WAVELETS FROM THE LOOP SCHEME

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ABSTRACT. A new wavelet-based geometric mesh compression algorithm was developed recently in the area of computer graphics by Khodakovsky, Schröder, and Sweldens in their interesting paper [23]. The new wavelets used in [23] were designed from the Loop scheme by using ideas and methods of [26, 27], where orthogonal wavelets with exponential decay and pre-wavelets with compact support were constructed. The wavelets have the same smoothness order as that of the basis function of the Loop scheme around the regular vertices which has a continuous second derivative; the wavelets also have smaller supports than those wavelets obtained by constructions in [26, 27] or any other compactly supported biorthogonal wavelets derived from the Loop scheme (e.g. [11, 12]). Hence, the wavelets used in [23] have a good time frequency localization. This leads to a very efficient geometric mesh compression algorithm as proposed in [23]. As a result, the algorithm in [23] outperforms several available geometric mesh compression schemes used in the area of computer graphics. However, it remains open whether the shifts and dilations of the wavelets form a Riesz basis of $L_2(\mathbb{R}^2)$. Riesz property plays an important role in any wavelet-based compression algorithm and is critical for the stability of any wavelet-based numerical algorithms. We confirm here that the shifts and dilations of the wavelets used in [23] for the regular mesh, as expected, do indeed form a Riesz basis of $L_2(\mathbb{R}^2)$ by applying the more general theory established in this paper.

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1. Introduction

The main objective of this paper is to show that the wavelets, which are derived from the Loop Scheme and used in computer graphics by Khodakovsky, Schröder, and Sweldens in [23], are indeed Riesz wavelets, i.e., the shifts and dilations of the wavelets form a Riesz basis of $L_2(\mathbb{R}^2)$. This is done by establishing a general theory for the construction of high-dimensional Riesz wavelets with short support from a multiresolution analysis generated by multivariate refinable functions such as box splines.

We start with some necessary concepts and notations. Let $\Psi := \{\psi^1, \ldots, \psi^r\}$ be a finite set of functions in $L_2(\mathbb{R}^s)$. A dyadic system $X(\Psi)$ generated by Ψ is defined to be

$$X(\Psi) := \{\psi^\ell_{j,k} \ : \ j \in \mathbb{Z}, k \in \mathbb{Z}^s, \ell = 1, \ldots, r\},$$

where

$$\psi^\ell_{j,k} := 2^{js/2}\psi^\ell(2^j \cdot -k).$$

Recall that a system $X(\Psi)$ is a *Riesz basis* of $L_2(\mathbb{R}^s)$ if there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^s} |c^\ell_{j,k}|^2 \leqslant \Big\| \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^s} c^\ell_{j,k}\psi^\ell_{j,k}\Big\|^2_{L_2(\mathbb{R}^s)} \leqslant C_2 \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^s} |c^\ell_{j,k}|^2 \tag{1.1}$$

for all $\{c^\ell_{j,k}\} \in \ell_2(\{1,\ldots,r\}\times\mathbb{Z}\times\mathbb{Z}^s)$, and the linear span of $X(\Psi)$ is dense in $L_2(\mathbb{R}^s)$. The generators $\psi^1, \ldots, \psi^r$ are called *Riesz wavelets* if $X(\Psi)$ is a Riesz basis of $L_2(\mathbb{R}^s)$. Wavelets are normally obtained from a refinable function via the multiresolution analysis. A function $\phi : \mathbb{R}^s \mapsto \mathbb{C}$ is *refinable* if ϕ satisfies the refinement equation

$$\phi = 2^s \sum_{k\in\mathbb{Z}^s} a(k)\phi(2\cdot -k), \tag{1.2}$$

where $a : \mathbb{Z}^s \mapsto \mathbb{C}$ is a sequence on $\mathbb{Z}^s$, called the *refinement mask* for ϕ. The *Fourier series* or symbol $\hat{u}$ of a sequence u on $\mathbb{Z}^s$ is defined as

$$\hat{u}(\xi) := \sum_{k\in\mathbb{Z}^s} u(k)e^{-ik\cdot\xi}, \qquad \xi \in \mathbb{R}^s, \tag{1.3}$$

where $k\cdot\xi$ denotes the inner product between the vectors k and ξ in $\mathbb{R}^s$. Now we can rewrite (1.2) in the frequency domain as follows:

$$\hat{\phi}(\xi) = \hat{a}(\xi/2)\hat{\phi}(\xi/2), \tag{1.4}$$

where the Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined as $\hat{f}(\xi) := \int_{\mathbb{R}^s} f(t)e^{-i\xi\cdot t}\,dt$ and can be naturally extended to tempered distributions. Throughout the paper we assume that $\hat{\phi}(0) = 1$ and $\hat{a}(0) = 1$.

For a given refinable function $\phi \in L_2(\mathbb{R}^s)$ such that its Fourier transform is continuous at the origin with $\hat{\phi}(0) = 1$, define V_0 to be the smallest closed subspace of $L_2(\mathbb{R}^s)$ generated by $\phi(\cdot - k)$, $k \in \mathbb{Z}$. Then, V_0 is a shift invariant subspace of $L_2(\mathbb{R}^s)$. Let $V_j := \{f(2^j\cdot) : f \in V_0\}$, for $j \in \mathbb{Z}$. Then, the sequence of spaces V_j, $j \in \mathbb{Z}$ forms a multiresolution analysis (MRA) generated by ϕ, i.e., (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$; (ii) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}^s)$ and (iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ (see e.g. [1] and [20]).

For a function $\phi \in L_2(\mathbb{R})$, we say that the shifts of ϕ are *stable* (or simply ϕ is stable) if there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{k\in\mathbb{Z}^s} |c_k|^2 \leqslant \Big\| \sum_{k\in\mathbb{Z}^s} c_k\phi(\cdot - k)\Big\|^2_{L_2(\mathbb{R})} \leqslant C_2 \sum_{k\in\mathbb{Z}^s} |c_k|^2$$

for all finitely supported sequences $\{c_k\}_{k\in\mathbb{Z}^s}$.

An important family of refinable functions consists of the box splines. For a given $s \times n$ (direction) matrix Ξ of full rank with integer entries and $n \geqslant s$, the Fourier transform of its associated box spline M_Ξ is given by

$$\widehat{M_\Xi}(\xi) := \prod_{k\in\Xi} \frac{1-e^{-ik\cdot\xi}}{ik\cdot\xi}, \qquad \xi \in \mathbb{R}^s, \tag{1.5}$$

where $k \in \Xi$ means that k is a column vector of Ξ and k goes through all the columns of Ξ once and only once. The box spline M_Ξ is refinable and its refinement mask is given by

$$\widehat{a}(\xi) = \prod_{k\in\Xi} \frac{1+e^{-ik\cdot\xi}}{2}. \tag{1.6}$$

Box splines are symmetric (to some point in $\mathbb{Z}^s/2$) and belong to $C^{m(\Xi)-1}$, where $m(\Xi)+1$ is the minimum number of columns that can be discarded from Ξ to obtain a matrix of rank $< s$. Furthermore, it was shown in [8] that the shifts of a box spline are stable, whenever the matrix Ξ is a unimodular matrix, that is, every basis of columns from Ξ has determinant ± 1. When Ξ is a $1 \times n$ row vector with all its components being 1, the box spline M_Ξ is the well-known B-spline of order n with the Fourier series of its mask being $2^{-n}(1+e^{-i\xi})^n$ and the shifts of M_Ξ are stable. The reader should consult [3] for more details on box splines.

Let ϕ be a refinable function with a refinement mask a that generates an MRA. In general, a non-redundant dyadic wavelet system from the multiresolution analysis generated by ϕ in dimension s has $2^s - 1$ generators $\psi^1, \ldots, \psi^{2^s-1}$ which are derived from the refinable function ϕ in the following way:

$$\widehat{\psi^\ell}(\xi) := \widehat{b^\ell}(\xi/2)\hat{\phi}(\xi/2), \qquad \ell = 1, \ldots, 2^s - 1, \tag{1.7}$$

for some wavelet masks $b^1, \ldots, b^{2^s-1}$ on $\mathbb{Z}^s$. For dimension $s = 1$, a natural choice is

$$\widehat{b^1}(\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}, \qquad \xi \in \mathbb{R}. \tag{1.8}$$

Let ψ^1 be defined in (1.7) with the choice of b^1 in (1.8). It is well known ([9]) that when ϕ is an orthonormal refinable function (that is, the shifts of ϕ form an orthonormal system), the wavelet system $X(\{\psi^1\})$ forms an orthonormal basis of $L_2(\mathbb{R})$. More recently, it has been shown in [16] that when ϕ is the B-spline function of order n with mask $\hat{a}(\xi) = 2^{-n}(1+e^{-i\xi})^n$, $X(\{\psi^1\})$ is a Riesz basis in $L_2(\mathbb{R})$.

When the dimension s is 2 or 3, a simple and natural construction of orthonormal wavelets from a given symmetric orthonormal refinable function was provided in [27] where the wavelet masks are obtained from the refinement mask by multiplying an exponential to a proper shift of the refinement mask $\widehat{a}$. For example, when $s = 2$, the choice of [27] for wavelet masks b^1, b^2, b^3 is

$$\begin{aligned} \widehat{b^1}(\xi_1,\xi_2) &:= e^{-i(\xi_1+\xi_2)}\overline{\hat{a}(\xi_1+\pi,\xi_2)}, \\ \widehat{b^2}(\xi_1,\xi_2) &:= e^{-i\xi_2}\overline{\hat{a}(\xi_1,\xi_2+\pi)}, \\ \widehat{b^3}(\xi_1,\xi_2) &:= e^{-i\xi_1}\overline{\hat{a}(\xi_1+\pi,\xi_2+\pi)}. \end{aligned} \tag{1.9}$$

This choice of the wavelet masks leads to orthogonal wavelets with exponential decay when orthogonalized box splines are used (for details, see [26] and [27]). Let Ψ be defined in (1.7) using the wavelet masks give in (1.9) and the mask $\hat{a}$ of a bivariate box spline given in (1.6). It is quite natural to ask under what conditions the set Ψ generates a Riesz wavelet basis in $L_2(\mathbb{R}^2)$, due to (i) box splines generalize the B-splines, (ii) when the corresponding construction is applied to B-splines (univariate box splines), one obtains Riesz wavelets, and (iii) wavelet masks given in (1.9) are derived from the refinement mask in a similar way as its univariate counterpart.

This is one of our motivations here. However, the main motivation of this paper is to find Riesz wavelets with short support which can be used in computer graphics as described in the rest of this section.

Since we are interested in deriving Riesz wavelets with short support, we start with stable box splines. When $s = 2$, the only stable box splines are those with direction matrices based on the three directions $k_1 := (1, 0)$, $k_2 := (0, 1)$ and $k_1 + k_2 = (1, 1)$ (see e.g. [3] and [8]).

In order to have an efficient algorithm, symmetries of the refinement masks used in subdivision schemes are required in computer graphics. We say that G is a *symmetry group* on $\mathbb{Z}^s$ ([14]) if each element $E \in G$ is an $s \times s$ integer matrix with $|\det E| = 1$ and G forms a group under matrix multiplication. For example, the commonly used (regular) triangular mesh in computer graphics is invariant under the symmetry group D_6:

$$D_6 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}.$$

We say that a sequence a is G-*symmetric* if

$$a(Ek) = a(k) \qquad \forall\, k \in \mathbb{Z}^s \quad \text{and} \quad E \in G. \tag{1.10}$$

In order to have a D_6-symmetric refinement mask, we use those box splines such that the three directions are given with equal multiplicities r and centered at the origin. The centered box spline denoted by $\phi_{r,r,r}$ belongs to C^{2r-2} and its Fourier transform is

$$\widehat{\phi}_{r,r,r}(\xi_1, \xi_2) = \frac{\sin^r(\xi_1/2)}{(\xi_1/2)^r} \frac{\sin^r(\xi_2/2)}{(\xi_2/2)^r} \frac{\sin^r(\xi_1/2 + \xi_2/2)}{(\xi_1/2 + \xi_2/2)^r}. \tag{1.11}$$

Its refinement mask is given by

$$\hat{a}_{r,r,r}(\xi_1, \xi_2) := \cos^r\left(\frac{\xi_1}{2}\right) \cos^r\left(\frac{\xi_2}{2}\right) \cos^r\left(\frac{\xi_1 + \xi_2}{2}\right). \tag{1.12}$$

It is easy to see that the mask $a_{r,r,r}$ is D_6-symmetric.

It turns out that $\phi_{2,2,2}$ is the basis function of the Loop scheme in computer graphics; its refinement mask $a_{2,2,2}$ is supported on $[-2, 2]^2$ and is given by

$$a_{2,2,2} = \begin{bmatrix} 0 & 0 & 1/64 & 1/32 & 1/64 \\ 0 & 1/32 & 3/32 & 3/32 & 1/32 \\ 1/64 & 3/32 & \mathbf{5/32} & 3/32 & 1/64 \\ 1/32 & 3/32 & 3/32 & 1/32 & 0 \\ 1/64 & 1/32 & 1/64 & 0 & 0 \end{bmatrix}, \tag{1.13}$$

where $a_{2,2,2}(0) = 5/32$. The Loop scheme is an algorithm to generate a smooth subdivision surface from an initial triangular mesh (see [24]). The Loop scheme around the regular vertices is the subdivision scheme derived from the mask in (1.13). It is well-known that the basis function of the Loop scheme is C^2 around regular vertices and is only C^1 around irregular vertices ([2, 23, 24]). The Loop scheme is a widely used subdivision scheme in computer graphics to generate subdivision surfaces from initial triangular meshes (see e.g. [2, 23]). Taking into consideration the efficiency of implementation and visual quality of the reconstructed compressed mesh, and several other desirable properties of the Loop scheme, wavelets used in a mesh compression scheme are preferred to have small supports (see [23]). It seems natural to choose a wavelet system derived from a biorthogonal wavelet which is derived from $\phi_{2,2,2}$ as the primal refinable function and some dual refinable function $\tilde{\phi}$ of $\phi_{2,2,2}$. In order to reduce the complexity of implementation of the wavelet transform on a triangular mesh with an arbitrary topology, the support of the dual refinable function (in other words, the support of the dual mask) must be as small as possible (see [23]), since the complexity in general increases exponentially with respect to the support of a dual mask. The short support of a dual refinable function generally implies the short support of the wavelet functions which are used in the geometric mesh compression algorithm ([23]). However, as proved in [12], there is no dual refinable function $\tilde{\phi} \in L_2(\mathbb{R}^2)$ of $\phi_{2,2,2}$ such that the mask for $\tilde{\phi}$ can be supported on $[-4,4]^2$. Though a dual refinable function of $\phi_{2,2,2}$ with support $[-5,5]^2$, which is the smallest support for a possible dual refinable function of $\phi_{2,2,2}$ in $L_2(\mathbb{R}^2)$, is given in [12] and is constructed by the CBC algorithm introduced in [11], the implementation of such wavelet system using the dual refinable function with support $[-5,5]^2$ in [12] is very difficult and complicated if not formidable to be implemented.

One of the main ideas of [23] to overcome the above mentioned difficulties is to build up a wavelet system with short support by using the methods of [26] and [27] without using an explicit dual refinable function. By this, one still has a fast wavelet reconstruction algorithm, while the wavelet decomposition is obtained by solving a system of linear equations numerically. Since in applications the reconstruction is normally "online" that needs to be fast and decomposition is "off-line" whose speed is not as urgent and critical as the "online" counterpart, this choice is reasonable and feasible. The wavelets used in [23] are

$$\Psi_{2,2,2} := \{\psi^1, \psi^2, \psi^3\}, \tag{1.14}$$

where ψ^1, ψ^2 and ψ^3 are defined in (1.7) with $\widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ being the symbols of the sequences

$$b^1 = \begin{bmatrix} 0 & 0 & 1/64 & -1/32 & 1/64 \\ 0 & -1/32 & 3/32 & -3/32 & 1/32 \\ 1/64 & -3/32 & 5/32 & -3/32 & 1/64 \\ 1/32 & \mathbf{-3/32} & 3/32 & -1/32 & 0 \\ 1/64 & -1/32 & 1/64 & 0 & 0 \end{bmatrix},$$

$$b^2 = \begin{bmatrix} 0 & 0 & 1/64 & 1/32 & 1/64 \\ 0 & -1/32 & -3/32 & -3/32 & -1/32 \\ 1/64 & 3/32 & 5/32 & 3/32 & 1/64 \\ -1/32 & -3/32 & \mathbf{-3/32} & -1/32 & 0 \\ 1/64 & 1/32 & 1/64 & 0 & 0 \end{bmatrix},$$

$$b^3 = \begin{bmatrix} 0 & 0 & 1/64 & -1/32 & 1/64 \\ 0 & 1/32 & -3/32 & 3/32 & -1/32 \\ 1/64 & \mathbf{-3/32} & 5/32 & -3/32 & 1/64 \\ -1/32 & 3/32 & -3/32 & 1/32 & 0 \\ 1/64 & -1/32 & 1/64 & 0 & 0 \end{bmatrix},$$

and the underlying refinable function being $\phi_{2,2,2}$, the basis function of the Loop scheme given in (1.11) with its refinement mask in (1.13) (also see (1.12) with $r = 2$). We call $\Psi_{2,2,2}$ the wavelets based on the Loop scheme and $X(\Psi_{2,2,2})$ the wavelet system based on the Loop scheme. In Figure 1, the graphs of the basis function $\phi_{2,2,2}$ and the three wavelets in $\Psi_{2,2,2}$ based on the Loop scheme are given.

The wavelets $\Psi_{2,2,2}$ are applied in [23] to geometric mesh compression (around the regular vertices) in computer graphics with very impressive results and outperforms several other traditional mesh compression methods as demonstrated in the pioneering work of [23]. This is largely because the wavelets have sufficient smoothness (in C^2 around regular vertices) with small support. In other words, the system has a good time frequency localization which leads to an efficient compression scheme. Note that the wavelet functions ψ^1, ψ^2 and ψ^3 have supports in (but smaller than) $[-3/2, 5/2]^2$, $[-2,2] \times [-3/2, 5/2]$ and $[-3/2, 5/2] \times [-2,2]$, respectively; the

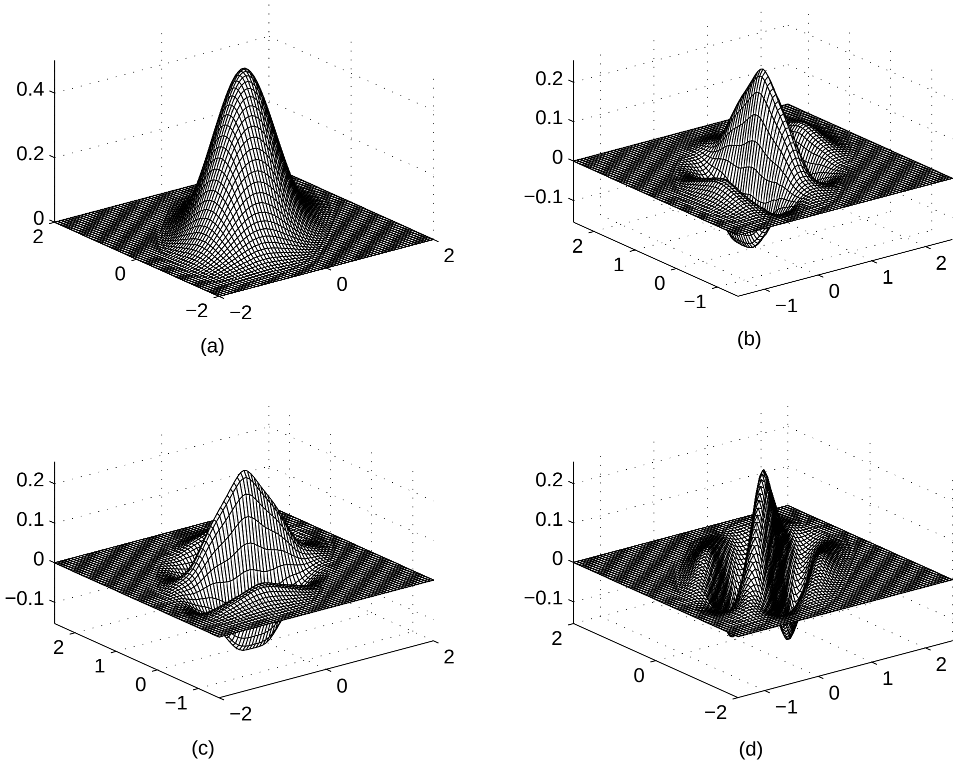

FIGURE 1. (a) is the graph of the basis function $\phi_{2,2,2}$ in the Loop scheme for the regular mesh. (b), (c), and (d) are the graphs of the three wavelet functions ψ^1, ψ^2, ψ^3 in (1.14), respectively. The dyadic wavelet system $X(\Psi_{2,2,2})$ forms a Riesz basis in $L_2(\mathbb{R}^2)$, where $\Psi_{2,2,2} := \{\psi^1, \psi^2, \psi^3\}$, as defined in (1.14).

areas of the support of the wavelets are the same and equal to the area of the refinable function $\phi_{2,2,2}$.

There always exist a few irregular (extraordinary) vertices, since it is impossible to model an arbitrary geometric topology by a triangular mesh with only regular vertices. How to design a wavelet system around the irregular vertices to obtain an efficient wavelet algorithm around them remains unresolved. The irregular vertices are handled indirectly in [23] by a careful and proper modification so that it does not impact the numerically computed condition numbers of the wavelet transform (see [23] for details). As a result, a progressive geometry compression algorithm and software have been developed in [23]. The software implementing the algorithm in [23] can be downloaded at

`http://www.multires.caltech.edu/software/pgc/`.

Using this, we reproduce two examples of [23] here to illustrate the compression quality of the algorithm. As pointed out in [23] that "For most models, our [Khodakovsky, Schröder and Sweldens] error is about four times smaller at comparable bit rates, a remarkable 12dB improvement [over the progressive CPM coder]". The detailed comparison with various algorithms can be found in [23]. The interested reader should consult [23] for more details.

While the wavelet system $X(\Psi_{2,2,2})$ is successfully used in computer graphics as illustrated above, it remains open whether the wavelet system $X(\Psi_{2,2,2})$ forms a Riesz basis of $L_2(\mathbb{R}^2)$. This is a challenging problem, theoretically, and relates to the numerical stability of the corresponding algorithms, practically. All these, together with our curiosities, lead to our adventure here. In Section 2, we will prove that $X(\Psi_{2,2,2})$ does indeed form a Riesz basis of $L_2(\mathbb{R}^s)$.

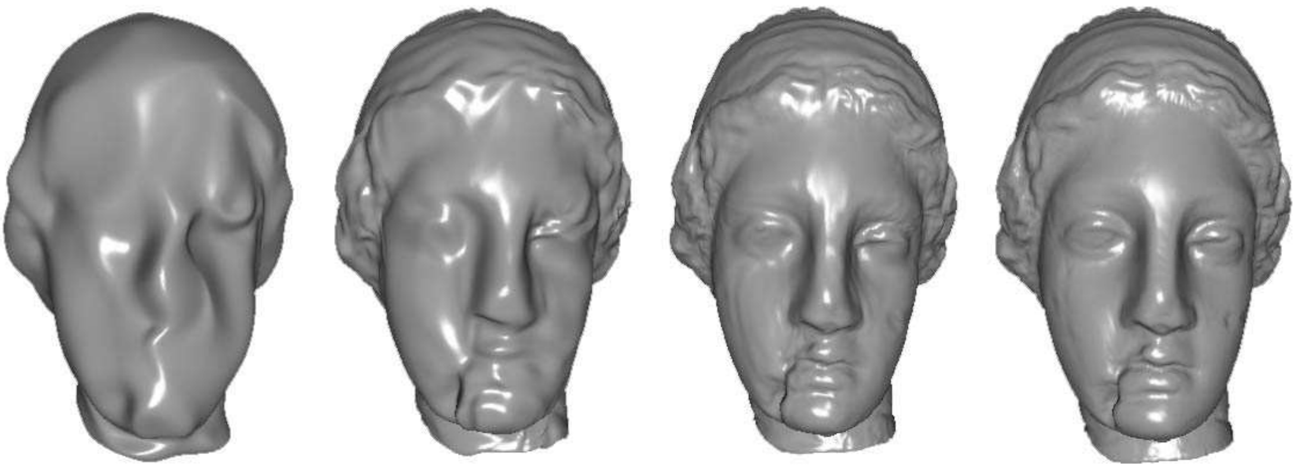

FIGURE 2. Partial bi-stream reconstructions from the progressive encoding in [23] of the Venus head model. From left to right, the file sizes given in bytes, relative L^2 reconstruction error and the PNSR are: ($520B$, 36.65×10^{-4}, 48.71), ($1528B$, 12.04×10^{-4}, 58.39), ($6802B$, 2.89×10^{-4}, 70.78), and ($26728B$, 0.812×10^{-4}, 81.74), respectively. The rightmost reconstruction is indistinguishable from the original.

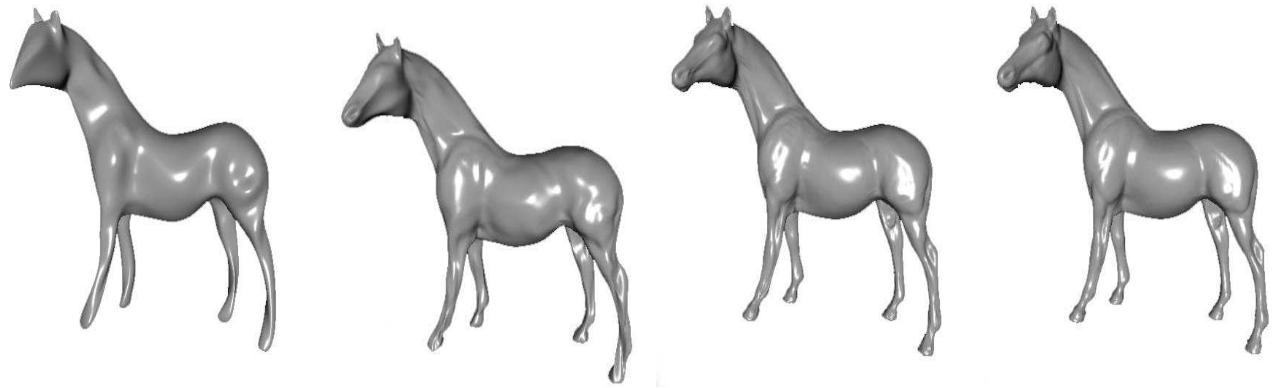

FIGURE 3. Partial bi-stream reconstructions from the progressive encoding in [23] of the horse model. From left to right, the file sizes given in bytes are: $337B$, $1149B$, $4188B$, and $14739B$, respectively. The rightmost reconstruction is indistinguishable from the original.

In Section 3, we shall present a general theory on Riesz wavelet bases in any dimension that is used in the proof of our main result in Section 2. Riesz wavelets have been investigated in the literature ([5, 21, 25]). Differences between our approach and the existing literature as well as what are the new contributions here to this topic will be discussed in this section.

2. Wavelets Based on the Loop Scheme

This section is to show that the system $X(\Psi_{2,2,2})$ is indeed a Riesz wavelet basis in $L_2(\mathbb{R}^2)$. In order to do so, let us first introduce some necessary concepts and notions here.

For $j \in \mathbb{Z}^s$, we denote δ_j the sequence on $\mathbb{Z}^s$ such that $\delta_j(j) = 1$ and $\delta_j(k) = 0$ for all $k \in \mathbb{Z}^s \backslash \{j\}$. In particular, we denote $\delta := \delta_0$. The convolution of two sequences is defined to be

$$[u * v](j) := \sum_{k \in \mathbb{Z}^s} u(k)v(j-k), \qquad u, v \in \ell_0(\mathbb{Z}^s),$$

where $\ell_0(\mathbb{Z}^s)$ denotes the space of all finitely supported sequences on $\mathbb{Z}^s$. Clearly, $\widehat{u*v} = \hat{u}\hat{v}$.

The space $\ell_p(\mathbb{Z}^s)$ with $1 \leqslant p \leqslant \infty$ consists of all sequences v on $\mathbb{Z}^s$ such that

$$\|v\|_{\ell_p(\mathbb{Z}^s)} := (\sum_{k\in\mathbb{Z}^s} |v(k)|^p)^{1/p} < \infty \quad \text{for } 1 \leqslant p < \infty$$

and $\|v\|_{\ell_\infty(\mathbb{Z}^s)} := \sup_{k\in\mathbb{Z}^s} |v(k)| < \infty$. For two sequences a and u on $\mathbb{Z}^s$, we define the following quantity

$$\rho(a,p,u) := \limsup_{n\to\infty} \|u * [S_a^n \delta]\|_{\ell_p(\mathbb{Z}^s)}^{1/n}, \qquad 1 \leqslant p \leqslant \infty, \tag{2.1}$$

where $S_a : \ell_p(\mathbb{Z}^s) \mapsto \ell_p(\mathbb{Z}^s)$ is the *subdivision operator* which is defined to be

$$[S_a v](j) := \sum_{k\in\mathbb{Z}^s} a(j-2k)v(k), \qquad j \in \mathbb{Z}^s \quad \text{and} \quad v \in \ell_p(\mathbb{Z}^s). \tag{2.2}$$

One can check that $\|S_a v\|_{\ell_p(\mathbb{Z}^s)} \leqslant \|a\|_{\ell_1(\mathbb{Z}^s)}\|v\|_{\ell_p(\mathbb{Z}^s)}$. So, the subdivision operator S_a is a bounded operator on $\ell_p(\mathbb{Z}^s)$ if $a \in \ell_1(\mathbb{Z}^s)$.

For $k \in \mathbb{Z}^s$ and $t \in \mathbb{R}^s$, we define

$$\nabla_k v := v - v(\cdot - k), \quad \nabla_t f := f - f(\cdot - t), \qquad v \in \ell_p(\mathbb{Z}^s), f \in L_p(\mathbb{R}^s). \tag{2.3}$$

Denote $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. For $\mu = (\mu_1,\ldots,\mu_s) \in \mathbb{N}_0^s$ and $\xi = (\xi_1,\ldots,\xi_s) \in \mathbb{R}^s$, $|\mu| := |\mu_1|+\cdots+|\mu_s|$, $\xi^\mu := \xi_1^{\mu_1}\cdots\xi_s^{\mu_s}$ and $\nabla^\mu := \nabla_{e_1}^{\mu_1}\cdots\nabla_{e_s}^{\mu_s}$, where e_j is the jth coordinate unit vector in $\mathbb{R}^s$. Note that $\nabla^\mu v = [\nabla^\mu\delta] * v$ and

$$\nabla^\mu f = [\nabla^\mu \delta] * f := \sum_{k\in\mathbb{Z}^s} [\nabla^\mu\delta](k) f(\cdot - k).$$

The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by $\partial_j f$. For $\mu = (\mu_1,\ldots,\mu_s) \in \mathbb{N}_0^s$, we denote $\partial^\mu := \partial_1^{\mu_1}\cdots\partial_s^{\mu_s}$.

Let $\Gamma := [0,1]^s \cap \mathbb{Z}^s$ be the set of all the vertices of the cube $[0,1]^s$, and Π_r be the space of all polynomials of total degree at most r. We say that a mask a satisfies the *sum rules* of order $r+1$ ([18]) with respect to the lattice $2\mathbb{Z}^s$ if

$$\sum_{k\in2\mathbb{Z}^s} a(\gamma+k)p(\gamma+k) = \sum_{k\in2\mathbb{Z}^s} a(k)p(k) \qquad \forall\, \gamma \in \Gamma \quad \text{and} \quad p \in \Pi_r. \tag{2.4}$$

In the frequency domain, a mask a satisfies the sum rules of order $r+1$ if and only if

$$\partial^\mu \hat{a}(\pi\gamma) = 0 \qquad \forall\, |\mu| \leqslant r \quad \text{and} \quad \gamma \in \Gamma\backslash\{0\}. \tag{2.5}$$

If a mask a satisfies the sum rules of order r but not $r+1$, then we define (see [14, 15]) the following quantity:

$$\nu_p(a) := s/p - s - \log_2 \max\{\rho(a,p,\nabla^\mu\delta) \ : \ |\mu| = r\}, \qquad 1 \leqslant p \leqslant \infty. \tag{2.6}$$

If $|\widehat{a_1}(\xi)| \leqslant |\widehat{a_2}(\xi)|$ for all $\xi \in \mathbb{R}^s$, it is easy to see that $\nu_2(a_1) \geqslant \nu_2(a_2)$ by the definition of $\nu_2(a)$ in (2.6) and the definition of $\rho(a,2,u)$ in (2.1). Indeed, this follows from

$$|\widehat{S_{a_1}^n\delta}(\xi)| = |\widehat{a_1}(\xi)\widehat{a_1}(2\xi)\cdots\widehat{a_1}(2^{n-1}\xi)| \leqslant |\widehat{a_2}(\xi)\widehat{a_2}(2\xi)\cdots\widehat{a_2}(2^{n-1}\xi)| = |\widehat{S_{a_2}^n\delta}(\xi)|$$

and consequently,

$$\begin{aligned}\|\nabla^\mu\delta * [S_{a_1}^n\delta]\|_{\ell_2(\mathbb{Z}^s)}^2 &= \frac{1}{(2\pi)^s}\int_{[-\pi,\pi]^s} |\widehat{\nabla^\mu\delta}(\xi)|^2 |\widehat{S_{a_1}^n\delta}(\xi)|^2\, d\xi \\ &\leqslant \frac{1}{(2\pi)^s}\int_{[-\pi,\pi]^s} |\widehat{\nabla^\mu\delta}(\xi)|^2 |\widehat{S_{a_2}^n\delta}(\xi)|^2\, d\xi = \|\nabla^\mu\delta * [S_{a_2}^n\delta]\|_{\ell_2(\mathbb{Z}^s)}^2.\end{aligned}$$

The quantity $\nu_p(a)$ plays an important role in the study of subdivision schemes and wavelets (see [15] and many references therein for detail). However, it is generally not easy to compute the quantity $\nu_p(a)$, except for $p = 2$. By taking into account symmetries, a numerical algorithm has been proposed in [14, Algorithm 2.1] for efficient computation of the quantity $\nu_2(a)$ for a finitely supported G-symmetric mask a. (See also [22] and [29] without using symmetry.)

In order to show that the wavelet system $X(\Psi_{2,2,2})$ forms a Riesz basis in $L_2(\mathbb{R}^2)$, we need the following result, which is a corollary of Theorem 3.2 and will be proven in Section 3.

Theorem 2.1. *Let $a, b^1, \ldots, b^{2^s-1}$ be finitely supported sequences on $\mathbb{Z}^s$ such that $\hat{a}(0) = 1$ and $\widehat{b^1}(0) = \cdots = \widehat{b^{2^s-1}}(0) = 0$. Let ϕ be the refinable function associated with mask a such that $\hat{\phi}(0) = 1$. Define*

$$M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi) := \begin{bmatrix} \hat{a}(\xi+\pi\gamma_0) & \hat{a}(\xi+\pi\gamma_1) & \cdots & \hat{a}(\xi+\pi\gamma_{2^s-1}) \\ \widehat{b^1}(\xi+\pi\gamma_0) & \widehat{b^1}(\xi+\pi\gamma_1) & \cdots & \widehat{b^1}(\xi+\pi\gamma_{2^s-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{b^{2^s-1}}(\xi+\pi\gamma_0) & \widehat{b^{2^s-1}}(\xi+\pi\gamma_1) & \cdots & \widehat{b^{2^s-1}}(\xi+\pi\gamma_{2^s-1}) \end{bmatrix}, \tag{2.7}$$

where $\{\gamma_0, \ldots, \gamma_{2^s-1}\} = [0,1]^s \cap \mathbb{Z}^s$ with $\gamma_0 := 0$. Assume that

(i) *$\det M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^s$,*
(ii) *$\nu_2(a) > 0$,*
(iii) *for $\hat{\mathring{a}}(\xi)$, the $(1,1)$-entry of the matrix $\overline{M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi)}^{-1}$, there exists a finitely supported sequence $\mathring{a}$ on $\mathbb{Z}^s$ such that $\hat{\mathring{a}}(0) = 1$, $\nu_2(\mathring{a}) > s/2$ and $|\hat{\mathring{a}}(\xi)| \leqslant |\hat{\mathring{a}}(\xi)|$ for all $\xi \in \mathbb{R}^s$.*

Then, $X(\Psi)$, where $\Psi := \{\psi^1, \ldots, \psi^{2^s-1}\}$ denotes the set of the wavelet functions defined in (1.7), is a Riesz basis in $L_2(\mathbb{R}^s)$.

Let a be the mask of a refinable function $\phi \in L_2(\mathbb{R}^2)$ such that $\overline{\hat{a}(\xi)} = \hat{a}(\xi)$ for all $\xi \in \mathbb{R}^2$. Let b^1, b^2 and b^3 be the wavelet mask defined in (1.9) via its Fourier series. Let $M_{[a,b^1,b^2,b^3]}$ be the matrix defined in (2.7) by masks a, b^1, b^2 and b^3. Then, it was proved in [26, 27] that

$$\begin{aligned} &M_{[a,b^1,b^2,b^3]}(\xi)\overline{M_{[a,b^1,b^2,b^3]}(\xi)}^T \\ &\qquad = \big[|\hat{a}(\xi_1,\xi_2)|^2 + |\hat{a}(\xi_1+\pi,\xi_2)|^2 + |\hat{a}(\xi_1,\xi_2+\pi)|^2 + |\hat{a}(\xi_1+\pi,\xi_2+\pi)|^2\big] I_4. \end{aligned}$$

Before we prove the main result of this section, we also need the following result later.

Lemma 2.2. *Let $g(\xi_1,\xi_2) := \cos(\xi_1/2)\cos(\xi_2/2)\cos(\xi_1/2+\xi_2/2)$ and*

$$c(\xi_1,\xi_2) := |g(\xi_1,\xi_2)|^2 + |g(\xi_1+\pi,\xi_2)|^2 + |g(\xi_1,\xi_2+\pi)|^2 + |g(\xi_1+\pi,\xi_2+\pi)|^2.$$

Then

$$7/16 \leqslant c(\xi_1,\xi_2) \leqslant 1; \quad \textit{for all} \quad \xi_1,\xi_2 \in \mathbb{R}. \tag{2.8}$$

Proof. Simplifying $c(\xi_1,\xi_2)$, one gets

$$\begin{aligned} c(\xi_1,\xi_2) = \cos^2\frac{\xi_1+\xi_2}{2}\Big[\cos^2\frac{\xi_1}{2}\cos^2\frac{\xi_2}{2} + \sin^2\frac{\xi_1}{2}\sin^2\frac{\xi_2}{2}\Big] \\ + \sin^2\frac{\xi_1+\xi_2}{2}\Big[\sin^2\frac{\xi_1}{2}\cos^2\frac{\xi_2}{2} + \cos^2\frac{\xi_1}{2}\sin^2\frac{\xi_2}{2}\Big]. \end{aligned}$$

This immediately leads to

$$0 < c(\xi_1,\xi_2) \leqslant \Big[\cos^2\frac{\xi_1}{2} + \sin^2\frac{\xi_1}{2}\Big]\Big[\cos^2\frac{\xi_2}{2} + \sin^2\frac{\xi_2}{2}\Big]\Big[\cos^2\frac{\xi_1+\xi_2}{2} + \sin^2\frac{\xi_1+\xi_2}{2}\Big] = 1.$$

Next, we compute $c_{min} := \min_{\xi_1,\xi_2\in\mathbb{R}} c(\xi_1,\xi_2)$. Since c is a π-periodic trigonometric polynomial,

$$c_{min} := \min_{\xi\in\mathbb{R}^2} c(\xi) = \min_{\xi\in[0,\pi]^2} c(\xi).$$

Let $x_1 := \sin^2(\xi_1/2)$, $x_2 := \sin^2(\xi_2/2)$. Then, $0 \leqslant x_1, x_2 \leqslant 1$. Using these to express $c(\xi_1, \xi_2)$ in terms of x_1 and x_2, one obtains that $c(\xi_1, \xi_2) = f(x_1, x_2)$, where

$$\begin{aligned} f(x_1, x_2) :=& \big[(1-x_1)(1-x_2) + x_1x_2 - 2\sqrt{x_1(1-x_1)x_2(1-x_2)}\big]\big[(1-x_1)(1-x_2) + x_1x_2\big] \\ &+ \big[x_1(1-x_1) + (1-x_1)x_2 + 2\sqrt{x_1(1-x_1)x_2(1-x_2)}\big]\big[x_1(1-x_2) + (1-x_1)x_2\big]. \end{aligned}$$

Note that $0 \leqslant x_1, x_2 \leqslant 1/2$ and $1/2 \leqslant (1-x_1), (1-x_2) \leqslant 1$. Since x_1 and $1-x_1$, as well as x_2 and $1-x_2$, are in symmetric position in f, we have

$$\min_{\xi \in [0,\pi]^2} c(\xi) = \min_{0 \leqslant x_1, x_2 \leqslant 1/2} f(x_1, x_2).$$

In order to find the minimum of f, we compute $\partial_1 f$ and $\partial_2 f$ as follows:

$$\begin{aligned} \partial_1 f = \frac{1-2x_1}{\sqrt{x_1(1-x_1)x_2(1-x_2)}} \Big[& 8x_1^2x_2^2 - 8x_1^2x_2 - 8x_1x_2^2 + 8x_1x_2 + x_2^2 - x_2 \\ & - (8x_1x_2 - 4x_1 - 4x_2 + 2)\sqrt{x_1(1-x_1)x_2(1-x_2)} \Big] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \partial_2 f = \frac{1-2x_2}{\sqrt{x_1(1-x_1)x_2(1-x_2)}} \Big[& 8x_1^2x_2^2 - 8x_1^2x_2 - 8x_1x_2^2 + 8x_1x_2 + x_1^2 - x_1 \\ & - (8x_1x_2 - 4x_1 - 4x_2 + 2)\sqrt{x_1(1-x_1)x_2(1-x_2)} \Big]. \end{aligned} \tag{2.10}$$

Next, we find all the solutions (x_1, x_2) such that $\partial_1 f(x_1, x_2) = \partial_2 f(x_1, x_2) = 0$ for $(x_1, x_2) \in (0, 1/2)^2$. Suppose that $\partial_1 f(x_1, x_2) = 0$ and $\partial_2 f(x_1, x_2) = 0$ for some $0 < x_1, x_2 < 1/2$. Then from (2.9), we have

$$8x_1^2x_2^2 - 8x_1^2x_2 - 8x_1x_2^2 + 8x_1x_2 + x_2^2 - x_2 - (8x_1x_2 - 4x_1 - 4x_2 + 2)\sqrt{x_1(1-x_1)x_2(1-x_2)} = 0 \tag{2.11}$$

and from (2.10), we have

$$8x_1^2x_2^2 - 8x_1^2x_2 - 8x_1x_2^2 + 8x_1x_2 + x_1^2 - x_1 - (8x_1x_2 - 4x_1 - 4x_2 + 2)\sqrt{x_1(1-x_1)x_2(1-x_2)} = 0. \tag{2.12}$$

Consequently, subtracting (2.11) from (2.12), we must have

$$x_2^2 - x_2 - x_1^2 + x_1 = 0.$$

Since $x_2^2 - x_2 - x_1^2 + x_1 = (x_2 - x_1)(x_1 + x_2 - 1)$, and since $0 < x_1, x_2 < 1/2$, we conclude that $x_2 = x_1$. But when $x_2 = x_1$, the identity in (2.11) becomes

$$x_1(x_1 - 1)(4x_1 - 3)(4x_1 - 1) = 0.$$

Note that $0 < x_1 < 1/2$. Hence, the above equation asserts that we must have $4x_1 - 1 = 0$, that is, $x_1 = 1/4$. This means that the only possible solution (x_1, x_2) such that $\partial_1 f(x_1, x_2) = \partial_2 f(x_1, x_2) = 0$ and $(x_1, x_2) \in (0, 1/2)^2$ is $x_1 = x_2 = 1/4$. Note that $\partial_1 f(1/4, 1/4) = \partial_2 f(1/4, 1/4) = 0$ and $f(1/4, 1/4) = 7/16$.

Finally, the values of f along the boundary of $[0, 1/2]^2$ are

$$f(x_1, 0) = (1-x_1)^2 + x_1^2, \quad f(x_1, 1/2) = 1/2, \quad f(0, x_2) = (1-x_2)^2 + x_2^2, \quad f(1/2, x_2) = 1/2.$$

So, the minimal value of f along the boundary of $[0, 1/2]^2$ is $1/2$. Altogether, we have

$$c_{min} := \min_{\xi \in \mathbb{R}^2} c(\xi) = \min_{\xi \in [0,\pi]^2} c(\xi) = \min_{0 \leqslant x_1, x_2 \leqslant 1/2} f(x_1, x_2) = f(1/4, 1/4) = 7/16,$$

which completes the proof. ∎

Finally, we prove the main theorem of this section which confirms that the system $X(\Psi_{2,2,2})$ derived from the Loop scheme and used in mesh compression algorithms in computer graphics in [23] indeed forms a Riesz basis for $L_2(\mathbb{R}^2)$ for the regular mesh.

Theorem 2.3. *The wavelet system $X(\Psi_{2,2,2})$ based on the Loop scheme is a Riesz basis for $L_2(\mathbb{R}^2)$.*

Proof. We verify that the conditions (i)-(iii) in Theorem 2.1 are satisfied. The underlying refinable function, from that $\Psi_{2,2,2}$ is constructed, is the box spline $\phi_{2,2,2}$ whose refinement mask is

$$\hat{a} := \hat{a}_{2,2,2}(\xi_1,\xi_2) = \cos^2(\xi_1/2)\cos^2(\xi_2/2)\cos^2(\xi_1/2+\xi_2/2).$$

It is known that $\nu_2(a_{2,2,2}) = 3.5 > 0$. This shows that (ii) of Theorem 2.1 holds.

Using the properties and definition of masks $\hat{a}$, $\hat{b}^1, \hat{b}^2$ and $\hat{b}^3$, one can check ([26, 27]) that

$$M_{[a,b^1,b^2,b^3]}(\xi)\overline{M_{[a,b^1,b^2,b^3]}(\xi)}^T = c_1(\xi)I_4,$$

where

$$c_1(\xi_1,\xi_2) := |\hat{a}(\xi_1,\xi_2)|^2 + |\hat{a}(\xi_1+\pi,\xi_2)|^2 + |\hat{a}(\xi_1,\xi_2+\pi)|^2 + |\hat{a}(\xi_1+\pi,\xi_2+\pi)|^2.$$

Note that $\hat{a}(\xi) = \hat{a}_{2,2,2}(\xi) = g(\xi)^2$, where $g(\xi_1,\xi_2) = \cos(\xi_1/2)\cos(\xi_2/2)\cos(\xi_1/2+\xi_2/2)$. It follows from Lemma 2.2 that

$$\begin{aligned}7/16 &\leqslant |g(\xi_1,\xi_2)|^2 + |g(\xi_1+\pi,\xi_2)|^2 + |g(\xi_1,\xi_2+\pi)|^2 + |g(\xi_1+\pi,\xi_2+\pi)|^2 \\ &\leqslant \left[|g(\xi_1,\xi_2)|^4 + |g(\xi_1+\pi,\xi_2)|^4 + |g(\xi_1,\xi_2+\pi)|^4 + |g(\xi_1+\pi,\xi_2+\pi)|^4\right]^{1/2} 4^{1/2} \\ &= 4^{1/2} c_1(\xi_1,\xi_2)^{1/2}.\end{aligned}$$

Therefore, we must have

$$\frac{49}{1024} = \left(\frac{7}{16}\right)^2 \frac{1}{4} \leqslant c_1(\xi_1,\xi_2) \leqslant 1, \quad \text{for all} \quad \xi_1,\xi_2 \in \mathbb{R}.$$

This shows that (i) of Theorem 2.1 holds.

For (iii), consider $\hat{\tilde{a}}(\xi)$, the $(1,1)$th-entry of the matrix $\overline{M_{[a,b^1,b^2,b^3]}(\xi)}^{-1}$. Then $\hat{\tilde{a}}(\xi) = \hat{a}(\xi)/c_1(\xi)$ from the above observation. Let $d(\xi) := 1 - c_1(\xi)$. Then $d(0) = 0$. Since $49/1024 \leqslant c_1(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^2$, $0 \leqslant d(\xi) \leqslant 1 - 49/1024 < 1$. This leads to

$$\frac{1}{c_1(\xi)} = \frac{1}{1-d(\xi)} = 1 + \sum_{j=1}^{m-1}[d(\xi)]^j + \sum_{j=m}^{\infty}[d(\xi)]^j = 1 + \sum_{j=1}^{m-1}[d(\xi)]^j + \frac{[d(\xi)]^m}{c_1(\xi)}.$$

Since $49/1024 \leqslant c_1(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^2$, we conclude that

$$1 + [d(\xi)]^m + \sum_{j=1}^{m-1}[d(\xi)]^j \leqslant \frac{1}{c_1(\xi)} \leqslant 1 + \frac{1024}{49}[d(\xi)]^m + \sum_{j=1}^{m-1}[d(\xi)]^j. \tag{2.13}$$

Define

$$\begin{aligned}\widehat{\tilde{a}^{right}}(\xi) &:= \hat{a}(\xi)\left[1 + \frac{1024}{49}[d(\xi)]^m + \sum_{j=1}^{m-1}[d(\xi)]^j\right], \\ \widehat{\tilde{a}^{left}}(\xi) &:= \hat{a}(\xi)\left[1 + [d(\xi)]^m + \sum_{j=1}^{m-1}[d(\xi)]^j\right].\end{aligned} \tag{2.14}$$

Then, $|\widehat{\tilde{a}^{left}}(\xi)| \leqslant |\hat{\tilde{a}}(\xi)| \leqslant |\widehat{\tilde{a}^{right}}(\xi)|$ for all $\xi \in \mathbb{R}^2$. Therefore, we have $\nu_2(\tilde{a}^{right}) \leqslant \nu_2(\tilde{a}) \leqslant \nu_2(\tilde{a}^{left})$. Choose $m = 4$. The masks $\tilde{a}^{right}$ and $\tilde{a}^{left}$ are supported on $[-18, 18]^2$ and D_6-symmetric. By [14, Algorithm 2.1], we have $\nu_2(\tilde{a}^{right}) \geqslant 1.155966$. In fact, if taking $m = 6$, we have the estimate

$$1.589879 \approx \nu_2(\tilde{a}^{right}) \leqslant \nu_2(\tilde{a}) \leqslant \nu_2(\tilde{a}^{left}) \approx 2.261579.$$

This shows that (iii) of Theorem 2.1 holds by taking $\mathring{a} = \tilde{a}^{right}$. Hence, by Theorem 2.1, we conclude that $X(\Psi_{2,2,2})$ is a Riesz basis in $L_2(\mathbb{R}^2)$. ■

3. High-dimensional Riesz Wavelets

This section is to build up a general theory which can be applied to more general setting and, as a consequence, Theorem 2.1 is a corollary of the main result of this section. For this, we shall extend some results on the real line of [16] to the ones for high dimensions. Although ideas here look similar to those of [16], the proofs are based on a totally different technique and are highly nontrivial since the approach in [16] is largely based on the factorization of the Fourier series of univariate masks which we do not have for the multivariate case any more. Since the proofs cannot be derived by a modification from those of [16], we include a complete proof here.

A function f on $\mathbb{R}^s$ has *polynomial decay* if

$$(1 + \|\cdot\|)^j f \in L_\infty(\mathbb{R}^s) \qquad \forall\, j \in \mathbb{N}$$

and f has *exponential decay* if there exists a positive number c such that $e^{c\|\cdot\|} f \in L_\infty(\mathbb{R}^s)$. If a function f has polynomial decay or exponential decay, then clearly $f \in L_p(\mathbb{R}^s)$ for all $1 \leqslant p \leqslant \infty$. Similarly, a sequence a on $\mathbb{Z}^s$ has polynomial decay if

$$\sup_{k \in \mathbb{Z}^s} (1 + \|k\|)^j |a(k)| < \infty \qquad \forall\, j \in \mathbb{N}.$$

It is easy to see that a sequence a has polynomial decay if and only if $\hat{a} \in C^\infty(\mathbb{R}^s)$.

The following Lemma generalizes [16, Lemma 3.1] and [15, Theorem 3.6], and plays a critical role later in our proof of the main results in this section.

Lemma 3.1. *Let $f \in L_\infty(\mathbb{R}^s)$ be a function having polynomial decay and m be a positive integer. Then, the following statements are equivalent:*

(i) $\partial^\mu \hat{f}(2\pi k) = 0$ *for all* $k \in \mathbb{Z}^s$ *and* $|\mu| < m$;
(ii) $\sum_{k \in \mathbb{Z}^s} k^\mu f(\cdot - k) = 0$ *for all* $|\mu| < m$;
(iii) *There exists a set of functions h_μ, $\mu \in \mathbb{N}_0^s$ and $|\mu| = m$, with each of them having polynomial decay such that*

$$f = \sum_{|\mu| = m} \nabla^\mu h_\mu. \tag{3.1}$$

Proof. The fact that (i) and (ii) are equivalent is well-known (see e.g. [15, Theorem 3.6]).

Suppose that f is of the form in (3.1) with h_μ having polynomial decay. Since $\hat{f}(\xi) = \sum_{|\mu|=m} \widehat{\nabla^\mu \delta}(\xi) \widehat{h_\mu}(\xi)$ and since $\partial^\nu \widehat{\nabla^\mu \delta}(2\pi k) = 0$ for all $|\nu| < m$, $|\mu| = m$, and $k \in \mathbb{Z}^s$, (iii) implies (i).

To complete the proof, it remains to show that (ii) implies (iii). This means that we need to find a set of functions h_μ, $\mu \in \mathbb{N}_0^s$ and $|\mu| = m$, with each of them having polynomial decay, such that (3.1) holds. This is done by first constructing functions $f^0, f^1, \ldots, f^s$ with $f^0 = f$ and $h_1, \ldots, h_s$ such that

(1) $f^1, \ldots, f^s, h^1, \ldots, h^s$ are functions with polynomial decay;
(2) For $j = 1, \ldots, s$, the support of f^j is contained inside $[0, m]^j \times \mathbb{R}^{s-j}$;

(3) $f^{j-1} = f^j + \nabla_{e_j}^m h^j$ for all $j = 1, \dots, s$;

(4) For $j = 0, \dots, s$, we have $\sum_{k\in\mathbb{Z}^s} k^\mu f^j(\cdot - k) = 0$ for all $|\mu| < m$.

In particular, we deduce from the above relation in (3) that

$$f = f^0 = \nabla_{e_1}^m h^1 + f^1 = \cdots = \nabla_{e_1}^m h^1 + \cdots + \nabla_{e_s}^m h^s + f^s. \tag{3.2}$$

Since f^s is a compactly supported function in $L_\infty(\mathbb{R}^s)$ and $\sum_{k\in\mathbb{Z}^s} k^\mu f^s(\cdot - k) = 0$ for all $|\mu| < m$, by [15, Theorem 3.6], there exist compactly supported functions $h_\mu \in L_\infty(\mathbb{R}^s)$, $|\mu| = m$ such that $f^s = \sum_{|\mu|=m} \nabla^\mu h_\mu$. Hence, (iii) follows directly from (3.2).

Next, we give details for the construction of $f^1, \dots, f^s$ and $h_1, \dots, h_s$ by assuming (ii). For f, define a sequence of functions $\{f_k\}_{k\in\mathbb{Z}^s}$ on $\mathbb{R}^s$ (in fact, each function f_k is supported on $[0,1)^s$) as follows:

$$f_k(x) := \begin{cases} f(x+k), & \text{if } x \in [0,1)^s, \\ 0, & \text{if } x \in \mathbb{R}^s \backslash [0,1)^s, \end{cases} \qquad k \in \mathbb{Z}^s. \tag{3.3}$$

Clearly, f can be reconstructed from $\{f_k\}_{k\in\mathbb{Z}^s}$ by the formula $f = \sum_{k\in\mathbb{Z}^s} f_k(\cdot - k)$. It is easy to see that the function f has polynomial decay if and only if the sequence $\{\|f_k\|_{L_\infty(\mathbb{R}^s)}\}_{k\in\mathbb{Z}^s}$ has polynomial decay. More precisely, for every $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, there exists a positive constant C_μ such that

$$\|f_k\|_{L_\infty(\mathbb{R}^s)} \leqslant C_{(\mu_1,\dots,\mu_s)}(1+|k_1|)^{-\mu_1} \cdots (1+|k_s|)^{-\mu_s}, \quad \text{for all} \quad k = (k_1, \dots, k_s) \in \mathbb{Z}^s. \tag{3.4}$$

By the definition of f_k in (3.3), we observe that (ii) is equivalent to

$$\sum_{k\in\mathbb{Z}^s} k^\mu f_k = 0, \quad \text{for all} \quad |\mu| < m,\ \mu \in \mathbb{N}_0^s. \tag{3.5}$$

Define functions $F^j_{(k_2,\dots,k_s)}$ by

$$F^j_{(k_2,\dots,k_s)} := \sum_{k_1\in\mathbb{Z}} k_1^j f_{(k_1,k_2,\dots,k_s)}, \qquad j = 0, \dots, m-1 \quad \text{and} \quad k_2, \dots, k_s \in \mathbb{Z}. \tag{3.6}$$

By the decay condition in (3.4), it is not difficult to verify that for every nonnegative integer j, $\{F^j_k\}_{k\in\mathbb{Z}^{s-1}}$ has polynomial decay. Next, let

$$f^1 := \sum_{k\in\mathbb{Z}^s} f^1_k(\cdot - k),$$

where the sequence $\{f^1_k\}_{k\in\mathbb{Z}^s}$ is derived as follows:

(1) $f^1_k = 0$ for $k = (k_1, k_2, \dots, k_s) \in \mathbb{Z}^s$ with $k_1 \notin \{0, \dots, m-1\}$ and $k_2, \dots, k_s \in \mathbb{Z}$;

(2) $\{f^1_{(0,k_2,\dots,k_s)}, \dots, f^1_{(m-1,k_2,\dots,k_s)}\}$ is the unique solution to the following system of linear equations:

$$\sum_{k_1=0}^{m-1} k_1^j f^1_{(k_1,k_2,\dots,k_s)} = F^j_{(k_2,\dots,k_s)}, \qquad j = 0, \dots, m-1. \tag{3.7}$$

Note that the coefficient matrix of the above linear system is a Vandermonde matrix and is independent of all the integers $k_2, \dots, k_s$. Since the sequence $\{F^j_k\}_{k\in\mathbb{Z}^{s-1}}$ has polynomial decay for every nonnegative integer j, we conclude that the sequence $\{f^1_k\}_{k\in\mathbb{Z}^s}$ has polynomial decay. This leads to that $f^1 := \sum_{k\in\mathbb{Z}^s} f^1_k(\cdot - k)$ has polynomial decay. By the definition of F^j_k and f^1_k, it follows from (3.7) that

$$\sum_{k_1\in\mathbb{Z}} k_1^j [f_{(k_1,k_2,\dots,k_s)} - f^1_{(k_1,k_2,\dots,k_s)}] = 0 \qquad \forall\, k_2, \dots, k_s \in \mathbb{Z} \quad \text{and} \quad j = 0, \dots, m-1$$

which is equivalent to

$$\sum_{k_1\in\mathbb{Z}} k_1^j [f-f^1](\cdot - k_1 e_1) = 0 \qquad \forall\, j = 0,\dots,m-1. \tag{3.8}$$

Now we define a function h^1 as in [16, Lemma 3.1] by

$$h^1 := \sum_{\ell=0}^{\infty} \frac{(\ell+m-1)!}{\ell!(m-1)!}[f-f^1](\cdot - \ell e_1).$$

Since $f-f^1$ has polynomial decay and (3.8) holds, by the same argument as in the proof of [16, Lemma 3.1], we see that h^1 has polynomial decay and $f-f^1 = \nabla_{e_1}^m h^1$. Since (ii) holds and

$$\widehat{\nabla_{e_1}^m h^1}(\xi_1,\dots,\xi_s) = (1-e^{-i\xi_1})^m \widehat{h^1}(\xi_1,\dots,\xi_s),$$

we must have

$$\partial^\mu \widehat{f^1}(2\pi k) = \partial^\mu \hat f(2\pi k) - \partial^\mu \widehat{\nabla_{e_1}^m h^1}(2\pi k) = 0 \qquad \forall\, |\mu|<m,\ k\in\mathbb{Z}^s,$$

that is,

$$\sum_{k\in\mathbb{Z}^s} k^\mu f^1(\cdot - k) = \sum_{k\in\mathbb{Z}^s} k^\mu [f(\cdot - k) - \nabla_{e_1}^m h^1(\cdot - k)] = 0 \qquad \forall\, |\mu|<m.$$

With f^1, we perform the same procedure with respect to the second dimension variable and continue the process inductively. Consequently, we obtain the required functions $f^1,\dots,f^s$ and $h^1,\dots,h^s$. This completes the proof. ∎

Before stating the major theorem of this section, we introduce the following: For $\Psi := \{\psi^1,\dots,\psi^r\} \in L_2(\mathbb{R}^s)$, we say that the wavelet system $X(\Psi)$ is a *Bessel sequence* in $L_2(\mathbb{R}^s)$ if there exists a positive constant C such that

$$\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^s} |\langle f, \psi_{j,k}^\ell\rangle|^2 \leqslant C\|f\|^2 \qquad \forall\, f\in L_2(\mathbb{R}^s),$$

where $\psi_{j,k}^\ell := 2^{js/2}\psi^\ell(2^j\cdot - k)$ and $\langle f,g\rangle := \int_{\mathbb{R}^s} f(t)\overline{g(t)}\,dt$ for $f,g\in L_2(\mathbb{R}^s)$.

For $\Psi := \{\psi^1,\dots,\psi^{2^s-1}\}$ and $\tilde\Psi := \{\tilde\psi^1,\dots,\tilde\psi^{2^s-1}\}$, we say that $(X(\Psi),X(\tilde\Psi))$ forms a pair of *biorthogonal wavelet bases* in $L_2(\mathbb{R}^s)$ if

(1) Each of $X(\Psi)$ and $X(\tilde\Psi)$ is a Bessel sequence in $L_2(\mathbb{R}^s)$,
(2) Both linear spans of $X(\Psi)$ and $X(\tilde\Psi)$ are dense in $L_2(\mathbb{R}^s)$,
(3) The biorthogonal relation holds:

$$\langle \psi_{j,k}^\ell, \tilde\psi_{j',k'}^{\ell'}\rangle = \delta(\ell-\ell')\delta(j-j')\delta(k-k'), \quad \forall\, \ell,\ell' = 1,\dots,2^s-1, \quad j,j'\in\mathbb{Z}, \quad k,k'\in\mathbb{Z}^s.$$

It is well known that the above three conditions for $(X(\Psi),X(\tilde\Psi))$ imply that each of $X(\Psi)$ and $X(\tilde\Psi)$ is a Riesz basis of $L_2(\mathbb{R}^s)$; hence the pair $(X(\Psi),X(\tilde\Psi))$ forms a biorthogonal Riesz basis of $L_2(\mathbb{R}^s)$.

For two given systems $X(\Psi)$ and $X(\tilde\Psi)$, assuming that they satisfy the biorthogonal relation in (3.10), the proof of the Riesz property of both systems is reduced to the proof of their Bessel properties. This is the approach taken by [5, 25]. Biorthogonal Riesz wavelets have been investigated in [5, 25] under the assumption that the mask a and all the wavelet filters $b^1,\dots,b^{2^s-1}$ having exponential decay. Using the Fredholm determinant theory, [5] presented a method for computing the L_2 smoothness of a refinable function with a mask of exponential decay. By an argument in [7] involving Jackson and Bernstein inequalities in approximation theory, it has been showed in [25] that the Bessel property of $X(\tilde\Psi)$ in (iv) of Theorem 3.2 holds if all the sequences involved have exponential decay and if $\nu_2(\mathring{a}) > 0$ for some finitely supported mask $\mathring{a}$

satisfying $|\hat{\tilde{a}}(\xi)| \leqslant |\hat{\mathring{a}}(\xi)| \forall \xi \in \mathbb{R}^s$ and $\hat{\mathring{a}}(0) = 1$. With the biorthogonal assumption in (3.10), together with the conditions $\hat{\phi}(0) = \hat{\tilde{\phi}}(0) = 1$, the Bessel properties imply the Riesz properties. When applying results in both papers to prove the Riesz properties for the systems give here, we need to check whether the biorthogonal relation (3.10) holds. This requires the convergence of the corresponding cascade algorithms for non-compactly supported masks. It turns out that this requires a stronger condition $\nu_2(\mathring{a}) > s/2$ (than the smoothness condition needed for Bessel properties) to make the proof work. In fact, we will show that this smoothness condition implies the biorthogonal relation in (3.10) for mask a and wavelet filters $b^1, \ldots, b^{2^s-1}$ having polynomial decay (or even weaker decay condition). This condition also guarantees that the function $\hat{\tilde{\phi}}$ has the desired decay in (3.22), which yields the Bessel property of $X(\tilde{\Psi})$ in (iv) of Theorem 3.2.

For functions $f, g \in L_2(\mathbb{R}^s)$, the *bracket product* $[f, g]$ is defined to be (see [19])

$$[f, g](\xi) := \sum_{k \in \mathbb{Z}^s} f(\xi + 2\pi k)\overline{g(\xi + 2\pi k)}, \qquad \xi \in \mathbb{R}^s.$$

For functions ϕ and $\tilde{\phi}$ in $L_2(\mathbb{R}^s)$, it is well-known that $\langle \phi, \tilde{\phi}(\cdot - k)\rangle = \delta(k)$ for all $k \in \mathbb{Z}^s$ if and only if $[\hat{\phi}, \hat{\tilde{\phi}}] = 1$. Moreover, the shifts of ϕ are stable if and only if $[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R}^s)$ and $[\hat{\phi}, \hat{\phi}]^{-1} \in L_\infty(\mathbb{R}^s)$ (see [19]).

Theorem 3.2. *Let a and $b^1, \ldots, b^{2^s-1}$ be sequences on $\mathbb{Z}^s$ with polynomial decay satisfying $\hat{a}(0) = 1$ and $\widehat{b^1}(0) = \cdots = \widehat{b^{2^s-1}}(0) = 0$. Assume that*

(i) *$\det M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^s$, where the matrix $M_{[a,b^1,\ldots,b^{2^s-1}]}$ is defined in (2.7).*

Let $[\hat{\tilde{a}}(\xi), \widehat{\tilde{b}^1}(\xi), \ldots, \widehat{\tilde{b}^{2^s-1}}(\xi)]$ be the first row of the matrix $\overline{M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi)}^{-1}$. Define

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi) \quad \text{and} \quad \hat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \hat{\tilde{a}}(2^{-j}\xi), \qquad \xi \in \mathbb{R}^s$$

and

$$\widehat{\psi^\ell}(\xi) := \widehat{b^\ell}(\xi/2)\hat{\phi}(\xi/2), \qquad \widehat{\tilde{\psi}^\ell}(\xi) := \widehat{\tilde{b}^\ell}(\xi/2)\hat{\tilde{\phi}}(\xi/2), \qquad \ell = 1, \ldots, 2^s - 1. \tag{3.9}$$

Assume that

(ii) *$\nu_2(a) > 0$ and $\nu_2(\tilde{a}) > 0$.*

Then all the functions $\phi, \tilde{\phi}$, $\Psi := \{\psi^1, \ldots, \psi^{2^s-1}\}$ and $\tilde{\Psi} := \{\tilde{\psi}^1, \ldots, \tilde{\psi}^{2^s-1}\}$ belong to $L_2(\mathbb{R}^s)$ and satisfy

$$\begin{aligned} &\langle \phi, \tilde{\phi}(\cdot - k)\rangle = \delta(k), \qquad \langle \psi^\ell, \tilde{\psi}^{\ell'}(\cdot - k)\rangle = \delta(k)\delta(\ell - \ell'), \\ &\langle \phi, \tilde{\psi}^\ell(\cdot - k)\rangle = 0, \qquad \langle \psi^\ell, \tilde{\phi}(\cdot - k)\rangle = 0 \qquad \forall\, k \in \mathbb{Z}^s,\ \ell, \ell' = 1, \ldots, 2^s - 1. \end{aligned} \tag{3.10}$$

If we further assume that

(iii) *$[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R}^s)$ and $[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}] \in L_\infty(\mathbb{R}^s)$,*

(iv) *$X(\Psi)$ and $X(\tilde{\Psi})$ are Bessel sequences in $L_2(\mathbb{R}^s)$,*

then $(X(\Psi), X(\tilde{\Psi}))$ forms a pair of biorthogonal wavelet bases in $L_2(\mathbb{R}^s)$. In particular, $X(\Psi)$ is a Riesz basis of $L_2(\mathbb{R}^s)$.

Proof. By the definition of $\tilde{a}$ and $\tilde{b}^1, \ldots, \tilde{b}^{2^s-1}$, it is easy to check that

$$M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi)\overline{M_{[\tilde{a},\tilde{b}^1,\ldots,\tilde{b}^{2^s-1}]}(\xi)}^T = I_{2^s}, \qquad \xi \in \mathbb{R}^s. \tag{3.11}$$

Since $\hat{a}(0) = 1$ and $\widehat{b^1}(0) = \cdots = \widehat{b^{2^s-1}}(0) = 0$, it follows from (3.11) that $\hat{\tilde{a}}(0) = 1$ and $\widehat{\tilde{b}^1}(0) = \cdots = \widehat{\tilde{b}^{2^s-1}}(0) = 0$. Furthermore, all $\tilde{a}, \tilde{b}^1, \ldots, \tilde{b}^{2^s-1}$ have polynomial decay, because $\det M_{[a,b^1,\ldots,b^{2^s-1}]}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^s$ and all $a, b^1, \ldots, b^{2^s-1}$ have polynomial decay. It follows from (3.11) that

$$\sum_{\gamma\in\Gamma} \hat{a}(\xi + \pi\gamma)\overline{\hat{\tilde{a}}(\xi + \pi\gamma)} = 1, \tag{3.12}$$

where $\Gamma := [0, 1]^s \cap \mathbb{Z}^s$. Suppose that a satisfies the sum rules of order m but not $m + 1$, and $\tilde{a}$ satisfies the sum rules of order $\tilde{m}$ but not $\tilde{m} + 1$. Since $\nu_2(a) > 0$ and $\nu_2(\tilde{a}) > 0$, by [15, Proposition 4.1], we must have $m > 0$ and $\tilde{m} > 0$.

The ideas of the following proof are similar to those in [16]. First, we prove that cascade algorithms associated with both masks a and $\tilde{a}$ converge in the space $L_2(\mathbb{R}^s)$. The convergence of the cascade algorithms, together with (3.12) and (iii), leads to that both refinable functions ϕ and $\tilde{\phi}$ are stable and their shifts form biorthogonal systems. Then, a standard argument on biorthogonal wavelets by using (3.11), one obtains that ϕ, Ψ, $\tilde{\phi}$ and $\tilde{\Psi}$ satisfy (3.10). (See e.g. [28] and [17]). Note that $\hat{\phi}(0) = \hat{\tilde{\phi}}(0) = 1$. By a standard argument on multiresolution analysis ([1]), wee see that both linear spans of $X(\Psi)$ and $X(\tilde{\Psi})$ are dense in $L_2(\mathbb{R}^s)$. Finally, if $X(\Psi)$ and $X(\tilde{\Psi})$ are Bessel, then they form a pair of biorthogonal wavelet bases of $L_2(\mathbb{R}^s)$. (See [6, 9, 17, 28]).

It remains to demonstrate that the cascade algorithms associated with masks a and $\tilde{a}$ converge in $L_2(\mathbb{R}^s)$. This is done by employing several techniques from [15]. Though the following proof looks similar to that in [16] for the univariate case, however, the proof there relies on the factorization of the masks which we cannot use here. For the sake of completeness, we present a proof without using factorization of the Fourier series of the masks.

The cascade algorithms associated with masks a and $\tilde{a}$ are defined iteratively via the cascade operators

$$Q_a f := 2^s \sum_{k\in\mathbb{Z}^s} a(k)f(2\cdot -k) \quad \text{and} \quad Q_{\tilde{a}} f := 2^s \sum_{k\in\mathbb{Z}^s} \tilde{a}(k)f(2\cdot -k), \qquad f \in L_2(\mathbb{R}^s).$$

The initial functions ϕ_0 and $\tilde{\phi}_0$ are chosen as follows: Let η be a tensor product Daubechies orthonormal refinable function in $C^{\max(m,\tilde{m})}(\mathbb{R}^s)$ such that

$$\hat{\eta}(0) = 1, \quad \partial^\mu \hat{\eta}(2\pi k) = 0 \quad \forall\, k \in \mathbb{Z}^s\backslash\{0\} \quad \text{and} \quad |\mu| < \max(m, \tilde{m}). \tag{3.13}$$

Since η is an orthonormal refinable function, we must have $\partial^\mu(\hat{\eta}\overline{\hat{\eta}})(0) = \delta(\mu)$ for all $|\mu| < \max(m, \tilde{m})$. Note that $\hat{a}(0) = \hat{\eta}(0) = 1$. By [15, Lemmas 2.2 and 3.4], there exists a finitely supported sequence c on $\mathbb{Z}^s$ such that

$$\hat{c}(0) = 1, \qquad 2^{-|\mu|}\partial^\mu[\hat{\tilde{a}}\hat{c}\hat{\eta}](0) = \partial^\mu[\hat{c}\hat{\eta}](0) \qquad \forall\, |\mu| < \max(m, \tilde{m}) \tag{3.14}$$

and $\hat{c}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^s$. Now we define

$$\widehat{\phi_0}(\xi) := \hat{\eta}(\xi)/\overline{\hat{c}(\xi)} \quad \text{and} \quad \widehat{\tilde{\phi}_0}(\xi) := \hat{c}(\xi)\hat{\eta}(\xi).$$

Since the sequence c is finitely supported and $\hat{c}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^s$, by the fact that η is compactly supported and continuous, we see that both ϕ_0 and $\tilde{\phi}_0$ have polynomial decay and in particular, $\phi_0, \tilde{\phi}_0 \in L_2(\mathbb{R}^s)$. It is straightforward to check that $[\widehat{\phi_0}, \widehat{\tilde{\phi}_0}] = [\hat{\eta}/\overline{\hat{c}}, \hat{\eta}\hat{c}] = [\hat{\eta}, \hat{\eta}] = 1$.

Let $g := Q_a\phi_0 - \phi_0$ and $\tilde{g} := Q_{\tilde{a}}\tilde{\phi}_0 - \tilde{\phi}_0$. Then their Fourier transforms are given by

$$\hat{g}(\xi) = \hat{a}(\xi/2)\hat{\eta}(\xi/2)/\overline{\hat{c}(\xi/2)} - \hat{\eta}(\xi)/\overline{\hat{c}(\xi)} \quad \text{and} \quad \hat{\tilde{g}}(\xi) = \hat{\tilde{a}}(\xi/2)\hat{c}(\xi/2)\hat{\eta}(\xi/2) - \hat{c}(\xi)\hat{\eta}(\xi). \tag{3.15}$$

Note that the function η is compactly supported and the sequences a and $\tilde{a}$ have polynomial decay. It follows from the above identities that both g and $\tilde{g}$ have polynomial decay. In the following, we show that

$$\partial^\mu \hat{g}(2\pi k) = 0 \qquad \forall\, |\mu| < m \quad \text{and} \quad k \in \mathbb{Z}^s \tag{3.16}$$

and

$$\partial^\mu \hat{\tilde{g}}(2\pi k) = 0 \qquad \forall\, |\mu| < \tilde{m} \quad \text{and} \quad k \in \mathbb{Z}^s. \tag{3.17}$$

Clearly, by (3.13) we see that $\partial^\mu \hat{g}(2\pi k) = \partial^\mu \hat{\tilde{g}}(2\pi k) = 0$ for all $|\mu| < \max(m, \tilde{m})$ and $k \in 2\mathbb{Z}^s \backslash \{0\}$. Since a and $\tilde{a}$ satisfy the sum rules of orders m and $\tilde{m}$, respectively, for every $k \in \mathbb{Z}^s \backslash [2\mathbb{Z}^s]$, we have

$$\partial^\mu \hat{a}(\pi k) = 0 \qquad \forall\, |\mu| < m \quad \text{and} \quad \partial^\nu \hat{\tilde{a}}(\pi k) = 0 \qquad \forall\, |\nu| < \tilde{m}.$$

Thus, for every $k \in \mathbb{Z}^s \backslash [2\mathbb{Z}^s]$, it follows from the above identities that $\partial^\mu \hat{g}(2\pi k) = 0$ for all $|\mu| < m$ and $\partial^\nu \hat{\tilde{g}}(2\pi k) = 0$ for all $|\nu| < \tilde{m}$.

In order to prove (3.16) and (3.17), it suffices to prove the case $k = 0$. By a direct calculation, it follows from (3.14) and (3.15) that for every $|\nu| < \max(m, \tilde{m})$,

$$\partial^\nu \hat{\tilde{g}}(0) = \partial^\nu [\hat{\tilde{a}}(\cdot/2)\hat{c}(\cdot/2)\hat{\eta}(\cdot/2) - \hat{c}\hat{\eta}](0) = 2^{-|\nu|}\partial^\nu [\hat{\tilde{a}}\hat{c}\hat{\eta}](0) - \partial^\nu [\hat{c}\hat{\eta}](0) = 0.$$

Therefore, (3.17) holds. On the other hand, it follows from (3.12) that

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) = 1 - \sum_{\gamma \in \Gamma \backslash \{0\}} \overline{\hat{a}(\xi + \pi\gamma)}\hat{\tilde{a}}(\xi + \pi\gamma),$$

that is,

$$\hat{a}(\xi) = [\overline{\hat{\tilde{a}}(\xi)}]^{-1} - [\overline{\hat{\tilde{a}}(\xi)}]^{-1} \sum_{\gamma \in \Gamma \backslash \{0\}} \hat{a}(\xi + \pi\gamma)\overline{\hat{\tilde{a}}(\xi + \pi\gamma)}.$$

Since a and $\tilde{a}$ satisfy the sum rules of order m and $\tilde{m}$, we must have $\partial^\mu [\overline{\hat{a}(\cdot + \pi\gamma)}\hat{\tilde{a}}(\cdot + \pi\gamma)](0) = 0$ for all $|\mu| < m + \tilde{m}$ and $\gamma \in \Gamma \backslash \{0\}$. Consequently, $\partial^\mu \hat{a}(0) = \partial^\mu [1/\overline{\hat{\tilde{a}}}](0)$ for all $|\mu| < m + \tilde{m}$. Since $\partial^\mu [\hat{\eta}\overline{\hat{\eta}}](0) = \delta(\mu)$ for all $|\mu| < \max(m, \tilde{m})$, we have $\partial^\mu \hat{\eta}(0) = \partial^\mu [1/\overline{\hat{\eta}}](0)$ for all $|\mu| < \max(m, \tilde{m})$. By Leibniz differentiation formula, it follows from (3.15) that for every $|\mu| < \max(m, \tilde{m})$,

$$\begin{aligned}
\partial^\mu \hat{g}(0) &= \partial^\mu \left[\frac{\hat{a}(\cdot/2)\hat{\eta}(\cdot/2)}{\overline{\hat{c}(\cdot/2)}} - \frac{\hat{\eta}}{\overline{\hat{c}}} \right](0) = \partial^\mu \left[\frac{1}{\overline{\hat{\tilde{a}}(\cdot/2)}\,\overline{\hat{\eta}(\cdot/2)}\,\overline{\hat{c}(\cdot/2)}} - \frac{1}{\overline{\hat{\eta}}\,\overline{\hat{c}}} \right](0) \\
&= \partial^\mu \overline{\left[\frac{\hat{\eta}\hat{c} - \hat{\tilde{a}}(\cdot/2)\hat{c}(\cdot/2)\hat{\eta}(\cdot/2)}{\hat{\tilde{a}}(\cdot/2)\hat{\eta}(\cdot/2)\hat{c}(\cdot/2)\hat{\eta}\hat{c}} \right]}(0) = 0,
\end{aligned}$$

where we used (3.14) in the last identity. Therefore, we conclude that (3.16) holds. Now by Lemma 3.1, there exist functions $h_\mu, |\mu| = m$ and $\tilde{h}_\nu, |\nu| = \tilde{m}$ with polynomial decay such that

$$g = \sum_{|\mu| = m} \nabla^\mu h_\mu \quad \text{and} \quad \tilde{g} = \sum_{|\nu| = \tilde{m}} \nabla^\nu \tilde{h}_\nu.$$

Consider $f_n = Q_a^n \phi_0$ and $\tilde{f}_n = Q_{\tilde{a}}^n \tilde{\phi}_0$. By induction, we have

$$\begin{aligned}
f_{n+1} - f_n = Q_a^n g &= 2^{sn} \sum_{k \in \mathbb{Z}^s} [S_a^n \delta](k) g(2^n \cdot -k) \\
&= 2^{sn} \sum_{|\mu| = m} \sum_{k \in \mathbb{Z}^s} [S_a^n \delta](k) [\nabla^\mu h_\mu](2^n \cdot -k) \\
&= 2^{sn} \sum_{|\mu| = m} \sum_{k \in \mathbb{Z}^s} [\nabla^\mu S_a^n \delta](k) h_\mu(2^n \cdot -k)
\end{aligned}$$

and similarly,

$$\tilde{f}_{n+1} - \tilde{f}_n = 2^{sn} \sum_{|\nu|=\tilde{m}} \sum_{k\in\mathbb{Z}^s} [\nabla^\nu S^n_{\tilde{a}}\delta](k)\tilde{h}_\nu(2^n \cdot -k).$$

Since both h_μ and $\tilde{h}_\nu$ are functions in $L_2(\mathbb{R}^s)$ with polynomial decay, we must have $[\hat{h}_\mu, \hat{h}_\mu]$ and $[\hat{\tilde{h}}_\nu, \hat{\tilde{h}}_\nu]$ belong to $L_\infty(\mathbb{R}^s)$ for all $|\mu| = m$ and $|\nu| = \tilde{m}$. Therefore, it follows from the above identities that there exists a positive constant C, depending only on $h_\mu, |\mu| = m$ and $\tilde{h}_\nu, |\nu| = \tilde{m}$, such that

$$\|f_{n+1} - f_n\|_{L_2(\mathbb{R}^s)} \leqslant C2^{sn/2} \sum_{|\mu|=m} \|\nabla^\mu S^n_a\delta\|_{\ell_2(\mathbb{Z}^s)} \qquad \forall\, n \in \mathbb{N} \tag{3.18}$$

and

$$\|\tilde{f}_{n+1} - \tilde{f}_n\|_{L_2(\mathbb{R}^s)} \leqslant C2^{sn/2} \sum_{|\nu|=\tilde{m}} \|\nabla^\nu S^n_{\tilde{a}}\delta\|_{\ell_2(\mathbb{Z}^s)} \qquad \forall\, n \in \mathbb{N}. \tag{3.19}$$

Since $\nu_2(a) > 0$ and $\nu_2(\tilde{a}) > 0$, by the definition of $\nu_2(a)$ and $\nu_2(\tilde{a})$, for any ρ such that $2^{-\min(\nu_2(a),\nu_2(\tilde{a}))} < \rho < 1$, there exists a positive constant C_1 such that

$$\|\nabla^\mu S^n_a\delta\|_{\ell_2(\mathbb{Z}^s)} \leqslant C_1\rho^n 2^{-sn/2} \quad \text{and} \quad \|\nabla^\nu S^n_{\tilde{a}}\delta\|_{\ell_2(\mathbb{Z}^s)} \leqslant C_1\rho^n 2^{-sn/2} \qquad \forall\, n \in \mathbb{N},\ |\mu| = m, |\nu| = \tilde{m}.$$

Consequently, by (3.18) and (3.19), we have

$$\|f_{n+1} - f_n\|_{L_2(\mathbb{R}^s)} \leqslant C_2\rho^n \quad \text{and} \quad \|\tilde{f}_{n+1} - \tilde{f}_n\|_{L_2(\mathbb{R}^s)} \leqslant C_2\rho^n \qquad \forall\, n \in \mathbb{N}, \tag{3.20}$$

where

$$C_2 := CC_1 \sum_{|\mu|=\max(m,\tilde{m})} 1 < \infty.$$

Therefore, both $\{f_n\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in $L_2(\mathbb{R}^s)$. Hence, both sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ must converge. Note that

$$\widehat{f_n}(\xi) = \widehat{\phi_0}(2^{-n}\xi) \prod_{j=1}^n \hat{a}(2^{-j}\xi) \quad \text{and} \quad \widehat{\tilde{f}_n}(\xi) = \widehat{\tilde{\phi}_0}(2^{-n}\xi) \prod_{j=1}^n \hat{\tilde{a}}(2^{-j}\xi).$$

Since $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$, by $\hat{a}, \hat{\tilde{a}} \in C^\infty(\mathbb{R}^s)$, we conclude that $\lim_{n\to\infty} \widehat{f_n}(\xi) = \hat{\phi}(\xi)$ and $\lim_{n\to\infty} \widehat{\tilde{f}_n}(\xi) = \hat{\tilde{\phi}}(\xi),\ \xi \in \mathbb{R}^s$. This implies that both sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ converge to ϕ and $\tilde{\phi}$ in a distribution sense respectively, which is weaker than the L_2-norm convergence. Consequently, we must have

$$\lim_{n\to\infty} \|f_n - \phi\|_{L_2(\mathbb{R}^s)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{f}_n - \tilde{\phi}\|_{L_2(\mathbb{R}^s)} = 0, \tag{3.21}$$

since $\{f_n\}$ and $\{\tilde{f}_n\}$ are Cauchy sequences in $L_2(\mathbb{R}^s)$.

Finally, we show that the shifts of ϕ and $\tilde{\phi}$ form biorthogonal systems. It is well-known (see e.g. [6, 9, 17, 28]) that it is equivalent to show $[\hat{\phi}, \hat{\tilde{\phi}}] = 1$. First, since $f_0 = \phi_0$ and $\tilde{f}_0 = \tilde{\phi}_0$, we have $[\hat{f}_0, \hat{\tilde{f}}_0] = [\hat{\phi}_0, \hat{\tilde{\phi}}_0] = 1$. It follows from (3.12) that $[\widehat{f_n}, \widehat{\tilde{f}_n}] = 1$ for all $n \in \mathbb{N}$ by induction. Therefore, it follows from (3.21) that $[\hat{\phi}, \hat{\tilde{\phi}}] = 1$. ∎

For $0 < \alpha \leqslant 1$ and a function $f \in L_p(\mathbb{R}^s)$, we say that f belongs to the *Lipschitz space* $\mathrm{Lip}(\alpha, L_p(\mathbb{R}^s))$ if there exists a positive constant C such that

$$\|f - f(\cdot - t)\|_{L_p(\mathbb{R}^s)} \leqslant C\|t\|^\alpha \qquad \forall\, t \in \mathbb{R}^s.$$

The L_p smoothness of a function $f \in L_p(\mathbb{R}^s)$ is measured by its L_p *critical smoothness exponent* $\nu_p(f)$ which is defined by

$$\nu_p(f) := \sup\{n + \alpha \quad : \quad \partial^\mu f \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}^s)) \quad \forall\, |\mu| = n\}.$$

If $f \in L_1(\mathbb{R}^s) \cap L_2(\mathbb{R}^s)$ is compactly supported, then it is known that $\nu_1(f) \geqslant \nu_2(f) \geqslant \nu_1(f) - s/2$. Furthermore, if a is a finitely supported refinement mask of a compactly supported refinable function ϕ, then $\nu_p(\phi) \geqslant \nu_p(a)$ for all $1 \leqslant p \leqslant \infty$ ([15, Page 69] and references therein).

We now prove Theorem 2.1.

Proof of Theorem 2.1. It follows from Theorem 3.2. Indeed, assuming that conditions (i)-(iii) in Theorem 2.1 hold, we show that conditions (i)-(iv) of Theorem 3.2 hold.

It is clear that (i) of Theorem 3.2 is implied by (i) of Theorem 2.1. We, then, prove that the assertions of (ii) -(iv) in Theorem 3.2 with respect to ϕ and $X(\Psi)$ hold under assumptions of Theorem 2.1. It is clear that assumptions (ii) of Theorem 3.2 follows from (ii) of Theorem 2.1. Since ϕ is compactly supported and all sequences $a, b^1, \dots, b^{2^s-1}$ are finitely supported, and since $\nu_2(a) > 0$ by the assumption of Theorem 2.1, the assertions of (iii) and (iv) with respect to ϕ and $X(\Psi)$ follow directly from [13, Theorems 2.2 and 2.3], i.e., $[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R}^s)$ and $X(\Psi)$ is Bessel in $L_2(\mathbb{R}^s)$.

Next, we prove that the assertions of (ii) -(iv) in Theorem 3.2 with respect to $\tilde{\phi}$ and $X(\tilde{\Psi})$ hold under the assumptions of Theorem 2.1. We prove it under a weaker assumption that $\nu_1(\mathring{a}) > s/2$ instead of $\nu_2(\mathring{a}) > s/2$, since the assumption $\nu_2(\mathring{a}) > s/2$ implies $\nu_1(\mathring{a}) > s/2$ by the fact that $\nu_1(\mathring{a}) \geqslant \nu_2(\mathring{a})$.

Since $\nu_1(\mathring{a}) > s/2$ and $\mathring{a}$ is finitely supported, we deduce that $\nu_2(\mathring{a}) \geqslant \nu_1(\mathring{a}) - s/2 > 0$. Since $\nu_2(\tilde{a}) \geqslant \nu_2(\mathring{a})$, (ii) of Theorem 3.2 holds. In order to complete the proof, we need to show that (iii) and (iv) of Theorem 3.2 hold for $\tilde{\phi}$ and $X(\tilde{\Psi})$, i.e., to show that $[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}] \in L_\infty(\mathbb{R}^s)$ and $X(\tilde{\Psi})$ is Bessel in $L_2(\mathbb{R}^s)$.

Let $\mathring{\phi}$ denote the refinable function associated with mask $\mathring{a}$. Since

$$\hat{\mathring{\phi}}(\xi) = \prod_{j=1}^{\infty} \hat{\mathring{a}}(2^{-j}\xi) \quad \text{and} \quad \hat{\tilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \hat{\tilde{a}}(2^{-j}\xi),$$

it follows from the assumption $|\hat{\tilde{a}}(\xi)| \leqslant |\hat{\mathring{a}}(\xi)|\ \forall\, \xi \in \mathbb{R}^s$ that $|\hat{\tilde{\phi}}(\xi)| \leqslant |\hat{\mathring{\phi}}(\xi)|$ for all $\xi \in \mathbb{R}^s$. Since $\nu_1(\mathring{a}) > s/2$, we must have $\nu_1(\mathring{\phi}) \geqslant \nu_1(\mathring{a}) > s/2$ and so, it is easy to see ([13, Proof of Theorem 2.2]) that for any $s/2 < \rho < \nu_1(\mathring{a})$, there is a constant $C > 0$ such that

$$|\hat{\tilde{\phi}}(\xi)| \leqslant |\hat{\mathring{\phi}}(\xi)| \leqslant C(1 + \|\xi\|)^{-\rho} \qquad \forall\, \xi \in \mathbb{R}^s. \tag{3.22}$$

Since $\rho > s/2$, it is evident that $[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}] \in L_\infty(\mathbb{R}^s)$. Note that $\widehat{b^1}(0) = \cdots = \widehat{b^{2^s-1}}(0) = 0$. By the definition of the wavelet functions $\psi^1, \dots, \psi^{2^s-1}$ in (3.9), there is a constant $C_1 > 0$ such that

$$|\widehat{\tilde{\psi}^\ell}(\xi)| \leqslant C_1 \|\xi\| \quad \text{and} \quad |\widehat{\tilde{\psi}^\ell}(\xi)| \leqslant C_1(1 + \|\xi\|)^{-\rho} \qquad \forall\, \xi \in \mathbb{R}^s,\ \ell = 1, \dots, 2^s - 1.$$

Now by [10, Propositions 2.6 and 3.5], $X(\tilde{\Psi})$ is Bessel in $L_2(\mathbb{R}^s)$. ■

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