CONVERGENCE OF MULTIDIMENSIONAL CASCADE ALGORITHM

W. LAWTON, S. L. LEE AND ZUOWEI SHEN

ABSTRACT. Necessary and sufficient conditions on the spectrum of the restricted transition operators are given for the convergence in $L^2(\mathbf{R^d})$ of the multidimensional cascade algorithm .

1. INTRODUCTION

This paper is a continuation of [8]. In [8] we obtained a complete characterization of stability and orthonormality of the shifts of a refinable function in terms of its refinenent mask. In this paper we present a complete characterization of the convergence in $L^2(\mathbf{R^d})$ of the multidimensional cascade algorithm with arbitrary dilation matrices M in terms of the mask.

For fixed integers $d \geq 1$ and $m \geq 2$, let M be a $d \times d$ dilation matrix with $|\det(M)| = m$. A dilation matrix is an integer matrix with all eigenvalues of modulus > 1. Let $\ell^2(\mathbf{Z^d})$, where $\mathbf{Z^d}$ is the set of all multi-integers, be the space of all square-summable sequences, and $L^2(\mathbf{R^d})$ the space of all square-integrable functions. The Fourier transform of $f \in L^2(\mathbf{R^d})$ will be denoted by $\widehat{f}$.

The subgroup $M\mathbf{Z^d}$ partitions $\mathbf{Z^d}$ into m distinct cosets $\gamma_k + M\mathbf{Z^d}$, $k = 0, 1, \ldots, m-1$, where $\gamma_0 = 0$. Let $(c(p))_{p\in\mathbf{Z^d}}$ be a finite real or complex sequence satisfying

$$\sum_{p\in\mathbf{Z^d}} c(p) = 1 \ . \tag{1.1}$$

The condition

$$\sum_{p\in\mathbf{Z^d}} c(\gamma_k + Mp) = \frac{1}{m} \quad \text{for } k = 0, 1, \ldots, m-1 \ , \tag{1.2}$$

1991 *Mathematics Subject Classification.* 41A15, 41A30, 42C05, 42C15.

Key words and phrases. cascade algorithm, subdivision algorithm, dilation matrix, refinement equation, transition operators, eigenvalues, Condition E, fundamental condition.

is called the fundamental condition, and a sequence which satisfies the fundamental condition will be called a *fundamental sequence.* It is clear that if c is fundamental, then c satisfies (1.1) .

The linear operator $T_c : L^2(\mathbf{R^d}) \to \mathbf{L^2}(\mathbf{R^d})$ defined by

$$(T_c\phi)(x) := \sum_{p\in\mathbf{Z^d}} m\ c(p)\phi(Mx-p), \quad \phi \in L^2(\mathbf{R^d}) \ , \tag{1.3}$$

is the *refinement operator* corresponding to the *refinement sequence* c. A fixed point of T_c is called a (M,c)*-refinable function.*

It is well-known that for c satisfying (1.1) there exists a unique compactly supported (M,c)-refinable distribution ϕ, whose Fourier transform $\widehat{\phi}$ is continuous at the origin and $\widehat{\phi}(0) = 1$. Furthermore $\widehat{\phi}$ admits the infinite product representation

$$\widehat{\phi}(u) = \prod_{j=1}^{\infty} \widehat{c}\left((M^T)^{-j}u\right), \quad u \in \mathbf{R^d} \ , \tag{1.4}$$

where

$$\widehat{c}(u) := \sum_{p\in\mathbf{Z^d}} c(p)e^{-ipu} \ ,$$

is the Fourier transform of the sequence c. Further, (1.4) shows that

$$\widehat{\phi}(u) = \widehat{c}\left((M^T)^{-1}u\right)\widehat{\phi}\left((M^T)^{-1}u\right) \ ,$$

which is equivalent to ϕ being (M,c)-refinable, i.e.

$$\phi(x) = \sum_{p\in\mathbf{Z^d}} m\ c(p)\phi(Mx-p), \quad x \in \mathbf{R^d} \ , \tag{1.5}$$

in the distribution sense.

Starting with a compactly supported function ϕ_0, we define for $n = 1, 2, \dots$,

$$\phi_n(x) := \sum_{p\in\mathbf{Z^d}} m\ c(p)\phi_{n-1}(Mx-p) \ . \tag{1.6}$$

The algorithm (1.6) is called the *cascade algorithm* for the refinement sequence c. We shall say that the *cascade algorithm converges* if the sequence ϕ_n converges. The cascade algorithm always converges to ϕ as a distribution whenever $\widehat{c}(0) = 1$. If $\{\phi_n\}$ is bounded in $L^2(\mathbf{R^d})$, then it converges to $\phi \in L^2(\mathbf{R^d})$ weakly. The cascade algorithm is related to the stationary subdivision scheme. For the case $M = 2I$ and arbitrary

d, it is the subject of intensive study by Cavaretta, Dahmen and Micchelli [1] in the context of curve and surface modelling and by Daubechies [3] (for $d = 1$) in wavelet construction. If the cascade algorithm converges for an initial $\phi_0 \in L^1(\mathbf{R^d})$, its limit equals $\widehat{\phi}_0(0)\,\phi$. The L^2-convergence of the cascade algorithm depends on the spectrum of the transition operator. To see how the operator arises, we define for any $\psi \in L^2(\mathbf{R})$, the sequence

$$[\psi](p) := \int_{\mathbf{R^d}} \psi(x)\overline{\psi(x-p)}\; dx, \quad p \in \mathbf{Z^d}\ . \tag{1.7}$$

For each $n = 0, 1, \ldots$, the sequence $[\phi_n]$ is finitely supported, and (1.7) and (1.6) give

$$[\phi_n](k) = \sum_{p\in\mathbf{Z^d}} m\ a(Mk-p)[\phi_{n-1}](p), \quad k \in \mathbf{Z^d}\ , \tag{1.8}$$

where

$$a(k) := \sum_{p\in\mathbf{Z^d}} c(p)\overline{c(p-k)}\ ,$$

is the *autocorrelation* of c. Further, if ϕ is (M, c)-refinable, then its autocorrelation

$$f(x) := \int_{\mathbf{R^d}} \phi(t)\overline{\phi(t-x)}\; dt$$

is the solution of the refinement equation

$$f(x) := \sum_{p\in\mathbf{Z^d}} m\ a(p)f(Mx-p)\ , \tag{1.9}$$

i.e. f is (M, a)-refinable, where a is the autocorrelation of c. Equation (1.8) can be written in the form

$$[\phi_n] := W_a[\phi_{n-1}]\ , \tag{1.10}$$

where $W_a : \ell^1(\mathbf{Z^d}) \to \ell^{\mathbf{2}}(\mathbf{Z^d})$ is a linear transformation such that for all $b \in \ell^2(\mathbf{Z^d})$,

$$(W_ab)(k) := \sum_{p\in\mathbf{Z^d}} m\ a(Mk-p)b(p), \quad k \in \mathbf{Z^d}. \tag{1.11}$$

For a general dilation matrix M and a compactly supported refinement sequence c, a subset $\Omega \subset \mathbf{Z^d}$ is called an *invariant support set* for the associated transition operator W_a if the following conditions are satisfied:

(i) Ω is finite,

(ii) for all sequences b with support in Ω, the support of $W_a b$ is also in Ω, and

(iii) the support of every finitely supported eigenvector of W_a corresponding to a nonzero eigenvalue is contained in Ω.

For the case $d = 1, m = 2$, and c is supported on $\{0, 1, \ldots, N\}$, the set $\Omega = \{-N_1 + 1, -N_1 + 2, \ldots, N_1 - 1\}$ is an invariant support set whenever $N_1 \geq N$ and $\ell(\Omega) = \mathbf{C^{2N_1-1}}$. The construction of invariant support sets for general dilation matrices is more complex. In [8] a constructive proof is developed, based on the facts that the spectral radius of $r(M^{-1}) < 1$ and the support of c is finite, that shows every finite subset of Z^d is contained in some invariant support set Ω. The construction chooses a vector norm on R^d such that the corresponding operator norm for M^{-1} is < 1 and the invariant support sets consist of balls having sufficiently large radius with respect to this norm. The class of all complex sequences which vanish outside an invariant support set Ω is denoted by $\ell(\Omega)$. Clearly $\ell(\Omega)$ is a finite dimensional vector subspace of $\ell^2(\mathbf{Z^d})$ that is invariant under the operator W_a and the nonzero eigenvalues and corresponding eigenvectors of the restriction of W_a to $\ell(\Omega)$ does not depend on the choice of Ω. Furthermore, since $W_a[\phi] = [\phi]$, the support of $[\phi]$ is contained in every invariant support set Ω and therefore $[\phi] \in \ell(\Omega)$. We shall choose an invariant support set that contains $[\phi_0]$ where ϕ_0 is the initial function in the cascade algorithm. Therefore, $\text{supp}([\phi_n]) \subset \Omega$, for all $n = 0, 1, \ldots$. The operator W_a restricted to $\ell(\Omega)$ will be called the *restricted transition operator*. The restricted operator can be represented by the following finite order matrix

$$W_a := (m\ a(Mp - q))_{p,q\in\Omega}\,.$$

Since only restricted transition operators are considered in this paper, we will identify the operator W_a with the matrix W_a in this paper. Clearly the convergence of the operator sequence $\{W_a^n\}$ is equivalent to the convergence of the matrix sequence $\{W_a^n\}$. It would be convenient at times to express (1.11) in the frequency domain. Taking the Fourier transforms of the sequences in (1.11) leads to

$$\widehat{W_a b}(u) = \sum_{k=0}^{m-1} \left|\widehat{c}\left((M^T)^{-1}(u + 2\pi\gamma_k)\right)\right|^2 \widehat{b}\left((M^T)^{-1}(u + 2\pi\gamma_k)\right)\ . \tag{1.12}$$

In one dimension with dilation 2, i.e. $d = 1$ and $M = 2$, Cohen and Daubechies [2] proved that if the condition that all eigenvalues of the restricted W_a on the invariant subspace $S := \{a \in \ell(\Omega) : \sum_p a(p) = 0\}$ are inside the unit circle is satisfied, then the cascade algorithm with the intial function $\phi_0(x) = sinc(x)$, converges to the refinable funtion ϕ in $L^2(\mathbf{R})$. They further proved that if ϕ is stable, then the restricted transition operator W_a satisfied above mentioned condition. Here, we also mentione that this condition was also used by Long and Chen in their studies of biorthogonality of a pair of masks in [9]. Jia [6] gave a characterization for L^p-convergence of the subdivision scheme by using L_p joint spectral radius, which include L_2-convergence as a special case. Jia's results are based on the work of Goodman Miccheli and Ward [5], where L^∞ joint spectral radius are used to characterize the L^∞-convergence of the subdivision scheme. Strang [10] proved that for any fundamental sequence c, the cascade algorithm converges strongly in $L^2(\mathbf{R})$ for any starting function ϕ_0 whose integer shifts form a partition of unity, if and only if the restricted W_a has a simple eigenvalue 1 and all its other eigenvalues lie inside the unit circle. We also remark here that Stang's result can be derived from Jia's result as well.

Following Strang [10], we shall say that the restricted transition operator W_a : $\ell(\Omega) \to \ell(\Omega)$ satisfies Condition E if it has a simple eigenvalue 1 and all its other eigenvalues lie inside the unit circle. Suppose that c satisfies the fundamental condition (1.2) and that ϕ is an (M, c)-refinable function in $L^2(\mathbf{R^d})$. For the case $M = 2I$, it is known (see [1]) that if ϕ is compactly supported and stable, i.e. its integer shifts form a Riesz basis, then the cascade algorithm converges strongly in $L^2(\mathbf{R^d})$. In [8], we proved that if ϕ is compactly supported and stable, then the restricted transition operator W_a satisfies Condition E. We further proved that if 1 is a simple eigenvalue of W_a and the Fourier transform of its corresponding eigenvector is nonvanishing, then ϕ is stable. This implies that if W_a satisfies Condition E and the Fourier transform of the eigenvector corresponding to eigenvalue 1 is nonvanishing, then ϕ is stable. However, neither Condition E nor the L^2- convergence of the cascade algorithm implies stability of ϕ. Hence, both conditions are weaker than the stability of the (M, c)-refinable function ϕ. It is of interest to know the relations between the strong convergence of the cascade algorithm and Condition E. The object of this paper is to prove that Condition E together with the fundamental condition on the refinement sequence are equivalent to the convergence of the multidimensional cascade algorithm in $L^2(\mathbf{R^d})$. This gives a complete characterization of the strong convergence in $L^2(\mathbf{R^d})$ of the cascade algorithm.

2. Statement of the Main Theorem and Example

Theorem 2.1. *Suppose that $c(u)$ satisfies the fundamental condition* (1.2). *Then the cascade algorithm for c converges strongly in $L^2(\mathbf{R^d})$ for any compactly supported initial function ϕ_0 satisfying*

$$\sum_{p\in\mathbf{Z^d}} \phi_0(x-p) = 1, \quad x \in \mathbf{R^d}\ , \tag{2.1}$$

if and only if the restricted transition operator W_a satisfies Condition E.

For the case $d = 1$ and $M = 2$, this result was proved in [10]. In fact we shall prove the following stronger result, which is our main theorem.

Theorem 2.2. *Suppose that c satisfies* (1.1) . *Then the cascade algorithm for c converges strongly in $L^2(\mathbf{R^d})$ for any compactly supported initial function ϕ_0 satisfying (2.1) if and only if the restricted transition operator W_a satisfies Condition E and c is fundamental.*

The following example shows that Condition E does not imply the fundamental condition for c, thus showing that both Conditions E and the fundamental condition on the filter sequence are required for strong convergence.

Example 1. Let c be the sequence with $c(0) = 1/2$, $c(1) = 0$, $c(2) = 1/2$ and $c(j) = 0$ for $j \neq 0, 1, 2$. Then c is obviously not fundamental, but it is easy to check the nonzero eigenvalues of the W_a are $1, 1/2, 1/2, 1/2$ (MATLAB computation) so it satisfies Condition E. The cascade algorithm with the initial function $\phi_0 = \chi_{[0,1]}$ converges to $(1/2)\chi_{[0,2]}$ which has smaller $L^2(\mathbf{R^d})$ norm so the convergence is weak and not strong.

3. Auxiliary results

We first describe conditions on c that imply the (M,c)-refinable function ϕ is in $L^2(\mathbf{R^d})$.

An eigenvalue λ of a matrix A is said to be *nondegenerate* if its algebraic multiplicity is equal to its geometric multiplicity. In this case, the Jordan block in the Jordan normal form of A corresponding to the eigenvalue λ is a diagonal submatrix with all diagonal entries equal to λ. The restricted transition operator W_a is said to satisfy the *extended Condition E* if its spectral radius $r(A) \leq 1$ and all its eigenvalues on the unit circle are nondegenerate.

Proposition 3.1. *If W_a satisfies the extended Condition E then the solution ϕ of the refinement equation* (1.5) *belongs to $L^2(\mathbf{R^d})$, and the cascade sequence ϕ_n converges weakly to ϕ in $L^2(\mathbf{R^d})$ for any compactly supported starting function ϕ_0 with $\widehat{\phi}_0(0) = 1$.*

Proof. If c satisfies (1.1), then the corresponding cascade algorithm converges in distribution to the unique compactly supported (M,c)-refinable distribution ϕ. The extended Condition E implies that the cascade sequence $\{\phi_n\}$ is bounded in $L^2(\mathbf{R^d})$. Hence there is a subsequence $\{\phi_{n_j}\}$ which converges weakly to a function g in $L^2(\mathbf{R^d})$. Therefore $\{\phi_{n_j}\}$ converges in the distribution sense to g, hence $g = \phi$. Furthermore, $\{\phi_n\}$ converge weakly to ϕ, otherwise it has a subsequence which converges weakly, and thus in the distribution sense, to another function, thus contradicting the fact $\{\phi_n\}$ converges in the distribution sense to ϕ. □

Suppose that the restricted transition operator W_a satisfies Condition E. Then the existence of the L^2 solution ϕ of the refinement equation (1.5) is assured by Proposition 3.1. Furthermore, the cascade sequence ϕ_n converges weakly to ϕ for any starting function ϕ_0. Therefore, in order to prove that ϕ_n converges strongly in $L^2(\mathbf{R^d})$, it is sufficient to show that $\|\phi_n\|_2 \to \|\phi\|_2$ as $n \to \infty$,

We shall first establish some auxiliary results.

Lemma 3.1. *A sequence c satisfies the fundamental condition* (1.2) *if and only if*

$$\widehat{c}\left((M^T)^{-1}(2\pi(\gamma_j))\right) = \begin{cases} 1 & \gamma_j = 0 \\ 0 & \gamma_j \neq 0 \ . \end{cases} \tag{3.1}$$

Proof. The cosets $\gamma_k + M\mathbf{Z^d}$, $k = 0, 1, \ldots, m-1$, where $\gamma_0 = 0$, form a finite abelian group G of order m. Define the function $f : G \to \mathbf{C}$ by

$$f(\gamma_k + M\mathbf{Z^d}) := \sum_{\mathbf{p} \in \gamma_\mathbf{k} + \mathbf{MZ^d}} \mathbf{c}(\mathbf{p}). \tag{3.2}$$

Clearly, c is fundamental if and only if f is the constant function $\frac{1}{m}$ on G. Now the Plancherel theorem for finite abelian groups ([4], page 219) states

$$f(\gamma_k + M\mathbf{Z^d}) = \sum_{\mathbf{h} \in \widehat{\mathbf{G}}} \widehat{\mathbf{f}}(\mathbf{h})\, \mathbf{h}(\gamma_\mathbf{k} + \mathbf{MZ^d}) \ , \tag{3.3}$$

where $\widehat{G}$ denotes the character group of G (set of homomophisms into the unit circle group in $\mathbf{C}$) and

$$\widehat{f}(h) := \frac{1}{m} \sum_{g \in G} f(g)\overline{h(g)} \tag{3.4}$$

is the Fourier transform of f. Since the characters on G are mutually orthogonal functions on G ([4], page 218) and the identity character is the constant function 1, it follows that c is fundamental if and only if $\widehat{f}$ equals $\frac{1}{m}$ on the identity character and equals 0 elsewhere.

Now, the characters on G are exactly the characters on $\mathbf{Z^d}$ that map every element of $M\mathbf{Z^d}$ to 1. Clearly these characters are precisely functions h on $\mathbf{Z^d}$ of the form

$$h_j(p) = e^{i2\pi\gamma_j^T M^{-1} p}, \quad , p \in \mathbf{Z^d}, \ \mathbf{j} = \mathbf{0}, \ldots, \mathbf{m} - \mathbf{1} \ .$$

Therefore, a direct computation gives

$$\widehat{f}(h_j) = \frac{1}{m} \sum_{k=0}^{m-1} e^{-i2\pi\gamma_j^T M^{-1}\gamma_k} \ f(\gamma_k + M\mathbf{Z^d}) = \frac{\mathbf{1}}{\mathbf{m}} \widehat{\mathbf{c}}\left((\mathbf{M^T})^{-\mathbf{1}}(\mathbf{2}\pi(\gamma_\mathbf{j}))\right) \ , \tag{3.5}$$

for $j = 0, \ldots, m-1$, and the proof is complete. □

Lemma 3.2. *Assume $\widehat{c}(0) = 1$. Then the sequence c is fundamental if and only if its autocorrelation sequence a is fundamental.*

Proof. The result follows from Lemma 3.1 and the fact $\widehat{a} = |\widehat{c}|^2$. □

Lemma 3.3. *If c satisfies the fundamental condition* (1.2) *and ϕ is the (M, c)-refinable distribution whose Fourier transform is continuous at the origin, then $\widehat{\phi}(2\pi p) = 0$ for $p \in \mathbf{Z^d} \backslash \{\mathbf{0}\}$.*

Proof. The result follows from Lemma 3.1 and the infinite product representation for $\widehat{\phi}$ in (1.4). □

Lemma 3.3 says that ϕ satisfies the Strang-Fix condition of order 1, which is equivalent to the following by the Poisson summation formula.

Corollary 3.1. *If c is fundamental and ϕ is a compactly supported (M, c)-refinable function normalized so that $\widehat{\phi}(0) = 1$, then the integer shifts of ϕ form a partition of unity, i.e.*

$$\sum_{p \in \mathbf{Z^d}} \phi(x - p) = 1, \text{ for all } x \in \mathbf{R^d} \ . \tag{3.6}$$

Lemma 3.4. *Suppose that c is fundamental and $(\phi_n)_{n=0}^{\infty}$ is the corresponding cascade sequence with starting function ϕ_0. If ϕ_0 satisfies* (2.1) *, then every ϕ_n satisfies*

$$\sum_{p \in \mathbf{Z^d}} \phi_n(x - p) = 1, \text{ for all } x \in \mathbf{R^d} \ . \tag{3.7}$$

Proof. Applying the cascade algorithm, after a change of variable and the order of summation, gives

$$\sum_{p \in \mathbf{Z^d}} \phi_n(x - p) = \sum_{r \in \mathbf{Z^d}} \sum_{p \in \mathbf{Z^d}} m \ c(r - Mp)\phi_{n-1}(Mx - r). \tag{3.8}$$

Since c is fundamental,

$$\sum_{p \in \mathbf{Z^d}} c(r - Mp) = \frac{1}{m} \ ,$$

for all $r \in \mathbf{Z^d}$. Therefore (3.8) yields

$$\sum_{p \in \mathbf{Z^d}} \phi_n(x - p) = \sum_{r \in \mathbf{Z^d}} \phi_{n-1}(Mx - r),$$

from which the result follows by induction. □

Lemma 3.5. *Suppose that a is fundamental. If $v = W_a b$, then*

$$\sum_{k\in\mathbf{Z^d}} v(k) = \sum_{k\in\mathbf{Z^d}} b(k)\ . \tag{3.9}$$

Further, if b is an eigenvector of W_a with eigenvalue $\lambda \neq 1$, then $\sum_{k\in\mathbf{Z^d}} b(k) = 0$.

Proof. By (1.12), the relation $v = W_a b$, is equivalent to

$$\widehat{v}(u) = \sum_{k=0}^{m-1} \left|\widehat{c}\left((M^T)^{-1}(u+2\pi\gamma_k)\right)\right|^2 \widehat{b}\left((M^T)^{-1}(u+2\pi\gamma_k)\right)\ . \tag{3.10}$$

Setting $u = 0$ in (3.10) and using (3.1) gives

$$\widehat{v}(0) = |\widehat{c}(0)|^2\widehat{b}(0) = \widehat{b}(0)\ , \tag{3.11}$$

which is equivalent to (3.9). If $v = \lambda b$, then (3.11) becomes

$$\lambda\widehat{b}(0) = \widehat{b}(0)\ ,$$

which implies that $\widehat{b}(0) = 0$, if $\lambda \neq 1$. □

4. Proof of Theorem 2.2 and Corollaries

In this section we prove our main result and derive several corollaries.

Recall that a sequence of matrices $\{W_a^n\}$ generated by a finite order matrix W_a converges if and only if the spectral radius $r(W_a) \leq 1$ and 1 is the only eigenvalue on the unit circle and 1 is nondegenerate. Furthermore the sequence $\{W_a^n\}$ converges if and only if for all $b \in \ell(\Omega)$, the sequence $\{W_a^n b\}$ converges. Since $W_a(\lim_n W_a^n b) = \lim_n W_a^n b$, the vector $\lim_n W_a^n b$ is an eigenvector of W_a corresponding to the eigenvalue 1. In particular, if W_a satisfies Condition E, then, for arbitrary $b \in \ell(\Omega)$, $\lim_n W_a^n b = \beta[\phi]$ for some constant β.

Proof. Suppose that the restricted W_a satisfies Condition E and that c is fundamental. The cascade sequence ϕ_n converges weakly to ϕ in $L^2(\mathbf{R^d})$. We need only to show that $||\phi_n||_2 \to ||\phi||_2$.

First, observe that since $\int \phi_0(x)dx = 1$, the relation (1.6) implies that

$$\int_{\mathbf{R^d}} \phi_n(x)\ dx = 1\ , \tag{4.1}$$

for all $n = 0, 1, \ldots$. Furthermore, multiplying both sides of (3.7) by $\overline{\phi_n(x)}$ and integrating yields

$$\sum_{p\in\mathbf{Z^d}} [\phi_n](p) = 1\ . \tag{4.2}$$

Since

$$\int_{\mathbf{R}} \phi(x)\ dx = 1\ ,$$

a similar argument using (3.6) gives

$$\sum_{p\in\mathbf{Z^d}} [\phi](p) = 1\ . \tag{4.3}$$

Recall that $[\phi]$ is an eigenvector of the restricted transition operator W_a with eigenvalue 1. Since W_a satisfies Condition E, $[\phi]$ is the unique eigenvector of the eigenvalue 1. Hence

$$\lim_{n\to\infty} [\phi_n] = \beta\ [\phi]\ . \tag{4.4}$$

The relations (4.2), (4.3) and (4.4) show that $\beta = 1$, so that (4.4) implies

$$||\phi||_2 = [\phi](0) = \lim_{n\to\infty} [\phi_n](0) = \lim_{n\to\infty} ||\phi_n||_2\ .$$

Hence ϕ_n converges to ϕ strongly in $L^2(R^d)$.

Conversely, suppose that $||\phi_n - \phi||_2 \to 0$ as $n \to \infty$ for all compactly supported ϕ_0 satisfying (2.1). Consider the functions $\phi_0^\alpha := \chi_{[-1/2,1/2]^d}(\cdot - \alpha)$, $\alpha \in \Omega$. Since $0 \in \Omega$, one of these functions is $\phi_0^0 = \chi_{[-1/2,1/2]^d}$. Each function ϕ_0^α has unit integral and satisfies (2.1). With ϕ_0^α as an initial function, the cascade algorithm for c generates a sequence $(\phi_n^\alpha)_{n=0}^\infty$ which converges in $L^2(\mathbf{R})$ to ϕ for each $\alpha \in \Omega$. Also for each $\alpha \in \Omega$ and $n = 0, 1, \ldots$, define a sequence b_n^α by

$$b_n^\alpha(k) := \int_{\mathbf{R^d}} \phi_n^0(x)\overline{\phi_n^\alpha(x-k)}dx, \quad k \in \Omega\ . \tag{4.5}$$

Then

$$b_n^\alpha = W_a b_{n-1}^\alpha \ \text{ and } \ b_0^\alpha = \delta(\cdot - \alpha) := \delta_\alpha\ ,$$

where $\delta(0)=1$ and $\delta(\beta)=0$ for $\beta\in\Omega\setminus\{0\}$. It follows that

$$b_n^\alpha = W_a^n b_0^\alpha = W_a^n \delta_\alpha, \quad \text{for all } n=0,1,\dots\ , \tag{4.6}$$

for any $\alpha\in\Omega$. Since $||\phi_n^\alpha-\phi||\to 0$ as $n\to\infty$ for every $\alpha\in\Omega$,

$$b_n^\alpha(j) := \int_{\mathbf{R^d}} \phi_n^0(x)\overline{\phi_n^\alpha(x-j)}\,dx \to \int_{\mathbf{R^d}} \phi(x)\overline{\phi(x-j)}\,dx = [\phi](j)\ , \tag{4.7}$$

as $n\to\infty$, all $j\in\Omega$. Combining (4.6) and (4.7) gives

$$W_a^n\delta_\alpha \to [\phi], \quad \text{as } n\to\infty\ , \tag{4.8}$$

for every $\alpha\in\Omega$. Since δ_α, $\alpha\in\Omega$, form a basis of $\ell(\Omega)$, it follows that for any sequence $b\in\ell(\Omega)$

$$W_a^n b\to\beta[\phi]\ , \text{ where } \beta=\sum_{p\in\Omega} b(p). \tag{4.9}$$

This implies that the matrix sequence $\{W_a^n\}$ converges. Hence, the spectral radius $r(W_a)\le 1$ and 1 is the only eigenvalue on the unit circle. Further, 1 is a nondegenerate eigenvalue of the matrix W_a. Therefore, to show that 1 is a simple eigenvalue of W_a, we only need to show that $[\phi]$ is the unique eigenvector of the eigenvalue 1 of W_a, up to a constant multiple. Let $b\in\ell(\Omega)$ be an eigenvector of W_a corresponding to the eigenvalue 1. Then

$$\lim_n W_a^n b = b = \beta[\phi].$$

Hence $[\phi]$ is the only eigenvector of W_a corresponding to the eigenvalue 1.

We now show that c is fundamental. By Lemma 3.2 it suffices to show that a is fundamental, where a is the autocorrelation of c. Recall that for any $\alpha\in\Omega$ and $\delta_\alpha := \delta(\cdot-\alpha)\in\ell(\Omega)$,

$$W_a^n\delta_\alpha \to [\phi], \quad \text{as } n\to\infty\ . \tag{4.10}$$

Let $v\in\ell(\Omega)$ be a left eigenvector of W_a with eigenvalue 1. Then

$$v^T W_a^n = v^T, \quad \text{for all } n=0,1,\dots\ . \tag{4.11}$$

Hence, for each $\alpha\in\Omega$,

$$v(\alpha) = (v^T W_a^n)\delta_\alpha = v^T(W_a^n\delta_\alpha), \quad \text{for all } n=0,1,\dots\ .$$

It follows from (4.10) that

$$v(\alpha) = v^T[\phi] = \sum_j v(j)\ [\phi](j) := \mu \text{ for some constant } \mu\ .$$

This means that $v = \mu e$ where $e = \sum_{\alpha\in\Omega} \delta_\alpha$. Hence $e^T W_a = e^T$, which is equivalent to

$$m \sum_p a(Mp - q) = 1, \quad \text{for all } q \in \Omega\ .$$

Therefore, a is fundamental. Hence c is fundamental. □

We observe from this proof that if the cascade algorithm converges for a compactly supported initial function ψ_0 with $\widehat{\psi}_0(0) = 1$ and $[\psi_0] = \delta$, then W_a satisfies Condition E and c is fundamental. Therefore Theorem 2.2 implies

Corollary 4.1. *Assume that $\widehat{c}(0) = 1$ and that the cascade algorithm converges for a compactly supported initial function ψ_0 with $\widehat{\psi}_0(0) = 1$ and $[\psi_0] = \delta$. Then the cascade algorithm converges for any initial function ϕ_0 satisfying (2.1).*

Recall that a function ϕ is stable if its shifts $\{\phi(x-p)\,|\,p \in Z^d\}$ form a Riesz basis of the subspace of $L^2(R^d)$ they span.

Corollary 4.2. *If a compactly supported (M, c)-refinable function ϕ is stable, then the cascade algorithm converges strongly in $L^2(\mathbf{R})$ for any compactly supported initial function ϕ_0 satisfying (2.1).*

Proof. Since ϕ is stable, by Corollary 2.1 of [8], we have that W_a satisfies E condition. Therefore we need only to show that c is fundamental. Since $\phi \in L^2(\mathbf{R^d})$ is compactly supported, $\phi \in L^1(\mathbf{R^d})$ hence $\widehat{\phi}(u) \to 0$ as $|u| \to \infty$. If ϕ is refinable, then for any $p \in 2\pi\mathbf{Z^d}/\{\mathbf{0}\}$,

$$\widehat{\phi}((M^T)^n p) = \widehat{\phi}(p) \prod_{j=1}^{n} \widehat{c}((M^T)^{n-j} p) = \widehat{\phi}(p)\ . \tag{4.12}$$

Letting $n \to \infty$, implies $\widehat{\phi}(p) = 0$ for $p \in 2\pi\mathbf{Z^d}/\{\mathbf{0}\}$. Therefore, for arbitrary $q \in \mathbf{Z^d}$,

$$\delta(\gamma_j) = \widehat{\phi}(2\pi\gamma_j + M2\pi q) = \widehat{c}((M^T)^{-1}2\pi\gamma_j)\widehat{\phi}((M^T)^{-1}2\pi\gamma_j + 2\pi q).$$

Then $\widehat{c}(0) = 1$ since $\widehat{\phi}(0) = 1$. Furthermore, since ϕ is stable, there exists $q' \in \mathbf{Z^d}$, such that $\widehat{\phi}((M^T)^{-1}2\pi\gamma_j + 2\pi q') \neq 0$. Therefore $\widehat{c}((M^T)^{-1}2\pi\gamma_j) = 0$ for $\gamma_j \neq 0$ Hence by Lemma 3.1 the sequence c is fundamental. □

Remark 1. *Suppose that c is a M-CQF satisfying the condition $\sum_{p\in\mathbf{Z^d}} c(p) = 1$. Here, a sequence c is called a M-CQF, if*

$$\sum_{k=0}^{m-1} \left|\widehat{c}\left(u + (M^T)^{-1}2\pi\gamma_k\right)\right|^2 = 1 \ .$$

If 1 is a simple eigenvalue of W_a, the integer shifts of ϕ are orthonormal [8], *hence ϕ is stable and the cascade sequence ϕ_n converges strongly to ϕ in $L^2(\mathbf{R^d})$. Therefore, if c is a M-CQF, then stability of ϕ, Condition E, and convergence of the cascade algorithm are equivalent.*

Corollary 4.3. *If $\widehat{c}(0) = 1$ and if the cascade algorithm for c converges strongly in L^∞ for any compactly supported initial function ϕ_0 satisfying (2.1), then c is fundamental and W_a satisfies Condition E.*

Proof. Since the (M, c)-refinable function ϕ and each function ϕ_n of the cascade sequence are compactly supported, if the sequence ϕ_n converges to ϕ in L^∞ then it also converges to ϕ is L^2 and the result follows by Theorem 2.2. □

Acknowledgement: S. L. Lee thanks Professor Gil Strang for making available to him a copy of the reprint [10] *when he was visiting MIT in Spring 1995. The results in* [10] *and in our previous paper* [8] *are the motivations for this work.*

References

[1] Cavaretta, A. S., W. Dahmen and C. A. Micchelli, *Stationery subdivision,* Memoir Amer. Math. Soc. 93(1991), 1 -186 .

[2] Cohen, A. and I. Daubechies , *A stability criterion for biorthogonal wavelet bases and their related subband coding scheme,* Duke Math. J. 68(1992), 313 - 335 .

[3] Daubechies, I., *Orthonormal bases of compactly supported wavelet,* Comm. Pure and Appl. Math. 41(1988), 909-996.

[4] Dym, H. and McKean, H. P., *Fourier Series and Integrals,* Academic Press, New york, 1972.

[5] Goodman, T. N. T. , C. A. Micchelli and J. Ward , *Spectral radius formulas for subdivision operators ,* in " Recent Advances in Wavelet Analysis, " L. L. Schumaker and G. Webb (eds), Academic Press, 1995, 335-360.

[6] Jia, R. Q., *Subdivision schemes in L^p spaces,* Advances in Computational Mathematics (1995).

[7] Lawton, W., *Multilevel properties of the wavelet-Galerkin operator,* J. Math. Phys. 32(1991), 1440 - 1443.

[8] Lawton, W., S. L. Lee and Zuowei Shen, *Stability and orthonormality of multidimensional refinable functions,* Preprint .

[9] Long, R. and D. R. Chen, *Biorthogonal wavelets bases on* $\mathbf{R}^{\mathbf{d}}$, Appl. Comp. Harmonic Anal. 2 (1995), 230-242.

[10] Strang, G., *Eigenvalues of* $(\downarrow 2)H$ *and convergence of cascade algorithm ,* Preprint.

Institute of System Sciences, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511