are the columns of the matrix V_n where

$$V_1 = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0 & 1 & \frac{3}{2} & 0 & -1 & -\frac{3}{2} \\ \\ 1 & 1 & \frac{1}{2} & -1 & -1 & -\frac{1}{2} \end{pmatrix},$$

$$V_3 = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & -\frac{4}{3} \\ \\ \frac{5}{3} & \frac{1}{3} & -\frac{5}{3} & -\frac{1}{3} \end{pmatrix}.$$

Theorem 5.1. *Suppose that b is a one-dimensional mask satisfying the zero condition* (5.4)*, and let c be the two-dimensional induced mask. Then for $M = M_n$, $n = 1, 2, 3$, the symbol $\widehat{\phi_{au|}}(u)$ is positive. Further, if b is a CQF, then c is also a M-CQF, and the corresponding (M, c)-refinable function is orthonormal.*

Proof. It is straightforward to check that for each of the three dilation matrices $M_n,\ n = 1, 2, 3$, the interior of K contains the closure of a fundamental domain for $\mathbf{Z^2} \subset \mathbf{R^2}$. Therefore, the symbol $\widehat{\phi_{au|}}(u) > 0$.

If b is a CQF, then c is a M-CQF because the intersection of $M\mathbf{Z^2}$ with the lattice points on the x_1-axis is exactly the set of even integers. Since the symbol is positive, by Corollary 2.3, the integer shifts of ϕ are orthonormal. □

We note that the results of Theorem 5.1 for the dilation matrix M_2 have been obtained by Cohen and Daubechies [4].

Remark 4. *The fact that $\widehat{\phi_{au|}}(u) > 0$ implies orthonormality was first proved in* [19]*. However, the proof in that paper was based on the Lebesque dominated convergence theorem and special properties of scaling tiles and was quite complicated. In* [19] *the refinable functions produced above were also constructed and mesh plots of some of these functions were produced. However, the report did not examine the zero set of*

$\widehat{\phi}$ and therefore did not actually prove that the refinable functions constructed had orthonormal shifts.

Remark 5. *For each of the dilation matrices $M_n, n = 1, 2, 3,$ and the mask induced by the Daubechies length 4 coefficients b, we have computed an invariant support set for the transition operator $W_{c_a u}$ and we computed the eigenvalues of the corresponding restricted transition operator. There are $49, 63,$ and 39 nonzero eigenvalues (counted with multiplicity) corresponding to $M_1, M_2,$ and M_3 respectively, and the eigenvalue 1 is simple in all cases. All the eigenvalues have modulus ≤ 1 and several, but not all, of these eigenvalues are negative integer powers of the eigenvalues of the corresponding dilation matrix. This is significant because the degree of smoothness of the refinable function implies the existence of a finite number of such eigenvalues. The corresponding eigenvectors can be constructed from the derivatives of the refinable function in the directions of the eigenvectors of the dilation matrix. The degree of smoothness of the refinable function also implies the existence of a continuous spectrum for the unrestricted transition operator that includes a continuous family of eigenvectors constructed from fractional derivative and integral operators. The discrete spectrum of the transition operator can easily be shown to coincide with the spectrum of the restricted transition operator. The significance of discrete eigenvalues that do not correspond to negative integer powers of the eigenvalues of the dilation matrices will be discussed in a subsequent paper.*

Remark 6. *The existence of negative integer powers of the eigenvalues of the dilation matrix in the spectrum of $W_{c_a u}$ is not sufficient for regularity of the corresponding refinable function. Indeed, the (M_2, c) refinable function ϕ constructed from Daubechies length 4 sequence is not continuous ([24], Example 5.2), and none of the longer filters leads to C^1 solutions ([4], Theorem 4.2).*

Remark 7. *Let V_0 be the closed shift invariant subspace generated by ϕ and let*

$$V_k := \{f(M^k \cdot) : f \in V_0\}.$$

Then $\{V_k\}$ *forms a multiresolution analysis of* $L^2(\mathbf{R^d})$, *by Remark 2.6 of* [13]. *We further remark that the construction of the corresponding wavelet for* ϕ *and* c *is straightforward, since* $|\det M| = 2$.

Acknowledgement

We thank an anonymous referee for the Remark 6 and some relevant references.

References

[1] Cavaretta, A. S., W. Dahmen and C. A. Micchelli, *Stationary subdivision,* Memoir Amer. Math. Soc. 93(1991), 1 -186 .

[2] Cohen A., *Ondelettes, analyses, multirésolution et traitement numérique du signal*, Ph.D. Thesis, Universitë Paris, Dauphine, 1990.

[3] Cohen, A. and I. Daubechies, *A stability criterion for biorthogonal wavelet bases and their subband coding scheme,* Duke Math. J. 68(1992), 313 - 335 .

[4] Cohen, A. and I. Daubechies, *Non-separable bidimensional wavelet bases,* No.1, Rev. Mat. Iberoamericana 9(1993), 51-137.

[5] Cohen, A., I. Daubechies and J. C. Feauveau, *Biorthogonal basis of compactly supported wavelets,* Comm. Pure and Appl. Math. 45(1992) 485 - 560.

[6] Daubechies, I., *Orthonormal bases of compactly supported wavelet,* Comm. Pure and Appl. Math. 41(1988) 909-996.

[7] Daubechies, I., *Ten lectures on wavelets*, CBMF conference Series in Applied Mathematics, vol. 61, SIAM, Philadephia, (1992).

[8] Eirola, T., *Sobolev characterization of solutions of dilation equations,* SIAM J. Math. Anal. 23 (1992), 1015 - 1030.

[9] Heller, P. N. and R. O. Wells, Jr., *The spectral theory of multiresolution operators and applications,* AWARE Technical Report, 1993.

[10] Hervé, L., *Construction et regularite des fonctions d' echelle,* SIAM J. Math. Anal. #5, 26(1995), to appear.

[11] Hogan T., *Stability and independence of the shifts of a multivariate refinable function*, preprint.

[12] Jia, Rong-Qing, Jian Zhong Wang, *Stability and linear independence associated with wavelet decompositions*, Proc. Amer. Math. Soc. 117 (1993), 1115-1124.

[13] Jia, Rong-Qing and Zuowei Shen, *Multiresolution and wavelets*, Proc. Edinburgh Math. Soc. 37 (1994), 271-300.

[14] Long, R. and D. Chen, *Biorthogonal wavelet bases on* $\mathbf{R^d}$, Appl. Comp. Harmonic Anal. 2(1995), 230 - 242.

[15] Lawton, W., S. L. Lee and Zuowei Shen, *Complete characterization of refinable splines*, Advances in Computational Math. 3 (1995), 137-145.

[16] Lawton, W., S.L. Lee and Zuowei Shen, *An algorithm for matrix extension and wavelet construction*, Math. Comp. (to appear).

[17] Lawton, W., *Necessary and sufficient conditions for constructing orthonormal wavelets*, J. Math. Phys. 32(1991), 52-61.

[18] Lawton, W., *Multilevel properties of the wavelet-Galerkin operator,* J. Math. Phys. 32(1991), 1440 - 1443.

[19] Lawton, W. and H. Resnikoff, *Multidimensional Wavelet Bases*, AWARE Report, 1991

[20] Riemenschneider, S. D. and Zuowei Shen, *Box splines, cardinal series, and wavelets*, Approximation Theory and Functional Analysis, C.K. Chui eds., Academic Press, New York, (1991), 133–149.

[21] Riemenschneider, S. D. and Zuowei Shen, *Wavelets and pre-wavelets in low dimensions*, J. Approx. Theory **71** (1992), 18–38.

[22] Strang, G., *Eigenvalues of* $(\downarrow 2)H$ *and convergence of the cascade algorithm,* preprint, Dept. of Math., MIT (1995).

[23] Villemoes, L. F., *Energy moments in time and frequency for two-scale difference equation solutions and wavelets,* SIAM J. Math. Anal. 23(1992), 1519 - 1543.

[24] Villemoes, L. F., *Continuity of nonseparable quincux wavelets,* Appl. Comput. Harmonic Anal. 1(1994), 180 - 187.

Institute of System Sciences,, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511