COMPACTLY SUPPORTED SYMMETRIC C^∞ WAVELETS WITH SPECTRAL APPROXIMATION ORDER

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Abstract. In this paper, we obtain symmetric C^∞ real-valued tight wavelet frames in $L_2(\mathbb{R})$ with compact support and the spectral frame approximation order. Furthermore, we present a family of symmetric compactly supported C^∞ orthonormal complex wavelets in $L_2(\mathbb{R})$. A complete analysis of nonstationary tight wavelet frames and orthonormal wavelet bases in $L_2(\mathbb{R})$ is given.

1. Introduction

In this paper, we are interested in symmetric compactly supported C^∞ tight wavelet frames with the spectral frame approximation order. Since it is impossible to achieve all these properties under the framework of stationary tight wavelet frames, it is natural for us to consider nonstationary tight wavelet frames, in particular, nonstationary tight wavelet frames derived from nonstationary multiresolution analysis by the new (nonstationary) unitary extension principle.

We start with a family of 2π-periodic trigonometric polynomials $\widehat{a_j}, j \in \mathbb{N}$, and their associated nonstationary refinable functions (or tempered distributions) $\phi_{j-1}, j \in \mathbb{N}$, defined by

$$\widehat{\phi_{j-1}}(\xi) := \widehat{a_j}(\xi/2)\widehat{\phi_j}(\xi/2) = \prod_{n=1}^{\infty} \widehat{a_{n+j-1}}(2^{-n}\xi), \qquad \xi \in \mathbb{R},\ j \in \mathbb{N}, \tag{1.1}$$

where the 2π-periodic trigonometric polynomials $\widehat{a_j}$, $j \in \mathbb{N}$, are called refinement *masks*. Here, the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R})$ used in this paper is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-it\xi}\, dt$ and can be naturally extended to square integrable functions and tempered distributions.

The stationary multiresolution analysis corresponds to the case that all the masks $\widehat{a_j}$ are the same; therefore, all the functions ϕ_j are the same and in particular, $\widehat{\phi_0}(\xi) = \widehat{a_1}(\xi/2)\widehat{\phi_1}(\xi/2)$. We say that a function $\phi : \mathbb{R} \mapsto \mathbb{C}$ is refinable with a 2π-periodic trigonometric polynomial refinement mask $\hat{a}$ if $\hat{\phi}(\xi) = \hat{a}(\xi/2)\hat{\phi}(\xi/2)$. The frame generators ψ^ℓ are generally obtained from the refinable function ϕ via $\widehat{\psi^\ell}(\xi) = \widehat{b^\ell}(\xi/2)\hat{\phi}(\xi/2)$ for some 2π-periodic trigonometric polynomials $\widehat{b^\ell}$ with some desirable properties.

A tight wavelet frame in $L_2(\mathbb{R})$ (in the stationary case) is generated by the integer translates and dyadic dilates of a finite set of elements in $L_2(\mathbb{R})$. More precisely, we say that $\{\psi^1, \ldots, \psi^L\}$ generates a (normalized) *tight wavelet frame* in $L_2(\mathbb{R})$ if

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{j,k}\rangle|^2 \qquad \forall\, f \in L_2(\mathbb{R}), \tag{1.2}$$

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where $\psi^\ell_{j,k} := 2^{j/2}\psi^\ell(2^j\cdot -k)$ and $\langle f,g\rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}\,dt$. As a redundant wavelet system, tight frame wavelet systems are easier to design and provide more flexibilities in applications than orthonormal wavelet bases, especially, in image inpainting (see [1, 2, 3, 4, 5] for details). Because of this, tight wavelet frames have been extensively studied in the literature, to only mention a few here, see [6, 7, 8, 14, 16, 18, 20, 21, 25, 26, 27, 34] and many references therein.

Tight wavelet frames obtained from refinable functions are of particular interest, due to their associated multiresolution structure (hence is called MRA-based) and fast frame algorithms. Constructions of tight wavelet frames from a refinable function can be done by the unitary extension principle (UEP) in [34]. In fact, many tight wavelet frames have been constructed in [6, 16, 34]. Later, by using the more general oblique extension principle (OEP) which is independently developed in [7, 14], more tight wavelet frames with various desirable properties have been obtained in [7, 14, 20, 25, 26, 27] and many other references therein.

For the stationary case, it is already pointed out in [13] that there does not exist a compactly supported refinable function ϕ with a 2π-periodic trigonometric polynomial refinement mask such that ϕ belongs to $C^\infty(\mathbb{R})$. Hence, it is impossible to obtain MRA-based compactly supported (stationary) tight wavelet frames in $L_2(\mathbb{R})$ whose generators are in $C^\infty(\mathbb{R})$. However, it is shown in [10] that by using the class of masks for orthonormal refinable functions of [12] whose integer shifts form an orthonormal system, one can obtain a family of nonstationary refinable functions such that every nonstationary refinable function belongs to $C^\infty(\mathbb{R})$ and its integer shifts still form an orthonormal system in $L_2(\mathbb{R})$. For this family of nonstationary refinable functions, a C^∞ nonstationary orthonormal wavelet basis in $L_2(\mathbb{R})$ is derived in [10]. In fact, ideas of generating a class of nonstationary refinable functions in $C^\infty(\mathbb{R})$ from a given family of masks for stationary refinable functions have already been discussed in [15, 33]. One such example is the *up-function* ([10, 15, 33]) generated from the family of masks for the B-splines. Let $\widehat{a_j}(\xi) = 2^{-j}(1+e^{-i\xi})^j$, $j\in\mathbb{N}$, be the mask for the B-spline of order j and define $\phi_{j-1}, j\in\mathbb{N}$, as in (1.1). Then all $\phi_{j-1}, j\in\mathbb{N}$, are compactly supported C^∞ functions. In particular, the function ϕ_0 is supported on $[0,2]$ (see [10, 15, 33]).

Motivated by the interesting work of Cohen and Dyn [10] and equipped with the pseudo-splines (a more general class of refinable functions containing B-splines, interpolatory refinable functions and Daubechies orthonormal refinable functions in [12] as special cases), together with the idea of unitary extension principle, we establish the analysis needed here for constructing nonstationary $C^\infty(\mathbb{R})$ tight wavelet frames in $L_2(\mathbb{R})$ with desirable properties, especially, the symmetry property, which cannot be achieved by real-valued orthonormal dyadic refinable functions. As we will see, the construction more or less follows the idea of the unitary extension principle for the stationary case, while the main analysis of this paper is somehow different from that of [10]. For example, in the orthonormal wavelet case, the approximation order of the truncated wavelet series in [10] is the same as that of the (nonstationary) multiresolution analysis, while in the tight wavelet frame case, they are different, even for the stationary case, as shown in [14].

Next, we briefly describe ideas of the construction of tight wavelet frames. Although one of our major objectives of this paper is to use the family of refinement masks for pseudo-splines to construct tight wavelet frames and to provide the corresponding analysis, the construction in this paper is given for the general setting.

We start with 2π-periodic measurable functions $\widehat{a_j}, j\in\mathbb{N}$, as a sequence of refinement masks. To make the idea of the unitary extension principle work, it is necessary to require that for every $j\in\mathbb{N}$, the mask $\widehat{a_j}$ should satisfy

$$|\widehat{a_j}(\xi)|^2 + |\widehat{a_j}(\xi+\pi)|^2 \leqslant 1, \qquad a.e.\ \xi\in\mathbb{R}. \tag{1.3}$$

Since we are only interested in compactly supported tight wavelet frames, it is natural to start with compactly supported refinable functions ϕ_j, which, in turn, require that the degrees of the trigonometric polynomials $\widehat{a_j}$ do not increase too fast. For a 2π-periodic trigonometric polynomial $\hat{a}$, we denote $\deg(\hat{a})$ the smallest nonnegative integer such that its Fourier coefficients of $\hat{a}$ vanish outside $[-\deg(\hat{a}), \deg(\hat{a})]$. We note that $\deg(\hat{a})$ defined here is somewhat slightly different from the usual definition of the degree of a trigonometric polynomial; $\deg(\hat{a})$ here is the minimal integer k such that $[-k,k]$ contains the support of the Fourier coefficients of both $\hat{a}$ and $\hat{a}(-\cdot)$. For ϕ_0 in (1.1) to be compactly supported, by a simple calculation, it is very natural to require ([10]) that

$$\sum_{j=1}^{\infty} 2^{-j} \deg(\widehat{a_j}) < \infty. \tag{1.4}$$

With (1.3) and (1.4), under the condition that $\sum_{j=1}^{\infty} |\widehat{a_j}(0)-1| < \infty$, it can be proven that all the corresponding refinable functions ϕ_{j-1} in (1.1) are well-defined compactly supported functions in $L_2(\mathbb{R})$.

Wavelet functions ψ_{j-1}^{ℓ}, $j \in \mathbb{N}$ and $\ell \in \{1,\dots,\mathfrak{J}_j\}$, are obtained from ϕ_j by

$$\widehat{\psi_{j-1}^{\ell}}(\xi) := \widehat{b_j^{\ell}}(\xi/2)\widehat{\phi_j}(\xi/2), \qquad \ell = 1,\dots,\mathfrak{J}_j, \tag{1.5}$$

where $\mathfrak{J}_j$ are positive integers and each $\widehat{b_j^{\ell}}$, $\ell = 1,\dots,\mathfrak{J}_j$, is called a (high-pass) wavelet mask. Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We say that $\{\phi_0\} \cup \{\psi_j^{\ell} \ : \ j \in \mathbb{N}_0, \ell = 1,\dots,\mathfrak{J}_{j+1}\}$ generates *a nonstationary tight wavelet frame* in $L_2(\mathbb{R})$ if

$$\{\phi_0(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi_{j;j,k}^{\ell} := 2^{j/2}\psi_j^{\ell}(2^j \cdot - k) \ : \ j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1,\dots,\mathfrak{J}_{j+1}\} \tag{1.6}$$

is a tight frame of $L_2(\mathbb{R})$, that is, the following holds

$$\|f\|_{L_2(\mathbb{R})}^2 = \sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathfrak{J}_{j+1}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j;j,k}^{\ell}\rangle|^2 \qquad \text{for all} \quad f \in L_2(\mathbb{R}). \tag{1.7}$$

We say that ψ_j^{ℓ} has ν *vanishing moments* if $\widehat{\psi_j^{\ell}}^{(n)}(0) = 0$ for all $n = 0,\dots,\nu-1$, where $\widehat{\psi_j^{\ell}}^{(n)}$ denotes the nth derivative of $\widehat{\psi_j^{\ell}}$. It is clear that (1.7) is equivalent to

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi_0(\cdot - k)\rangle \phi_0(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathfrak{J}_{j+1}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j;j,k}^{\ell}\rangle \psi_{j;j,k}^{\ell}, \qquad f \in L_2(\mathbb{R}). \tag{1.8}$$

The frame approximation operators $Q_n, n \in \mathbb{N}$, associated with the truncation of the tight wavelet frame in (1.6) at level n, are defined to be

$$Q_n(f) := \sum_{k\in\mathbb{Z}} \langle f, \phi_0(\cdot - k)\rangle \phi_0(\cdot - k) + \sum_{j=0}^{n-1}\sum_{\ell=1}^{\mathfrak{J}_{j+1}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j;j,k}^{\ell}\rangle \psi_{j;j,k}^{\ell}, \qquad f \in L_2(\mathbb{R}). \tag{1.9}$$

For $\nu \geqslant 0$, we denote $W_2^{\nu}(\mathbb{R})$ the Sobolev space of all functions $f \in L_2(\mathbb{R})$ such that

$$\|f\|_{W_2^{\nu}(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + |\xi|^{2\nu})|\hat{f}(\xi)|^2 \, d\xi < \infty. \tag{1.10}$$

The Sobolev seminorm $|f|_{W_2^{\nu}(\mathbb{R})}$ is defined to be

$$|f|_{W_2^{\nu}(\mathbb{R})}^2 := \int_{\mathbb{R}} |\xi|^{2\nu}|\hat{f}(\xi)|^2 \, d\xi, \qquad f \in W_2^{\nu}(\mathbb{R}). \tag{1.11}$$

The unitary extension principle provides a sufficient condition on the wavelet masks $\widehat{a_j}$ and $\widehat{b_j^{\ell}}$, $\ell = 1,\dots,\mathfrak{J}_j$, so that, with ψ_{j-1}^{ℓ} defined in (1.5), the wavelet system in (1.6) forms a tight frame in $L_2(\mathbb{R})$. Altogether, we have the following result on nonstationary tight wavelet frames in $L_2(\mathbb{R})$.

Theorem 1.1. *Let $\widehat{a_j}, j \in \mathbb{N}$, be 2π-periodic trigonometric polynomials with $\widehat{a_j}(0) = 1$ for all $j \in \mathbb{N}$. If* (1.3) *and* (1.4) *hold, letting ϕ_j and ψ^ℓ_{j-1}, $j \in \mathbb{N}$ and $\ell \in \{1, \ldots, \mathfrak{J}_j\}$, be defined in* (1.1) *and* (1.5)*, respectively, then*

(i) *All functions ϕ_{j-1}, $j \in \mathbb{N}$, are well-defined compactly supported functions in $L_2(\mathbb{R})$.*

(ii) *If $\widehat{b^\ell_j}$, $j \in \mathbb{N}$ and $\ell \in \{1, \ldots, \mathfrak{J}_j\}$, are 2π-periodic trigonometric polynomials satisfying*

$$|\widehat{a_j}(\xi)|^2 + \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b^\ell_j}(\xi)|^2 = 1 \quad \text{and} \quad \widehat{a_j}(\xi)\overline{\widehat{a_j}(\xi+\pi)} + \sum_{\ell=1}^{\mathfrak{J}_j} \widehat{b^\ell_j}(\xi)\overline{\widehat{b^\ell_j}(\xi+\pi)} = 0, \tag{1.12}$$

then the wavelet system in (1.6) *is a compactly supported tight wavelet frame in $L_2(\mathbb{R})$.*

(iii) *If, in addition to* (1.12)*, we assume that*

$$\deg(\widehat{a_j}) = O(j^\alpha 2^{\beta j}) \qquad \text{as} \quad j \to \infty \quad \text{for some} \quad \alpha \geqslant 0, 0 \leqslant \beta < 1 \tag{1.13}$$

and assume that there exist a positive number $\nu \in \frac{1}{2}\mathbb{N}$ and a positive integer N such that $1 - |\widehat{a_j}(\xi)|^2$ has a zero of order 2ν at $\xi = 0$ for all $j \geqslant N$, that is, for $j \geqslant N$,

$$|\widehat{a_j}(\xi)|^2 = 1 + O(|\xi|^{2\nu}), \qquad \xi \to 0, \tag{1.14}$$

then there exists a positive constant C, independent of f and n, such that

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant C n^{\nu\alpha} 2^{-\nu(1-\beta)n} |f|_{W_2^\nu(\mathbb{R})} \qquad \forall\, f \in W_2^\nu(\mathbb{R}) \quad \text{and} \quad n \geqslant N, \tag{1.15}$$

where the linear operators Q_n are defined in (1.9)*.*

Item (ii) of Theorem 1.1 is called the unitary extension principle for the nonstationary case. Theorem 1.1 will be proven in Section 4. As we shall see in Section 4, the main effort there is to prove Items (i) and (iii) of Theorem 1.1. To show Item (ii) of Theorem 1.1, one needs to show the convergence of the frame series in the right side of (1.8) to the function f in $L_2(\mathbb{R})$. When (1.13) holds, the convergence of the frame series follows from (iii) by observing that the masks in Item (ii) satisfy (1.14) for $\nu = 1/2$. Furthermore, a refined analysis establishes the convergence of the frame series even without assuming (1.13).

We further remark that (1.12) guarantees the multiresolution frame decomposition algorithm whose proof can be straightforwardly verified and is more or less known. In fact, Theorem 1.1 generalizes the unitary extension principle from the stationary case in [34] to the general nonstationary case. It is clear that, similar to the stationary case, for every fixed $j \in \mathbb{N}$, in order to construct a set of 2π-periodic trigonometric polynomials $\widehat{b^\ell_j}$, $\ell = 1, \ldots, \mathfrak{J}_j$, derived from the mask $\widehat{a_j}$ so that (1.12) is satisfied, the mask $\widehat{a_j}$ must satisfy (1.3). Hence, (1.3) is a necessary and sufficient condition to make (1.12) hold, as we shall see later in this section.

Nonstationary spline tight wavelet frames using the oblique extension principle (OEP) developed in [7, 14] have been systematically studied in Chui, He and Stöckler [8] recently. There, they considered even more general nonstationary setting, i.e., it is not even shift-invariant at each level. Since the oblique extension principle is a generalization of the unitary extension principle, the proof of [8] might be modified to prove Item (ii) of Theorem 1.1. However, this at most leads to the conclusion $Q_n(f) \to f$ in $L_2(\mathbb{R})$. Our approach of Item (ii) is beyond the proof of the tight frame property itself in (1.8). Instead, we analyze the approximation power of the truncated tight wavelet frame series as stated in Item (iii) of Theorem 1.1. As a consequence of this analysis, we obtain the tight frame property stated in Item (ii) of Theorem 1.1. Finally, we remark that a systematic study of general nonstationary wavelet frames that may not be MRA-based was given in [35].

Following [14], we say that a tight wavelet frame $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, \ldots, \mathfrak{J}_{j+1}\}$ provides *frame approximation order ν* if there exist a positive constant C, independent of f and n, and a positive integer N such that

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant C 2^{-\nu n} |f|_{W_2^\nu(\mathbb{R})} \qquad \forall\, f \in W_2^\nu(\mathbb{R}) \quad \text{and} \quad n \geqslant N. \tag{1.16}$$

We say that a tight wavelet frame provides *the spectral frame approximation order* if it provides frame approximation order ν for any positive integer ν. Here, we point out that for the frame approximation order discussed in this paper, the constant C in (1.15) of Theorem 1.1 and the constant C in (1.16) of Theorems 1.2 and 1.4 can be explicitly obtained.

In Section 4, we shall study when $Q_n(f)$ approaches f with an approximation order ν, as $n \to \infty$. As a consequence, we prove Item (ii) of Theorem 1.1 and (1.8) by showing that $Q_n(f) \to f$ in $L_2(\mathbb{R})$ as $n \to \infty$ for every $f \in L_2(\mathbb{R})$, provided that the conditions in Theorem 1.1 are satisfied. The approximation order of $Q_n(f)$ was not studied in [10], since it was not needed there. In fact, since only orthonormal wavelet systems were considered in [10], the associated operators Q_n become orthogonal projections and attain the approximation order provided by the nonstationary multiresolution analysis. Therefore, one only needs to understand the conditions under which $Q_n(f) \to f$ in $L_2(\mathbb{R})$ as $n \to \infty$ in the orthonormal wavelet case. Nevertheless, our approach here applies to this special case as well and simplifies the conditions given in [10]. The approximation order of $Q_n(f)$ was not studied in [8] either, since it is a more challenging problem in its more general setting of [8]. For the stationary case, it is evident that (1.13) holds with $\alpha = \beta = 0$ and consequently, the notion of the frame approximation power in (1.15) agrees with that of the frame approximation order in (1.16). However, we shall present an example of nonstationary tight wavelet frames derived from the up-function (see Theorem 1.3) to demonstrate that (1.15) holds with $\nu = 2$, $\alpha = 1$ and $\beta = 0$, while (1.16) fails for any $\nu > 0$, that is to say, this particular nonstationary tight wavelet frame has a "weak" frame approximation order 2 in the sense of (1.15) but it does not have any "strong" frame approximation order in the sense of (1.16).

Finally, we note that the 2π-periodic trigonometric polynomial wavelet masks $\widehat{b_j^\ell}$, $j \in \mathbb{N}$ and $\ell \in \{1, \dots, \mathfrak{J}_j\}$, can be constructed from the masks $\widehat{a_j}$ by many ways provided that the refinement masks $\widehat{a_j}, j \in \mathbb{N}$, satisfy (1.3). Here is one of such constructions modified from the stationary case of [6] (also c.f. [16, 25, 26]). For every $j \in \mathbb{N}$, from the mask $\widehat{a_j}$ with real coefficients and satisfying (1.3), define

$$\begin{aligned} \widehat{b_j^1}(\xi) &:= e^{-i\xi}\overline{\widehat{a_j}(\xi+\pi)}, \\ \widehat{b_j^2}(\xi) &:= 2^{-1}[A_j(\xi) + e^{-i\xi}\overline{A_j(\xi)}], \\ \widehat{b_j^3}(\xi) &:= 2^{-1}[A_j(\xi) - e^{-i\xi}\overline{A_j(\xi)}], \end{aligned} \tag{1.17}$$

where A_j is a π-periodic trigonometric polynomial with real coefficients such that

$$|A_j(\xi)|^2 = 1 - |\widehat{a_j}(\xi)|^2 - |\widehat{a_j}(\xi+\pi)|^2.$$

Then, $\widehat{a_j}, \widehat{b_j^1}, \widehat{b_j^2}$ and $\widehat{b_j^3}$, $j \in \mathbb{N}$, satisfy (1.12) with $\mathfrak{J}_j = 3$. Furthermore, the corresponding wavelets defined by (1.5) using masks in (1.17) are symmetric or antisymmetric whenever ϕ_j is symmetric.

After establishing Theorem 1.1, we focus on constructing nonstationary $C^\infty(\mathbb{R})$ wavelets derived from a family of refinement masks for pseudo-splines. Pseudo-splines (of type I) were first introduced in [14] and [37] to improve the approximation order of truncated tight wavelet frame series for the tight wavelet frame system obtained by the unitary extension principle. The pseudo-splines in [14] are generally not symmetric. The pseudo-splines of type II are symmetric and were introduced in [16]. Since we are aiming at constructing symmetric tight wavelet frames, we will use pseudo-splines of type II. For positive integers $m, l \in \mathbb{N}$, throughout the paper we denote

$$P_{m,l}(x) := \sum_{j=0}^{l-1} \binom{m+j-1}{j} x^j = \sum_{j=0}^{l-1} \frac{(m+j-1)!}{j!(m-1)!} x^j, \qquad x \in \mathbb{R}. \tag{1.18}$$

The masks for pseudo-splines of type II with order (m, l) ([16]) are given by

$$\widehat{a_{m,l}}(\xi) := \cos^{2m}(\xi/2) P_{m,l}(\sin^2(\xi/2)), \qquad m \in \mathbb{N},\ l = 1, \dots, m. \tag{1.19}$$

Since it is evident that $\widehat{a_{m,l}}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$, the mask $\widehat{a^I_{m,l}}$, for the pseudo-spline of type I with order (m,l) introduced in [14] and [37], is obtained by taking square root of the mask $\widehat{a_{m,l}}$ in (1.19) for the pseudo-spline of type II with order (m,l) using the Fejér-Riesz lemma such that

$$|\widehat{a^I_{m,l}}(\xi)|^2 = \widehat{a_{m,l}}(\xi), \qquad \xi \in \mathbb{R}. \tag{1.20}$$

While the pseudo-splines of type II and their masks in (1.19) are symmetric, their type I counterparts usually do not have symmetry. For the case $l=1$, the corresponding refinable pseudo-splines are B-splines for both types. For the case $l=m$, the corresponding refinable pseudo-spline ϕ of type I with mask $\widehat{a^I_{m,m}}$ in (1.20) has orthonormal integer shifts (i.e., $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L_2(\mathbb{R})$), and the corresponding refinable pseudo-spline ϕ of type II with mask $\widehat{a_{m,m}}$ in (1.19) is interpolatory (i.e., $\phi(0)=1$ and $\phi(k)=0$ for all $k \in \mathbb{Z}\backslash\{0\}$). It is easy to verify that the condition in (1.3) is satisfied for all the masks for pseudo-splines of type I and type II (e.g., see [14, 16]).

Our construction here employs masks $\widehat{a_{m,l}}$ in (1.19) for pseudo-splines of type II, since we are interested in constructing symmetric tight wavelet frames. We have the following result on symmetric C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with compact support and the spectral frame approximation order.

Theorem 1.2. *Let $\widehat{a_j} := \widehat{a_{m_j,l_j}}$ be defined in (1.19), where $1 \leqslant l_j \leqslant m_j$ and m_j ($j \in \mathbb{N}$) are positive integers satisfying*

$$\lim_{j\to\infty} m_j = \infty \quad \text{and} \quad \sum_{j=1}^{\infty} 2^{-j} m_j < \infty. \tag{1.21}$$

For $j \in \mathbb{N}$, define ϕ_{j-1} as in (1.1) and ψ^1_{j-1}, ψ^2_{j-1}, and ψ^3_{j-1} as in (1.5) with the wavelet masks $\widehat{b^1_j}$, $\widehat{b^2_j}$ and $\widehat{b^3_j}$ being derived from $\widehat{a_j}$ in (1.17). Then

1. *Each nonstationary refinable function ϕ_j, $j \in \mathbb{N}_0$, is a compactly supported C^∞ real-valued function that is symmetric about the origin: $\phi_j(-\cdot) = \phi_j$.*
2. *Each wavelet function ψ^ℓ_j, $\ell = 1,2,3$ and $j \in \mathbb{N}_0$, is a compactly supported C^∞ function with l_{j+1} vanishing moments and satisfies $\psi^\ell_j(1-\cdot) = \psi^\ell_j$ for $\ell = 1,2$ and $\psi^3_j(1-\cdot) = -\psi^3_j$.*
3. *The system $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j,k} := 2^{j/2}\psi^\ell_j(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1,2,3\}$ is a compactly supported symmetric C^∞ tight wavelet frame in $L_2(\mathbb{R})$.*
4. *If in addition $\liminf_{j\to\infty} l_j/m_j > 0$, then the tight wavelet frame in Item (3) has the spectral frame approximation order.*

The simplest choice in Theorem 1.2 is $m_j = l_j = j$ for all $j \in \mathbb{N}$, for which the condition in (1.21) is evidently satisfied and $\liminf_{j\to\infty} l_j/m_j = 1 > 0$. Therefore, by Theorem 1.2, we have a symmetric C^∞ tight wavelet frame in $L_2(\mathbb{R})$ with compact support and the spectral frame approximation order. Of course, the claims in Theorem 1.2 also hold if one chooses $m \approx \rho_1 j$ and $l_j \approx \rho_2 j$ for all $j \in \mathbb{N}$ with some fixed positive numbers ρ_1 and ρ_2. In order to have refinable functions ϕ_j, $j \in \mathbb{N}$, in (1.1) with as small as possible support, one should choose a sequence $\{m_j\}_{j=1}^\infty$ so that m_j goes to ∞ as slow as possible. This is one of our motivations to choose a general integer m_j instead of the standard choice $m_j = j$ for our setup. We point out that such a strategy has already been considered by Cohen [9]. We also mention that all the claims in Theorem 1.2 hold, except possibly for the symmetry property, if one chooses the masks $\widehat{a_j} := \widehat{a^I_{m_j,l_j}}$ in (1.20) for the pseudo-splines of type I instead of type II in Theorem 1.2.

It is clear that the frame approximation order in (1.16) implies (1.15). For the stationary case, it is evident that (1.13) holds with $\alpha = \beta = 0$ and consequently, the notion of the frame approximation power in (1.15) agrees with that of the frame approximation order in (1.16). However, as illustrated by the following result, they could be quite different in the case of nonstationary tight wavelet frames.

Theorem 1.3. *Let $\widehat{a_j}(\xi) := 2^{-j}(1+e^{-i\xi})^j$, $j \in \mathbb{N}$, be the masks for the up-function, in other words, we take $m_j := j$ and $l_j := 1$ in Theorem 1.2. For $j \in \mathbb{N}$, define ϕ_{j-1} as in (1.1) and ψ^1_{j-1}, ψ^2_{j-1}, and ψ^3_{j-1} as in (1.5) with the wavelet masks $\widehat{b^1_j}$, $\widehat{b^2_j}$ and $\widehat{b^3_j}$ being derived from $\widehat{a_j}$ in (1.17). Then*

(i) $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1,2,3\}$ *is a compactly supported symmetric C^∞ tight wavelet frame in $L_2(\mathbb{R})$ and each ψ_j has one vanishing moment.*

(ii) *There exists a positive constant C, independent of f and n, such that*

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant Cn^2 2^{-2n} |f|_{W_2^2(\mathbb{R})} \qquad \forall\, f \in W_2^2(\mathbb{R}) \quad \text{and} \quad n \geqslant 2, \tag{1.22}$$

where the linear operators Q_n are defined in (1.9).

(iii) *The nonstationary tight wavelet frame in (i) does not have any frame approximation order; i.e., for any given $\nu > 0$, there does not exist a positive constant C such that (1.16) is satisfied.*

The Daubechies orthogonal masks $\widehat{a^I_{j,j}}$ in (1.20) with real coefficients for the pseudo-splines of type I with order (j,j) have been considered in [10] (also see [9] for the general case $\widehat{a^I_{m_j,m_j}}$) to obtain C^∞ compactly supported (nonstationary) orthonormal refinable functions, from which (nonstationary) orthonormal wavelets with the spectral approximation order are derived in [9, 10]. However, it is well-known ([13]) that such Daubechies orthogonal masks $\widehat{a^I_{j,j}}$, having real coefficients and obtained from $\widehat{a_{j,j}}$ via the Fejér-Riesz lemma in (1.20), are not symmetric (except $j = 1$) and therefore, all the associated nonstationary refinable functions $\phi_j, j \in \mathbb{N}_0$, are not symmetric. One way to achieve symmetry is to split the masks $\widehat{a_{j,j}}$ into masks similar to $\widehat{a^I_{j,j}}$ in (1.20) but allowing complex-valued coefficients (see [30]). Examples of symmetric orthonormal complex wavelets were first constructed in [30] in the above way from Daubechies orthogonal masks of odd orders. Recently, symmetric orthonormal complex-valued wavelets have been systematically studied in Han [23].

Let $P_{j,j}$ be the polynomial defined in (1.18). For an odd integer j, one can always construct ([23, Lemma 6 and Section 2]) two polynomials P^r_j and P^i_j with real coefficients such that

$$P_{j,j}(x) = [P^r_j(x)]^2 + [P^i_j(x)]^2, \qquad x \in \mathbb{R} \quad \text{with} \quad P^r_j(0) = 1, \quad P^i_j(0) = 0. \tag{1.23}$$

Now define

$$\widehat{a^S_j}(\xi) := e^{i(j-1)\xi/2} 2^{-j}(1+e^{-i\xi})^j [P^r_j(\sin^2(\xi/2)) + iP^i_j(\sin^2(\xi/2))]. \tag{1.24}$$

It is easy to check ([23, Lemma 3]) that $|\widehat{a^S_j}(\xi)|^2 = |\widehat{a^I_{j,j}}(\xi)|^2 = \widehat{a_{j,j}}(\xi)$ and the integer shifts of the stationary refinable function associated with the mask $\widehat{a^S_j}$ are orthonormal. Using this family of masks, one can obtain symmetric C^∞ orthonormal complex wavelets with compact support. We summarize the above discussion into the following result.

Theorem 1.4. *Let $m_j, j \in \mathbb{N}$, be positive odd integers such that (1.21) holds. Take $\widehat{a_j}(\xi) := \widehat{a^S_{m_j}}$, where $\widehat{a^S_{m_j}}$ is defined in (1.24). Define*

$$\widehat{\psi_{j-1}}(2\xi) := e^{-i\xi}\overline{\widehat{a_j}(\xi+\pi)}\widehat{\phi_j}(\xi), \qquad j \in \mathbb{N},$$

where $\phi_j, j \in \mathbb{N}_0$, are defined in (1.1). Then

(1) *Each refinable function $\phi_j, j \in \mathbb{N}_0$, is a compactly supported C^∞ complex-valued function such that $\phi_j(1-\cdot) = \phi_j$ and $\{\phi_j(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L_2(\mathbb{R})$.*

(2) *Each wavelet function $\psi_j, j \in \mathbb{N}_0$, is a compactly supported C^∞ complex-valued function such that $\psi_j(1-\cdot) = -\psi_j$ and ψ_j has m_{j+1} vanishing moments.*

(3) $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k} := 2^{j/2}\psi_j(2^j \cdot - k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ *is a compactly supported symmetric C^∞ orthonormal basis of $L_2(\mathbb{R})$ and has the spectral approximation order.*

This paper is organized as follows. In Section 2, we shall discuss nonstationary cascade algorithms and some properties of nonstationary refinable functions. In particular, we study the initial functions in a nonstationary cascade algorithm and provide a sufficient condition for the convergence of a nonstationary cascade algorithm in a Sobolev space $W_2^\nu(\mathbb{R})$. As a consequence, we obtain a characterization for nonstationary orthonormal wavelet bases in $L_2(\mathbb{R})$. In Section 3, we shall study the frame approximation order of a nonstationary tight wavelet frame. The proofs to Theorems 1.1, 1.2, 1.3, and 1.4 will be given in Section 4.

2. Nonstationary Cascade Algorithms and Refinable Functions

In this section, we first discuss the existence of L_2-solutions of nonstationary refinable functions for a general set of masks satisfying (1.3). In fact, this follows from the following result, proven in this section,

$$[\widehat{\phi_j},\widehat{\phi_j}](\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\phi_j}(\xi+2\pi k)|^2 \leqslant 1, \qquad a.e.\, \xi\in\mathbb{R} \quad \forall\, j\in\mathbb{N}_0 \tag{2.1}$$

for all masks satisfying (1.3), provided that the infinite products in (1.1) exist almost everywhere. The above inequality plays a critical role in our study of nonstationary tight wavelet frames and their frame approximation orders.

The question when the refinable functions are in Sobolev spaces is discussed next. In fact, we prove it as a consequence of the convergence of the cascade algorithm in various Sobolev spaces when $\widehat{a_j}$, $j\in\mathbb{N}$, are masks of pseudo-splines. The proof is done in the Fourier domain with the initial function whose Fourier transform is the characteristic function of $[-\pi,\pi]$. We, then, prove that when the cascade algorithm converges for one initial function, it converges for a large class of functions. Although a similar result is well-known for the stationary case, it is not straightforward for the case of nonstationary cascade algorithms. However, this result is important in computer aided geometric design, because it results in a compactly supported function in each iteration of the cascade algorithm that generates a curve from a finitely supported sequence of points to approximate the underlying curve. Hence, it is desirable to prove the convergence of a cascade algorithm with a compactly supported initial function instead of an infinitely supported band-limited function in computer aided geometric design ([17]).

For the nonstationary refinable functions $\phi_j, j\in\mathbb{N}_0$, defined by masks for pseudo-splines as in Theorem 1.2, one could use the same techniques developed in [10] to show that the cascade algorithm converges for the special initial function whose Fourier transform is the characteristic function of $[-\pi,\pi]$ that leads to $\phi_j\in C^\infty(\mathbb{R})$ for all $j\in\mathbb{N}_0$. But our discussion on nonstationary cascade algorithms in this section will supplement the results in [10] on nonstationary cascade algorithms. We use the results in [10] whenever they can be directly applied e.g. Lemma 2.1 and at the same time develop our own results to achieve our goal with a systematic and comprehensive approach. We also believe that some results (e.g., Theorem 2.4, Lemmas 2.2 and 2.7) derived in this section have their own values in addition to be used to prove Theorem 2.8 in this section.

2.1. L_2-Solutions.

We start with a basic property about the point-wise convergence of the infinite product in (1.1). A sufficient condition for the convergence of the infinite product in (1.1) has been established in the following lemma by Cohen and Dyn in [10, Theorem 2.1]:

Lemma 2.1. *Let $\widehat{a_j}, j\in\mathbb{N}$, be 2π-periodic trigonometric polynomials such that $\sup_{j\in\mathbb{N}}\|\widehat{a_j}\|_{L_\infty(\mathbb{R})}<\infty$. If* (1.4) *holds and $\sum_{j=1}^\infty|\widehat{a_j}(0)-1|<\infty$, then the infinite product in* (1.1) *converges uniformly on every compact set of $\mathbb{R}$ and all $\phi_j, j\in\mathbb{N}_0$, in* (1.1) *are well-defined compactly supported tempered distributions.*

Next, we consider when $\phi_j\in L_2(\mathbb{R})$, $j\in\mathbb{N}_0$, provided that the infinite products in (1.1) exist almost everywhere. In order to investigate the frame approximation order of a nonstationary tight wavelet frame, we establish (2.1), which is the following lemma.

Lemma 2.2. *Let $\widehat{a_j}, j \in \mathbb{N}$, be 2π-periodic measurable functions satisfying* (1.3) *for each $j \in \mathbb{N}$. Assume that for every $j \in \mathbb{N}_0$, $\widehat{\phi_j}(\xi) := \lim_{N\to\infty} \prod_{n=1}^N \widehat{a_{n+j}}(2^{-n}\xi)$ is well-defined for almost every $\xi \in \mathbb{R}$; that is, the infinite product in* (1.1) *exists for almost every point in $\mathbb{R}$. Then* (2.1) *holds and consequently, $\phi_j \in L_2(\mathbb{R})$ with $\|\phi_j\|_{L_2(\mathbb{R})} \leqslant 1$ for every $j \in \mathbb{N}_0$.*

Proof. It suffices to prove the case $j = 0$, since the proof of the general case $j \in \mathbb{N}_0$ is the same. Note that $\widehat{\phi_0}(\xi) = \lim_{n\to\infty} \prod_{j=1}^n \widehat{a_j}(2^{-j}\xi)$ for almost every $\xi \in \mathbb{R}$. For any fixed positive integer K, we have

$$\sum_{k=-K}^{K} |\widehat{\phi_0}(\xi + 2\pi k)|^2 = \lim_{n\to\infty} \sum_{k=-K}^{K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2, \qquad a.e.\ \xi \in \mathbb{R}. \tag{2.2}$$

Let N be the smallest positive integer such that $N > 1 + \log_2 K$. Then we have $[-K, K] \subseteq [-K, 2^N - 1 - K]$. Consequently, for all $n \geqslant N$, we have $[-K, K] \subseteq [-K, 2^n - 1 - K]$ and

$$\sum_{k=-K}^{K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2 \leqslant \sum_{k=-K}^{2^n-1-K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2. \tag{2.3}$$

Let $L_\infty(\mathbb{T}) := \{f \in L_\infty(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic}\}$. The transition operator $T_j : L_\infty(\mathbb{T}) \to L_\infty(\mathbb{T})$ is defined for each $f \in L_\infty(\mathbb{T})$:

$$[T_j f](\xi) := |\widehat{a_j}(\xi/2)|^2 f(\xi/2) + |\widehat{a_j}(\xi/2 + \pi)|^2 f(\xi/2 + \pi), \qquad \xi \in \mathbb{R}.$$

Observing that $|[T_j f](\xi)| \leqslant \left[|\widehat{a_j}(\xi/2)|^2 + |\widehat{a_j}(\xi/2 + \pi)|^2\right] \|f\|_{L_\infty(\mathbb{R})}$, by (1.3), we deduce that

$$\|T_j f\|_{L_\infty(\mathbb{R})} \leqslant \|f\|_{L_\infty(\mathbb{R})} \left\||\widehat{a_j}(\cdot/2)|^2 + |\widehat{a_j}(\cdot/2 + \pi)|^2\right\|_{L_\infty(\mathbb{R})} \leqslant \|f\|_{L_\infty(\mathbb{R})}. \tag{2.4}$$

By induction on n, we can verify (e.g., [20, Lemma 2.1]) that

$$\sum_{k=-K}^{2^n-1-K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2 = [T_1 T_2 \cdots T_{n-1} T_n 1](\xi), \qquad \xi \in \mathbb{R}. \tag{2.5}$$

Now it follows from (2.3) and (2.4) that for $n \geqslant N$ and for almost every $\xi \in \mathbb{R}$,

$$\sum_{k=-K}^{K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2 \leqslant [T_1 T_2 \cdots T_{n-1} T_n 1](\xi) \leqslant \|T_1 T_2 \cdots T_{n-1} T_n 1\|_{L_\infty(\mathbb{R})} \leqslant 1.$$

That is, we have

$$\sum_{k=-K}^{K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2 \leqslant 1 \qquad a.e.\ \xi \in \mathbb{R},\ n \geqslant N.$$

It follows from the above inequality and (2.2) that for any fixed positive integer K,

$$\sum_{k=-K}^{K} |\widehat{\phi_0}(\xi + 2\pi k)|^2 = \lim_{n\to\infty} \sum_{k=-K}^{K} \prod_{j=1}^{n} \left|\widehat{a_j}\big(2^{-j}(\xi + 2\pi k)\big)\right|^2 \leqslant 1, \qquad a.e.\ \xi \in \mathbb{R}.$$

Taking $K \to \infty$ in the above inequality, we conclude that (2.1) is true for $j = 0$.

Since (2.1) implies that $\|\widehat{\phi_j}\|^2_{L_2(\mathbb{R})} = \int_{-\pi}^{\pi} [\widehat{\phi_j}, \widehat{\phi_j}](\xi)\, d\xi \leqslant \int_{-\pi}^{\pi} 1\, d\xi = 2\pi$, by Plancherel's theorem, it follows that $\|\phi_j\|_{L_2(\mathbb{R})} \leqslant 1$. ∎

As Lemma 2.1 states, the assumption that, for $j \in \mathbb{N}_0$, $\widehat{\phi_j}(\xi) := \lim_{N\to\infty} \prod_{n=1}^N \widehat{a_{n+j}}(2^{-n}\xi)$ is well-defined for almost every $\xi \in \mathbb{R}$, required in Lemma 2.2, is satisfied whenever the conditions $\widehat{a_j}(0) = 1$, $j \in \mathbb{N}$, and (1.4) hold. In other words, Lemma 2.2 says that if the masks $\widehat{a_j}$, $j \in \mathbb{N}$, satisfy (1.3), (1.4) and $\widehat{a_j}(0) = 1$, then the corresponding nonstationary refinable functions $\phi_j \in L_2(\mathbb{R})$, $j \in \mathbb{N}_0$.

Since the approximation property of $\phi_j, j \in \mathbb{N}_0$, discussed in this paper depends only on ϕ_j for large enough j, without loss of generality, throughout the paper we shall assume that the normalization condition $\widehat{a_j}(0) = 1$ holds for all $j \in \mathbb{N}$. In fact, if the conclusion in Lemma 2.1 holds, since $\prod_{n=1}^{\infty} \widehat{a_{n+j}}(0)$ converges and is nonzero for sufficiently large j, then we can replace $\widehat{\phi_j}$ and $\widehat{a_j}$ with $\widehat{\phi_j}/\widehat{\phi_j}(0)$ and $\widehat{a_j}(\xi)/\widehat{a_j}(0)$, respectively.

2.2. **Cascade Algorithms.** A cascade algorithm is often used to study various properties of refinable functions and is closely related to a subdivision scheme in computer aided geometric design for generating smooth curves ([10, 15, 19, 24]). For a given sequence of masks $\{\widehat{a_j}\}_{j=1}^{\infty}$, starting with an initial function $f \in L_2(\mathbb{R})$, one computes a sequence of cascade functions f_n by

$$\widehat{f_n}(\xi) := \hat{f}(2^{-n}\xi) \prod_{j=1}^{n} \widehat{a_j}(2^{-j}\xi), \qquad \xi \in \mathbb{R},\, n \in \mathbb{N}. \tag{2.6}$$

If $\lim_{n\to\infty} \hat{f}(2^{-n}\xi) = 1$ and $\hat{\phi}(\xi) := \lim_{n\to\infty} \prod_{j=1}^{n} \widehat{a_j}(2^{-j}\xi)$ exists for almost every $\xi \in \mathbb{R}$, then by (2.6) it is evident that $\lim_{n\to\infty} \widehat{f_n}(\xi) = \hat{\phi}(\xi)$ for almost every $\xi \in \mathbb{R}$.

The cascade algorithm is closely related to another algorithm, called a subdivision scheme which we define next. For a sequence $u : \mathbb{Z} \mapsto \mathbb{C}$, we denote $\hat{u}$ its Fourier series as $\hat{u}(\xi) := \sum_{k\in\mathbb{Z}} u(k)e^{-ik\xi}$. In particular, by δ we denote the *Dirac sequence* on $\mathbb{Z}$ such that $\delta(0) = 1$ and $\delta(k) = 0$ for $k \in \mathbb{Z}\backslash\{0\}$. That is, $\hat{\delta} = 1$. For a sequence u and a mask a, the subdivision operator S_a maps the sequence u into a new sequence $S_a u$ on $\mathbb{Z}$ which is determined by $\widehat{S_a u}(\xi) = 2\hat{a}(\xi)\hat{u}(2\xi)$. In fact, the product $2^n \prod_{j=1}^{n} \widehat{a_j}(2^{n-j}\xi)$ is the Fourier series of the subdivision sequence $S_{a_n} S_{a_{n-1}} \cdots S_{a_2} S_{a_1}\delta$. More precisely, it follows from (2.6) that the cascade sequence $\{f_n\}_{n=1}^{\infty}$ and the subdivision sequence $\{S_{a_n} S_{a_{n-1}} \cdots S_{a_2} S_{a_1}\delta\}_{n=1}^{\infty}$ are related by

$$f_n = \sum_{k\in\mathbb{Z}} [S_{a_n} S_{a_{n-1}} \cdots S_{a_2} S_{a_1}\delta](k) f(2^n \cdot -k), \qquad n \in \mathbb{N}. \tag{2.7}$$

Recall that $f \in W_2^{\nu}(\mathbb{R})$ if $\|f\|^2_{W_2^{\nu}(\mathbb{R})} := \int_{\mathbb{R}} (1 + |\xi|^{2\nu})|\hat{f}(\xi)|^2\, d\xi < \infty$. For a sequence of masks $\{\widehat{a_j}\}_{j=1}^{\infty}$ and an initial function $f \in W_2^{\nu}(\mathbb{R})$, we say that the (nonstationary) cascade algorithm associated with masks $\{\widehat{a_j}\}_{j=1}^{\infty}$ and an initial function f converges in the Sobolev space $W_2^{\nu}(\mathbb{R})$ if $f_n \in W_2^{\nu}(\mathbb{R})$ for all $n \in \mathbb{N}$ and the sequence $\{f_n\}_{n=1}^{\infty}$ is convergent in $W_2^{\nu}(\mathbb{R})$. Many (but not all) functions in $W_2^{\nu}(\mathbb{R})$ can serve as an initial function in a cascade algorithm. One popular and natural choice of an initial function f in computer aided geometric design is from the B-spline functions, since they are compactly supported functions of piecewise polynomials. Hence, it is easy to compute the values of the underlying approximating function. However, to analyze the convergence of the cascade algorithm in the frequency domain, the sinc function $f(x) = \frac{\sin(\pi x)}{\pi x}$, that is, $\widehat{f} = \chi_{[-\pi,\pi]}$, the characteristic function of the interval $[-\pi, \pi]$, is a more natural choice (see e.g. [9, 11, 10, 13, 22]). Our analysis will show that for the cascade algorithm generated by pseudo-spline masks with the sinc function being the initial function converges in $W_2^{\nu}(\mathbb{R})$. To make sure that this cascade algorithm also converges in $W_2^{\nu}(\mathbb{R})$ when the initial seed is replaced by splines, we prove a more general result as follows: a cascade algorithm converges in $W_2^{\nu}(\mathbb{R})$ for one initial seed with stable integer shifts, it converges in $W_2^{\nu}(\mathbb{R})$ for a class of initial functions. As we will see that the proof is more technical than the stationary case, because a stationary refinable function is a fixed point of a stationary cascade algorithm while this is no longer the case for the nonstationary case.

Before proceeding further, let us introduce the following notation. For $\nu \in \mathbb{R}$ and $f \in L_2(\mathbb{R})$, we define

$$[\hat{f}, \hat{f}]_{\nu}(\xi) := \frac{1}{|\xi|^{2\nu}} \sum_{k\in\mathbb{Z}} |\hat{f}(\xi + 2\pi k)|^2 |\xi + 2\pi k|^{2\nu}, \qquad \xi \in \mathbb{R} \tag{2.8}$$

and

$$\{\hat{f},\hat{f}\}_\nu(\xi) := \frac{1}{|\xi|^{2\nu}} \sum_{k\in\mathbb{Z}\setminus\{0\}} |\hat{f}(\xi+2\pi k)|^2 |\xi+2\pi k|^{2\nu}, \qquad \xi\in\mathbb{R}. \tag{2.9}$$

Clearly, we have $[\hat{f},\hat{f}] := [\hat{f},\hat{f}]_0 = \sum_{k\in\mathbb{Z}} |\hat{f}(\cdot+2\pi k)|^2$ and

$$[\hat{f},\hat{f}]_\nu(\xi) = \{\hat{f},\hat{f}\}_\nu(\xi) + |\hat{f}(\xi)|^2. \tag{2.10}$$

Following [19, 22], we introduce the set $\mathcal{F}_\nu$ of initial functions in a cascade algorithm as

$$\begin{aligned}\mathcal{F}_\nu := \Big\{ f\in W_2^\nu(\mathbb{R}) \ :\ \lim_{n\to\infty} \hat{f}(2^{-n}\xi)=1, \quad \lim_{n\to\infty}\{\hat{f},\hat{f}\}_\nu(2^{-n}\xi)=0, \quad a.e.\, \xi\in[-\pi,\pi],\\ [\hat{f},\hat{f}]_\nu \in L_\infty([-\pi,\pi])\Big\}.\end{aligned} \tag{2.11}$$

The following result will be needed later whose proof is rather simple and therefore is omitted.

Lemma 2.3. *Let $f\in\mathcal{F}_\nu$ for some $\nu\geqslant 0$. Then $[\hat{f},\hat{f}](\xi)\leqslant[\hat{f},\hat{f}]_\nu(\xi)$ for almost every $\xi\in[-\pi,\pi]$, (consequently, $[\hat{f},\hat{f}]\in L_\infty(\mathbb{R})$), and*

$$\lim_{n\to\infty}[\hat{f},\hat{f}](2^{-n}\xi) = \lim_{n\to\infty}[\hat{f},\hat{f}]_\nu(2^{-n}\xi) = 1, \qquad a.e.\ \xi\in\mathbb{R}. \tag{2.12}$$

We say that the integer shifts of a function $f\in L_2(\mathbb{R})$ is *stable* in $L_2(\mathbb{R})$ if there exists a positive constant C such that

$$C^{-1}\leqslant[\hat{f},\hat{f}](\xi)\leqslant C, \qquad a.e.\,\xi\in\mathbb{R}. \tag{2.13}$$

Now we state the following result on an initial function with stable integer shifts in a nonstationary cascade algorithm.

Theorem 2.4. *Let $\widehat{a_j}, j\in\mathbb{N}$, be 2π-periodic measurable functions such that*

$$\widehat{f_\infty}(\xi) := \lim_{n\to\infty}\prod_{j=1}^n \widehat{a_j}(2^{-j}\xi)$$

exists for almost every $\xi\in\mathbb{R}$. For a function $f\in\mathcal{F}_\nu$ with $\nu\geqslant 0$ and stable integer shifts, define $f_n, n\in\mathbb{N}$, by (2.6). Assume that $\{f_n\}_{n=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$. Then it converges to f_∞ in $W_2^\nu(\mathbb{R})$. Furthermore, for every $g\in\mathcal{F}_\nu$, the sequence of functions $g_n, n\in\mathbb{N}$, defined by

$$\widehat{g_n}(\xi) := \hat{g}(2^{-n}\xi)\prod_{j=1}^n\widehat{a_j}(2^{-j}\xi), \qquad \xi\in\mathbb{R},\ n\in\mathbb{N}, \tag{2.14}$$

converges to f_∞ in $W_2^\nu(\mathbb{R})$.

Proof. By the definition of f_n in (2.6), we deduce that

$$\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{f_n}(\xi)|^2\,d\xi = \int_{\mathbb{R}}\chi_{[-\pi,\pi]}(2^{-n}\xi)\Big([\hat{f},\hat{f}](2^{-n}\xi)+|\xi|^{2\nu}[\hat{f},\hat{f}]_\nu(2^{-n}\xi)\Big)\prod_{j=1}^n|\widehat{a_j}(2^{-j}\xi)|^2\,d\xi.$$

That is, (2.6) implies

$$\|f_n\|^2_{W_2^\nu(\mathbb{R})} := \int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{f_n}(\xi)|^2\,d\xi = \int_{\mathbb{R}}F_n(\xi)\,d\xi \tag{2.15}$$

with

$$F_n(\xi) := \Big([\hat{f},\hat{f}](2^{-n}\xi)+|\xi|^{2\nu}[\hat{f},\hat{f}]_\nu(2^{-n}\xi)\Big)\chi_{[-\pi,\pi]}(2^{-n}\xi)\prod_{j=1}^n|\widehat{a_j}(2^{-j}\xi)|^2, \qquad \xi\in\mathbb{R}.$$

Similarly, by (2.14), we deduce that

$$\|g_n\|^2_{W_2^\nu(\mathbb{R})} := \int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{g_n}(\xi)|^2\,d\xi = \int_{\mathbb{R}}G_n(\xi)\,d\xi \tag{2.16}$$

with

$$G_n(\xi) := \Big([\hat g,\hat g](2^{-n}\xi)+|\xi|^{2\nu}[\hat g,\hat g]_\nu(2^{-n}\xi)\Big)\chi_{[-\pi,\pi]}(2^{-n}\xi)\prod_{j=1}^n|\widehat{a_j}(2^{-j}\xi)|^2.$$

On the one hand, since $g\in\mathcal{F}_\nu$, by the definition of $\mathcal{F}_\nu$ in (2.11) and Lemma 2.3, we see that there exists a positive constant C_1 such that $[\hat g,\hat g](\xi)\leqslant[\hat g,\hat g]_\nu(\xi)\leqslant C_1$ for almost every $\xi\in[-\pi,\pi]$. By Lemma 2.3, it follows from (2.13) that $C^{-1}\leqslant[\hat f,\hat f]_\nu(\xi)$ for almost every $\xi\in[-\pi,\pi]$ and

$$[\hat g,\hat g](\xi)\leqslant C_1\leqslant CC_1[\hat f,\hat f](\xi)\quad\text{and}\quad[\hat g,\hat g]_\nu(\xi)\leqslant C_1\leqslant CC_1[\hat f,\hat f]_\nu(\xi),\qquad a.e.\,\xi\in[-\pi,\pi].$$

Now it follows from the above inequalities that

$$|G_n(\xi)|\leqslant CC_1F_n(\xi),\qquad a.e.\,\xi\in\mathbb{R}\quad\text{and}\quad n\in\mathbb{N}. \tag{2.17}$$

On the other hand, by $f,g\in\mathcal{F}_\nu$ and Lemma 2.3, since $\lim_{n\to\infty}\prod_{j=1}^n\widehat{a_j}(2^{-j}\xi)=\widehat{f_\infty}(\xi)$ for almost every $\xi\in\mathbb{R}$, we see that $\lim_{n\to\infty}F_n(\xi)=\lim_{n\to\infty}G_n(\xi)=(1+|\xi|^{2\nu})|\widehat{f_\infty}(\xi)|^2$ for almost every $\xi\in\mathbb{R}$. Since $\{f_n\}_{n=1}^\infty$ is a convergent sequence in $W_2^\nu(\mathbb{R})$ and $\lim_{n\to\infty}\widehat{f_n}(\xi)=\widehat{f_\infty}(\xi)$ for almost every $\xi\in\mathbb{R}$, we have $f_\infty\in W_2^\nu(\mathbb{R})$ and $\lim_{n\to\infty}\|f_n-f_\infty\|_{W_2^\nu(\mathbb{R})}=0$. In particular, we have $\lim_{n\to\infty}\|f_n\|^2_{W_2^\nu(\mathbb{R})}=\|f_\infty\|^2_{W_2^\nu(\mathbb{R})}$. By (2.15), we have $\|f_n\|^2_{W_2^\nu(\mathbb{R})}=\int_{\mathbb{R}}F_n(\xi)\,d\xi$. Therefore, we conclude that $\lim_{n\to\infty}\int_{\mathbb{R}}F_n(\xi)\,d\xi=\|f_\infty\|^2_{W_2^\nu(\mathbb{R})}$. Now by (2.17) and the generalized Lebesgue dominated convergence theorem, it follows from (2.16) and $\lim_{n\to\infty}G_n(\xi)=(1+|\xi|^{2\nu})|\widehat{f_\infty}(\xi)|^2$ for almost every $\xi\in\mathbb{R}$ that

$$\lim_{n\to\infty}\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{g_n}(\xi)|^2\,d\xi=\lim_{n\to\infty}\int_{\mathbb{R}}G_n(\xi)\,d\xi=\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{f_\infty}(\xi)|^2\,d\xi. \tag{2.18}$$

But we also have

$$(1+|\xi|^{2\nu})|\widehat{g_n}(\xi)-\widehat{f_\infty}(\xi)|^2\leqslant2(1+|\xi|^{2\nu})\Big[|\widehat{g_n}(\xi)|^2+|\widehat{f_\infty}(\xi)|^2\Big].$$

By (2.18), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}}2(1+|\xi|^{2\nu})\Big[|\widehat{g_n}(\xi)|^2+|\widehat{f_\infty}(\xi)|^2\Big]\,d\xi=4\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{f_\infty}(\xi)|^2\,d\xi<\infty.$$

Now by the generalized Lebesgue dominated convergence theorem again, we conclude that

$$\lim_{n\to\infty}\|g_n-f_\infty\|^2_{W_2^\nu(\mathbb{R})}=\lim_{n\to\infty}\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{g_n}(\xi)-\widehat{f_\infty}(\xi)|^2\,d\xi=\int_{\mathbb{R}}(1+|\xi|^{2\nu})\lim_{n\to\infty}|\widehat{g_n}(\xi)-\widehat{f_\infty}(\xi)|^2\,d\xi=0,$$

since $\lim_{n\to\infty}\widehat{g_n}(\xi)=\widehat{f_\infty}(\xi)$ for almost every $\xi\in\mathbb{R}$. This completes the proof. ■

As shown in the next result, the conditions in (2.11) and (2.13) are not very restrictive. In fact, the sinc function and all B-spline functions belong to $\mathcal{F}_\nu$ for some ν.

Proposition 2.5. *If $f(x):=\frac{\sin(\pi x)}{\pi x}$, then $f\in\mathcal{F}_\nu$ for all $\nu\geqslant0$ and* (2.13) *holds, where $\mathcal{F}_\nu$ is defined in* (2.11)*. For the B-spline B_m of order m, that is, $\widehat{B_m}(\xi)=\left(\frac{1-e^{-i\xi}}{i\xi}\right)^m$, then $B_m\in\mathcal{F}_\nu$ and* (2.13) *holds for all $0\leqslant\nu<m-1/2$. Note that $B_m\notin W_2^{m-1/2}(\mathbb{R})$.*

Proof. Letting $\nu\geqslant0$, we note that $\hat f=\chi_{[-\pi,\pi]}$ and therefore, $f\in W_2^\nu(\mathbb{R})$. For $\xi\in(-\pi,\pi)$, we have $\hat f(\xi+2\pi k)=0$ for all $k\in\mathbb{Z}\backslash\{0\}$ and hence,

$$\{\hat f,\hat f\}_\nu(\xi):=\frac{1}{|\xi|^{2\nu}}\sum_{k\in\mathbb{Z}\backslash\{0\}}|\hat f(\xi+2\pi k)|^2|\xi+2\pi k|^{2\nu}=0,\qquad\xi\in(\pi,\pi)\backslash\{0\}.$$

Therefore, it is evident that $\lim_{n\to\infty}\hat f(2^{-n}\xi)=1$ and $\lim_{n\to\infty}\{\hat f,\hat f\}_\nu(2^{-n}\xi)=0$ for almost every $\xi\in[-\pi,\pi]$. Moreover, by (2.10), we have

$$[\hat f,\hat f]_\nu(\xi)=|\hat f(\xi)|^2+\{\hat f,\hat f\}_\nu(\xi)=|\hat f(\xi)|^2=1,\qquad\xi\in(-\pi,\pi)\backslash\{0\}.$$

Hence, $[\hat{f},\hat{f}]_\nu \in L_\infty([-\pi,\pi])$. So, $f \in \mathcal{F}_\nu$. (2.13) is obviously true by $[\hat{f},\hat{f}](\xi)=1$ for almost every $\xi \in \mathbb{R}$.

Letting $0 \leqslant \nu < m-1/2$, from $\widehat{B_m}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^m$, we have $\lim_{j\to\infty} \widehat{B_m}(2^{-j}\xi) = \widehat{B_m}(0)=1$, $B_m \in W_2^\nu(\mathbb{R})$, and $|\widehat{B_m}(\xi)|^2 = \frac{\sin^{2m}(\xi/2)}{(\xi/2)^{2m}}$. Now a simple computation shows that:

$$\{\widehat{B_m},\widehat{B_m}\}_\nu(\xi) \leqslant 2^{2(m-\nu)} \sin^{2(m-\nu)}(\xi/2) \sum_{k\in\mathbb{Z}\setminus\{0\}} |\xi+2\pi k|^{-2(m-\nu)}.$$

Since $0 \leqslant \nu < m-1/2$, we have $2(m-\nu)>1$. Hence, for all $\xi \in [-\pi,\pi]$, the series $\sum_{k\in\mathbb{Z}\setminus\{0\}} |\xi + 2\pi k|^{-2(m-\nu)}$ uniformly converges and therefore, it is uniformly bounded. Now it follows from the above inequality that $\{\widehat{B_m},\widehat{B_m}\}_\nu(\xi)$ is uniformly bounded on $[-\pi,\pi]$ and

$$\lim_{n\to\infty} \{\widehat{B_m},\widehat{B_m}\}_\nu(2^{-n}\xi) \leqslant \lim_{n\to\infty} \sin^{2(m-\nu)}(2^{-n-1}\xi) = 0 \qquad \forall\, \xi \in [-\pi,\pi].$$

Now it follows from (2.10) that $[\widehat{B_m},\widehat{B_m}]_\nu \in L_\infty([-\pi,\pi])$, since $\widehat{B_m} \in L_\infty(\mathbb{R})$ and $\{\widehat{B_m},\widehat{B_m}\}_\nu \in L_\infty([-\pi,\pi])$. It is well-known that (2.13) holds for B_m. ∎

The following result provides us a sufficient condition on the convergence of a nonstationary cascade algorithm in a Sobolev space $W_2^\nu(\mathbb{R})$. As we will see, this result is sufficient to study the convergence of nonstationary cascade algorithms with masks for pseudo-splines.

Proposition 2.6. *Let $\widehat{a_j}$ and $\widehat{b_j}$ ($j \in \mathbb{N}$) be 2π-periodic measurable functions such that for all $j \in \mathbb{N}$,*

$$|\widehat{a_j}(\xi)| \leqslant |\widehat{b_j}(\xi)|, \qquad a.e.\ \xi \in \mathbb{R}. \tag{2.19}$$

Let $\eta \in W_2^\nu(\mathbb{R})$ such that $\lim_{j\to\infty}\hat{\eta}(2^{-j}\xi)=1$ for almost every $\xi\in\mathbb{R}$. Define

$$\widehat{f_n}(\xi) := \hat{\eta}(2^{-n}\xi)\prod_{j=1}^n \widehat{a_j}(2^{-j}\xi) \quad and \quad \widehat{g_n}(\xi) := \hat{\eta}(2^{-n}\xi)\prod_{j=1}^n \widehat{b_j}(2^{-j}\xi), \qquad \xi \in \mathbb{R}.$$

Assume that $\widehat{f_\infty}(\xi) := \lim_{n\to\infty}\prod_{j=1}^n \widehat{a_j}(2^{-j}\xi)$ and $\widehat{g_\infty}(\xi) := \lim_{n\to\infty}\prod_{j=1}^n \widehat{b_j}(2^{-j}\xi)$ are well-defined for almost every $\xi\in\mathbb{R}$. Then, $\lim_{n\to\infty}\|g_n - g_\infty\|_{W_2^\nu(\mathbb{R})}=0$ implies $\lim_{n\to\infty}\|f_n-f_\infty\|_{W_2^\nu(\mathbb{R})}=0$. In particular, suppose that there are a positive integer J and a 2π-periodic measurable function $\hat{b}$ such that

$$|\widehat{a_j}(\xi)| \leqslant |\hat{b}(\xi)|, \qquad a.e.\ \xi\in\mathbb{R},\ \forall\, j>J \quad and \quad \widehat{a_j}\in L_\infty(\mathbb{R}), \qquad 1\leqslant j\leqslant J. \tag{2.20}$$

For $n\in\mathbb{N}$, define $\widehat{h_n}(\xi) := \hat{\eta}(2^{-n}\xi)\prod_{j=1}^n \hat{b}(2^{-j}\xi)$. If $\{h_n\}_{n=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$, then f_n converges to f_∞ in $W_2^\nu(\mathbb{R})$, i.e., $\lim_{n\to\infty}\|f_n-f_\infty\|_{W_2^\nu(\mathbb{R})}=0$.

Proof. The assumption that the functions $\widehat{f_\infty}$ and $\widehat{g_\infty}$ are well-defined implies that $\lim_{n\to\infty}\widehat{f_n}(\xi) = \widehat{f_\infty}(\xi)$ and $\lim_{n\to\infty}\widehat{g_n}(\xi)=\widehat{g_\infty}(\xi)$ for almost every $\xi\in\mathbb{R}$.

By assumption, $\{g_n\}_{n=1}^\infty$ converges to g_∞ in $W_2^\nu(\mathbb{R})$, together with the fact that $\lim_{n\to\infty}\widehat{g_n}(\xi) = \widehat{g_\infty}(\xi)$ for almost every $\xi\in\mathbb{R}$, we must have $g_\infty\in W_2^\nu(\mathbb{R})$ and $\lim_{n\to\infty}\|g_n-g_\infty\|_{W_2^\nu(\mathbb{R})}=0$. In particular, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{g_n}(\xi)|^2\,d\xi = \int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{g_\infty}(\xi)|^2\,d\xi < \infty.$$

Denote $\eta_n(\xi) := 2(1+|\xi|^{2\nu})\big[|\widehat{g_n}(\xi)|^2 + |\widehat{g_\infty}(\xi)|^2\big]$. It follows from the above identity that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\eta_n(\xi)\,d\xi = 4\int_{\mathbb{R}}(1+|\xi|^{2\nu})|\widehat{g_\infty}(\xi)|^2\,d\xi < \infty. \tag{2.21}$$

By (2.19), it follows from the definition of $\widehat{g_n}$ and $\widehat{f_n}$ that $|\widehat{f_n}(\xi)| \leqslant |\widehat{g_n}(\xi)|$ for almost every $\xi\in\mathbb{R}$. Since we have $\lim_{n\to\infty}\widehat{f_n}(\xi) = \widehat{f_\infty}(\xi)$ for almost every $\xi\in\mathbb{R}$, we also have $|\widehat{f_\infty}(\xi)| \leqslant |\widehat{g_\infty}(\xi)|$ for almost every $\xi \in \mathbb{R}$. Consequently, by $g_\infty, g_n \in W_2^\nu(\mathbb{R})$ for all $n \in \mathbb{N}$, we have $f_\infty, f_n \in W_2^\nu(\mathbb{R})$ for all $n \in \mathbb{N}$. Moreover, we have

$$(1+|\xi|^{2\nu})|\widehat{f_n}(\xi)-\widehat{f_\infty}(\xi)|^2 \leqslant 2(1+|\xi|^{2\nu})\big[|\widehat{f_n}(\xi)|^2+|\widehat{f_\infty}(\xi)|^2\big] \leqslant \eta_n(\xi), \qquad a.e.\ \xi \in \mathbb{R}.$$

By (2.21) and the generalized Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned}\lim_{n\to\infty} \|f_n - f_\infty\|^2_{W_2^\nu(\mathbb{R})} &= \lim_{n\to\infty} \int_{\mathbb{R}} (1+|\xi|^{2\nu})|\widehat{f_n}(\xi)-\widehat{f_\infty}(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}} \lim_{n\to\infty} (1+|\xi|^{2\nu})|\widehat{f_n}(\xi)-\widehat{f_\infty}(\xi)|^2\, d\xi = 0.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|g_n - g_\infty\|_{W_2^\nu(\mathbb{R})} = 0$ implies $\lim_{n\to\infty} \|f_n - f_\infty\|_{W_2^\nu(\mathbb{R})} = 0$.

If (2.20) holds, for $n > J$ we deduce that for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned}|\widehat{f_n}(\xi)| &= \left[\prod_{j=1}^{J} |\widehat{a_j}(2^{-j}\xi)|\right] |\hat{\eta}(2^{-n}\xi)| \prod_{j=J+1}^{n} |\widehat{a_j}(2^{-j}\xi)| \\ &\leqslant C|\hat{\eta}(2^{-n}\xi)| \prod_{j=J+1}^{n} |\hat{b}(2^{-j}\xi)| = C|\widehat{h_{n-J}}(2^{-J}\xi)|,\end{aligned}$$

where $C := \prod_{j=1}^{J} \|\widehat{a_j}\|_{L_\infty(\mathbb{R})} < \infty$. Since $\{h_n\}_{n=1}^\infty$ is convergent in $W_2^\nu(\mathbb{R})$, it is evident that $\{2^J h_{n-J}(2^J\cdot)\}_{n=J+1}^\infty$ is also convergent in $W_2^\nu(\mathbb{R})$. Note that the Fourier transform of $2^J h_{n-J}(2^J\cdot)$ is $\widehat{h_{n-J}}(2^{-J}\cdot)$. Now by the generalized Lebesgue dominated convergence theorem, we conclude that $\{f_n\}_{n=1}^\infty$ is also convergent in $W_2^\nu(\mathbb{R})$, that is, we have $\lim_{n\to\infty} \|f_n - f_\infty\|_{W_2^\nu(\mathbb{R})} = 0$. ∎

Let $\{\widehat{a_j}\}_{j=1}^\infty$ be a sequence of 2π-periodic measurable functions. Define $\{f_n\}_{n=1}^\infty$ by

$$\widehat{f_n}(\xi) := \chi_{[-\pi,\pi]}(2^{-n}\xi) \prod_{j=1}^{n} \widehat{a_j}(2^{-j}\xi), \qquad \xi \in \mathbb{R}, n \in \mathbb{N}, \tag{2.22}$$

where $\chi_{[-\pi,\pi]}$ denotes the characteristic function of the interval $[-\pi, \pi]$. This can be understood as a representation of the nonstationary cascade algorithm associated with the masks $\{\widehat{a_j}\}_{n=1}^\infty$ in the frequency domain. Due to Theorem 2.4, we say that a nonstationary cascade algorithm associated with masks $\{\widehat{a_j}\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$ if the sequence $\{f_n\}_{n=1}^\infty$ in (2.22) converges in $W_2^\nu(\mathbb{R})$. Note that the initial function here in (2.22) is the sinc function $f(x) = \frac{\sin(\pi x)}{\pi x}$ since $\hat{f}(\xi) = \chi_{[-\pi,\pi]}$. Similarly, we say that a stationary cascade algorithm associated with a mask $\hat{a}$ converges in $W_2^\nu(\mathbb{R})$ if the cascade algorithm associated with $\{\widehat{a_j}\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$ with $\widehat{a_j} = \hat{a}$ for all $j \in \mathbb{N}$.

Basically, Proposition 2.6 says that if (2.19) holds and if the nonstationary cascade algorithm associated with $\{\widehat{b_j}\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$, then so does the nonstationary cascade algorithm associated with $\{\widehat{a_j}\}_{j=1}^\infty$. Similarly, if (2.20) holds and the stationary cascade algorithm associated with mask $\hat{b}$ converges in $W_2^\nu(\mathbb{R})$, then Proposition 2.6 says that the nonstationary cascade algorithm associated with masks $\{\widehat{a_j}\}_{j=1}^\infty$ must converge in $W_2^\nu(\mathbb{R})$.

The convergence of a stationary cascade algorithm associated with a finitely supported mask can be verified easily by calculating the spectrum of the transition operator. Let $\hat{a}$ be a 2π-periodic trigonometric polynomial with $\hat{a}(0) = 1$. Write $\hat{a}(\xi) = (1+e^{-i\xi})^m \hat{c}(\xi)$ for some nonnegative integer m and some 2π-periodic trigonometric polynomial $\hat{c}(\xi)$ with $\hat{c}(\pi) \neq 0$. Write $|\hat{c}(\xi)|^2 = \sum_{k=-K}^{K} c_k e^{-ik\xi}$, where K is some nonnegative integer. Denote $\rho(\hat{a})$ the spectral radius of the square matrix $(c_{2j-k})_{-K\leqslant j,k\leqslant K}$ and define $\nu_2(\hat{a}) := -1/2 - \log_2 \sqrt{\rho(\hat{a})}$. It is known ([19, Theorem 4.3 and Proposition 7.2] and [22, Theorem 2.1]) that the stationary cascade algorithm associated with a 2π-periodic trigonometric polynomial mask $\hat{a}$ converges in $W_2^\nu(\mathbb{R})$ if and only if $\nu_2(\hat{a}) > \nu$. Moreover, $\phi \in W_2^\nu(\mathbb{R})$ for all $0 \leqslant \nu < \nu_2(\hat{a})$, where ϕ is the nontrivial compactly supported refinable function associated with mask $\hat{a}$ such that $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Moreover, $\phi \notin W_2^\nu(\mathbb{R})$ for $\nu > \nu_2(\hat{a})$ whenever the integer shifts of ϕ are stable. See [17, 19, 22, 24, 32, 36] and many references therein on the convergence of stationary cascade algorithms.

2.3. **Convergence of Cascade Algorithms with Pseudo-Spline Masks.** We show that the nonstationary cascade algorithm associated with masks for pseudo-splines in Theorem 1.2 converges in $W_2^\nu(\mathbb{R})$ for arbitrary $\nu \geqslant 0$. We first prove the following lemma which is not only used in the proof of the convergence of the nonstationary cascade algorithms associated with pseudo-spline refinement masks, but also plays an important role in our proof of Theorem 1.2 and the spectral frame approximation order.

Lemma 2.7. *Let $\widehat{a_{m,l}}$ be the refinement mask for pseudo-spline of type II with order (m,l) in (1.19). For positive integers m_1, m_2 and l_2 such that $1 \leqslant m_1 \leqslant m_2$ and $1 \leqslant l_2 \leqslant m_2$, the following inequality holds*

$$|\widehat{a_{m_2,l_2}}(\xi)| \leqslant |\widehat{a^I_{m_2,l_2}}(\xi)| \leqslant |\widehat{b_{m_1}}(\xi)| \qquad \forall\, \xi \in \mathbb{R} \quad \textit{with} \quad \widehat{b_{m_1}}(\xi) := 2(1+e^{-i\xi})^{-1}\widehat{a^I_{m_1,m_1}}(\xi). \tag{2.23}$$

Note that $\widehat{b_{m_1}}$ is uniquely determined by $\widehat{a^I_{m_1,m_1}}(\xi) = 2^{-1}(1+e^{-i\xi})\widehat{b_{m_1}}(\xi)$.

Proof. Since $|\widehat{a_{m,l}}(\xi)| \leqslant 1$ for all $\xi \in \mathbb{R}$ and $1 \leqslant l \leqslant m$, it follows from the relation $|\widehat{a^I_{m,l}}(\xi)|^2 = \widehat{a_{m,l}}(\xi)$ and (1.18) that $|\widehat{a_{m,l}}(\xi)| \leqslant |\widehat{a^I_{m,l}}(\xi)| \leqslant |\widehat{a^I_{m,m}}(\xi)|$ for all $\xi \in \mathbb{R}$ and $1 \leqslant l \leqslant m$. Now in order to prove (2.23), it suffices to prove that $\widehat{a_{m_2,m_2}}(\xi) = |\widehat{a^I_{m_2,m_2}}(\xi)|^2 \leqslant |\widehat{b_{m_1}}(\xi)|^2$ for all $\xi \in \mathbb{R}$ and $1 \leqslant m_1 \leqslant m_2$.

Setting $x = \sin^2(\xi/2)$, by the definition of $\widehat{a_{m_2,m_2}}$ in (1.19), we see that $\widehat{a_{m_2,m_2}}(\xi) \leqslant |\widehat{b_{m_1}}(\xi)|^2$ for all $\xi \in \mathbb{R}$ and $1 \leqslant m_1 \leqslant m_2$ is equivalent to

$$(1-x)^{m_2} P_{m_2,m_2}(x) \leqslant (1-x)^{m_1-1} P_{m_1,m_1}(x) \qquad \forall\, x \in [0,1] \quad \text{and} \quad 1 \leqslant m_1 \leqslant m_2. \tag{2.24}$$

By the definition of $P_{m,l}$ in (1.18), we deduce that

$$\begin{aligned}
(1-x)^{m+1} P_{m+1,m+1}(x) &= (1-x)^{m+1} \sum_{j=0}^{m} \frac{(m+j)!}{j!m!} x^j = (1-x)^m \sum_{j=0}^{m} \frac{(m+j)!}{j!m!}(x^j - x^{j+1}) \\
&= (1-x)^m \left[\sum_{j=0}^{m} \frac{(m+j)!}{j!m!} x^j - \sum_{j=1}^{m+1} \frac{(m+j-1)!}{(j-1)!m!} x^j \right] \\
&= (1-x)^m \left[1 - \frac{(2m)!}{m!m!} x^{m+1} + \sum_{j=1}^{m} \left(\frac{(m+j)!}{j!m!} - \frac{(m+j-1)!}{(j-1)!m!} \right) x^j \right] \\
&= (1-x)^m \left[1 - \frac{(2m)!}{m!m!} x^{m+1} + \sum_{j=1}^{m} \frac{(m+j-1)!}{j!(m-1)!} x^j \right].
\end{aligned}$$

That is, we have

$$\begin{aligned}
(1-x)^{m+1} P_{m+1,m+1}(x) &= (1-x)^m \left[\frac{(2m-1)!}{m!(m-1)!} x^m - \frac{(2m)!}{m!m!} x^{m+1} + \sum_{j=0}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^j \right] \\
&= (1-x)^m \left[\frac{(2m-1)!}{m!(m-1)!} x^m (1-2x) + P_{m,m}(x) \right] \\
&= (1-x)^m P_{m,m}(x) - \frac{(2m-1)!}{m!(m-1)!} x^m (1-x)^m (2x-1) \qquad \forall\, m \in \mathbb{N}.
\end{aligned}$$

Consequently, for $x \in [1/2, 1]$ and $m_2 \geqslant m_1$, we have $2x - 1 \geqslant 0$ and

$$(1-x)^{m_2} P_{m_2,m_2}(x) \leqslant (1-x)^{m_2-1} P_{m_2-1,m_2-1}(x) \leqslant \cdots \leqslant (1-x)^{m_1} P_{m_1,m_1}(x) \leqslant (1-x)^{m_1-1} P_{m_1,m_1}(x).$$

Therefore, (2.24) holds for $x \in [1/2, 1]$.

It remains to prove (2.24) for all $x \in [0, 1/2]$. Since

$$(1-x)^{m_2} P_{m_2,m_2}(x) \leqslant (1-x)^{m_2} P_{m_2,m_2}(x) + x^{m_2} P_{m_2,m_2}(1-x) = 1,$$

in order to prove (2.24) for all $x \in [0, 1/2]$, now it suffices to show that

$$1 \leqslant (1-x)^{m_1-1} P_{m_1,m_1}(x) \qquad \forall\, x \in [0, 1/2]. \tag{2.25}$$

Note that

$$\begin{aligned} 1 &= (1-x)(1-x)^{m_1-1} P_{m_1,m_1}(x) + x x^{m_1-1} P_{m_1,m_1}(1-x) \\ &= (1-x)^{m_1-1} P_{m_1,m_1}(x) - x\big[(1-x)^{m_1-1} P_{m_1,m_1}(x) - x^{m_1-1} P_{m_1,m_1}(1-x)\big], \end{aligned}$$

from which we have

$$(1-x)^{m_1-1} P_{m_1,m_1}(x) = 1 + x\big[(1-x)^{m_1-1} P_{m_1,m_1}(x) - x^{m_1-1} P_{m_1,m_1}(1-x)\big].$$

In order to prove (2.25), by the above identity, it suffices to prove that

$$(1-x)^{m_1-1} P_{m_1,m_1}(x) \geqslant x^{m_1-1} P_{m_1,m_1}(1-x) \qquad \forall\, x \in [0, 1/2]. \tag{2.26}$$

Note that for $x \in [0, 1/2]$, we have $0 \leqslant x/(1-x) \leqslant 1$. By the definition of $P_{m_1,m_1}(x) := \sum_{j=0}^{m_1-1} \binom{m_1+j-1}{j} x^j$, we have

$$1 = \frac{1}{1} \geqslant \frac{\binom{m_1}{1} x}{\binom{m_1}{1}(1-x)} \geqslant \frac{\binom{m_1+1}{2} x^2}{\binom{m_1+1}{2}(1-x)^2} \geqslant \cdots \geqslant \frac{\binom{2m_1-2}{m_1-1} x^{m_1-1}}{\binom{2m_1-2}{m_1-1}(1-x)^{m_1-1}} = \frac{x^{m_1-1}}{(1-x)^{m_1-1}}, \quad x \in [0, 1/2].$$

But for positive numbers a, b, c, d, it is easy to see that $\frac{a}{b} \geqslant \frac{c}{d}$ implies $\frac{a}{b} \geqslant \frac{a+c}{b+d} \geqslant \frac{c}{d}$. Now it follows from the above inequalities that

$$1 \geqslant \frac{P_{m_1,m_1}(x)}{P_{m_1,m_1}(1-x)} = \frac{\sum_{j=0}^{m_1-1} \binom{m_1+j-1}{j} x^j}{\sum_{j=0}^{m_1-1} \binom{m_1+j-1}{j} (1-x)^j} \geqslant \frac{x^{m_1-1}}{(1-x)^{m_1-1}}, \qquad x \in [0, 1/2],$$

from which we see that (2.26) holds. ■

Next we establish the following result on C^∞ nonstationary refinable functions. In fact, we prove that the nonstationary cascade algorithm associated with the masks for pseudo-splines of type I or type II in Theorem 1.2 converges in $W_2^\nu(\mathbb{R})$ for any $\nu \geqslant 0$.

Theorem 2.8. *Let $\widehat{a_j}$ be the mask for pseudo-spline of type I or type II with order (m_j, l_j) where $1 \leqslant l_j \leqslant m_j$ and $j \in \mathbb{N}$ are positive integers such that (1.21) holds. Then for every $n \in \mathbb{N}_0$, the nonstationary cascade algorithm (2.22) associated with $\{\widehat{a_{j+n}}\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$ for any $\nu \geqslant 0$. Consequently, the nonstationary refinable functions $\phi_j, j \in \mathbb{N}_0$, in (1.1) must be well-defined compactly supported $C^\infty(\mathbb{R})$ functions.*

Proof. Since $\widehat{a_j}(\xi) = \widehat{a_{m_j,l_j}}(\xi)$ or $\widehat{a_j}(\xi) = \widehat{a^I_{m_j,l_j}}(\xi)$, it is easy to see that $\deg(\widehat{a_j}) \leqslant 2m_j$ and $\widehat{a_j}(0) = 1$. Therefore, by our assumption in (1.21), we see that

$$\sum_{j=1}^\infty 2^{-j} \deg(\widehat{a_j}) \leqslant 2 \sum_{j=1}^\infty 2^{-j} m_j < \infty.$$

Moreover, we have $|\widehat{a_j}(\xi)| \leqslant 1$ for all $j \in \mathbb{N}$ and $\xi \in \mathbb{R}$. So, the condition of Lemma 2.1 is satisfied. By Lemma 2.1, we conclude that $\phi_j, j \in \mathbb{N}_0$, are well-defined compactly supported tempered distributions.

Let $\widehat{a_{j,j}}$ be the refinement mask for the pseudo-spline of type II with order (j, j). It was proved by Daubechies [12, 13] that $\lim_{j\to\infty} \nu_2(\widehat{a_{j,j}}) = \lim_{j\to\infty} \nu_2(\widehat{a^I_{j,j}}) = \infty$. Hence, there exists a positive integer J such that $\nu_2(\widehat{a^I_{J,J}}) \geqslant \nu+2$. By $\lim_{j\to\infty} m_j=\infty$, there exists a positive integer N such that

$$m_j \geqslant J \quad \text{and} \quad \nu_2(\widehat{a_j}) \geqslant \nu_2(\widehat{a^I_{J,J}}) \geqslant \nu+2 \qquad \forall\, j \geqslant N. \tag{2.27}$$

Let $\hat b$ be the unique 2π-periodic trigonometric polynomial such that $\widehat{a^I_{J,J}}(\xi)=2^{-1}(1+e^{-i\xi})\hat b(\xi)$. By the definition of $\nu_2(\hat b)$ and (2.27), it is straightforward to see that $\nu_2(\hat b)=\nu_2(\widehat{a^I_{J,J}})-1 \geqslant \nu+1>\nu$. Therefore, the stationary cascade algorithm associated with the mask $\hat b$ converges in $W_2^\nu(\mathbb{R})$ (see [19, Theorem 4.3]). On the other hand, by (2.23) of Lemma 2.7, since $m_j \geqslant J$ for $j \geqslant N$, we deduce that

$$|\widehat{a_j}(\xi)| \leqslant |\widehat{a^I_{m_j,l_j}}(\xi)| \leqslant |\hat b(\xi)| \qquad \forall\, \xi\in\mathbb{R}, j\geqslant N.$$

Since the stationary cascade algorithm associated with the mask $\hat b$ converges in $W_2^\nu(\mathbb{R})$, by Proposition 2.6, the nonstationary cascade algorithm associated with masks $\{\widehat{a_j}\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$. Therefore, we have $\phi_0\in W_2^\nu(\mathbb{R})$ for all $\nu\geqslant 0$. That is, ϕ_0 is a compactly supported C^∞ function. The same proof works for every ϕ_n and for the nonstationary cascade algorithm associated with masks $\{\widehat{a_{n+j}}\}_{j=1}^\infty$. ■

In computer aided geometric design, it is of interest to consider the convergence of a subdivision scheme and a cascade algorithm in $C^\kappa(\mathbb{R})$, the space of functions with the κth continuous derivative, instead of the Sobolev space $W_2^\nu(\mathbb{R})$. When a cascade algorithm is implemented in the space domain as given in (2.7), the initial function is often chosen to be a compactly supported function such as spline functions of order ν with $\nu>\kappa+1/2$. This is indeed true and can be proven easily by the imbedding theorems of Sobolev spaces:

Corollary 2.9. *Let $\widehat{a_j}$ ($j\in\mathbb{N}$) be the masks satisfying the conditions given in Theorem 2.8. Let κ be any nonnegative integer and let f be a compactly supported initial function in $W_2^\nu\cap\mathfrak{F}_\nu$ with $\nu>\kappa+1/2$. Then the nonstationary cascade algorithm defined by* (2.7) *associated with $\{\widehat{a_{n+j}}\}_{j=1}^\infty$ converges in $C^\kappa(\mathbb{R})$.*

Proof. The convergence of the cascade algorithm in $W_2^\nu(\mathbb{R})$ defined by (2.7) follows from Proposition 2.6 and Theorem 2.8. Since all f_n ($n\in\mathbb{N}_0$) and ϕ_0 are supported inside some compact set, it follows from the imbedding theorem that the sequence f_n also converges in $C^\kappa(\mathbb{R})$. ■

2.4. **Orthogonality.** For the stationary case, it is well-known that a compactly supported refinable function ϕ with a 2π-periodic trigonometric polynomial $\hat a$ such that the integer shifts of ϕ form an orthonormal system, if and only if, its refinement mask $\hat a$ satisfies

$$|\hat a(\xi)|^2+|\hat a(\xi+\pi)|^2=1, \qquad a.e.\ \xi\in\mathbb{R}$$

and the corresponding stationary cascade algorithm converges in $L_2(\mathbb{R})$ (that is, $\nu_2(\hat a)>0$, see [19, 22]). Furthermore, if one chooses the wavelet function ψ by $\hat\psi(2\xi):=e^{-i\xi}\overline{\hat a(\xi+\pi)}\hat\phi(\xi)$, then the wavelet system generated by ψ forms an orthonormal basis in $L_2(\mathbb{R})$. For example, see [11, 12, 13, 19, 22, 24, 31, 32, 36] and references therein. It turns out that this is also true for the nonstationary case, as a consequence of the proof of Theorem 2.4.

Theorem 2.10. *Let $\widehat{a_j}, j\in\mathbb{N}$, be 2π-periodic measurable functions such that for every $j\in\mathbb{N}_0$, $\widehat{\phi_j}(\xi):=\lim_{N\to\infty}\prod_{n=1}^N\widehat{a_{n+j}}(2^{-n}\xi)$ is well-defined for almost every $\xi\in\mathbb{R}$. Then the integer shifts of ϕ_j form an orthonormal system in $L_2(\mathbb{R})$ for all $j\in\mathbb{N}$, i.e.,*

$$\langle\phi_j(\cdot-k),\phi_j\rangle:=\int_{\mathbb{R}}\phi_j(x-k)\overline{\phi_j(x)}\,dx=\delta(k) \qquad \forall\, k\in\mathbb{Z} \quad and \quad j\in\mathbb{N}_0, \tag{2.28}$$

where $\delta(0)=1$ and $\delta(k)=0$ for all $k\neq 0$, if and only if

(1) *All the masks* $\widehat{a_j}$ *satisfy*

$$|\widehat{a_j}(\xi)|^2 + |\widehat{a_j}(\xi+\pi)|^2 = 1, \qquad a.e.\ \xi \in \mathbb{R}, \quad \forall\, j \in \mathbb{N}_0. \tag{2.29}$$

(2) *The nonstationary cascade algorithm associated with masks* $\{a_{n+j}\}_{n=1}^\infty$ *converges in* $L_2(\mathbb{R})$ *for large enough* $j \in \mathbb{N}_0$.

Moreover, if (1.4) *and* (2.28) *hold, define* $\widehat{\psi_{j-1}}(\xi) := e^{-i\xi/2}\overline{\widehat{a_j}(\xi/2+\pi)}\widehat{\phi_j}(\xi/2)$ *for* $j \in \mathbb{N}$*, then* $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ *is an orthonormal basis of* $L_2(\mathbb{R})$.

Proof. It is known that (2.28) holds for each j if and only if $[\widehat{\phi_j}, \widehat{\phi_j}](\xi) = 1$, $a.e.\,\xi \in \mathbb{R}$.

Assume that (1) and (2) hold. Then $[\widehat{\phi_j}, \widehat{\phi_j}](\xi) = 1$, $a.e.\, x \in \mathbb{R}$ for all $j \in \mathbb{N}_0$ can be proved by a similar argument as in the stationary case (see [11, 13, 22, 24, 32]). We omit the detail here.

The necessity part is proven as follows. If (2.28) holds, then $[\widehat{\phi_j}, \widehat{\phi_j}] = 1$ for all $j \in \mathbb{N}_0$. Now (2.29) can be verified by the same argument as in the stationary case. So, Item (1) holds. Next, we prove Item (2), that is, the sequence $\{f_N\}_{N=1}^\infty$, defined by

$$\widehat{f_N}(\xi) := \chi_{[-\pi,\pi]}(2^{-N}\xi)\prod_{n=1}^{N}\widehat{a_{n+j}}(2^{-n}\xi), \qquad N \in \mathbb{N},$$

converges in $W_2^\nu(\mathbb{R})$ for every $j \in \mathbb{N}_0$. By the relation $\widehat{\phi_j}(\xi) = \widehat{a_{j+1}}(\xi/2)\widehat{\phi_{j+1}}(\xi/2)$, we deduce that

$$\widehat{\phi_j}(\xi) = \widehat{\phi_{N+j}}(2^{-N}\xi)\prod_{n=1}^{N}\widehat{a_{n+j}}(2^{-n}\xi), \qquad N \in \mathbb{N} \quad \text{and} \quad j \in \mathbb{N}_0. \tag{2.30}$$

Applying (2.30) and $[\widehat{\phi_N}, \widehat{\phi_N}] = 1$, one obtains that

$$\|\widehat{\phi_j}\|^2_{L_2(\mathbb{R})} = \int_{\mathbb{R}} |\widehat{f_N}(\xi)|^2\, d\xi, \qquad N \in \mathbb{N}.$$

In particular, we have $\lim_{N\to\infty}\int_{\mathbb{R}} |\widehat{f_N}(\xi)|^2\, d\xi = \|\widehat{\phi_j}\|^2_{L_2(\mathbb{R})} < \infty$. Note that

$$|\widehat{f_N}(\xi) - \widehat{\phi_j}(\xi)|^2 \leqslant 2\Big[|\widehat{f_N}(\xi)|^2 + |\widehat{\phi_j}(\xi)|^2\Big] =: \eta_N(\xi), \qquad \xi \in \mathbb{R}$$

and

$$\lim_{N\to\infty}\int_{\mathbb{R}} \eta_N(\xi)\, d\xi = 4\int_{\mathbb{R}} |\widehat{\phi_j}(\xi)|^2\, d\xi < \infty.$$

By $\lim_{N\to\infty}\widehat{f_N}(\xi) = \widehat{\phi_j}(\xi)$ for almost every $\xi \in \mathbb{R}$ and the generalized Lebesgue dominated convergence theorem, we conclude that $\lim_{N\to\infty}\|\widehat{f_N} - \widehat{\phi_j}\|_{L_2(\mathbb{R})} = 0$. So, for every $j \in \mathbb{N}_0$, the cascade algorithm associated with masks $\{\widehat{a_{n+j}}\}_{n=1}^\infty$ converges in $L_2(\mathbb{R})$.

If (2.28) holds, by the definition of ψ_j and (2.29), then it is easy to check that $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ is an orthonormal system of $L_2(\mathbb{R})$. By Theorem 1.1, we see that $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ is a tight frame in $L_2(\mathbb{R})$. Therefore, $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ is an orthonormal basis of $L_2(\mathbb{R})$. ∎

3. The Approximation Order of the Truncated Frame Series

In this section, we shall study the approximation property of a nonstationary tight wavelet frame, i.e., the frame approximation properties of the operators Q_n in (1.9). The approximation operators Q_n and their approximation order for a given stationary tight wavelet frame have been extensively studied in [14]. The approximation operators Q_n provide a simple approximation scheme for a given tight wavelet frame, and have close links to the frame decomposition and reconstruction algorithms for a tight wavelet frame (see e.g. [14]). Moreover, their approximation order determines the accuracy of the truncation operators and is not necessarily equal to the best approximation order provided by the underlying nonstationary multiresolution analysis.

Since the approximation operators Q_n provide a simple approximation scheme for a given tight wavelet frame, they are often used in various applications. For example, in [1, 2, 3, 4, 5] where the (stationary) tight wavelet frame based algorithms for high/supper resolution image reconstructions, image inpainting, and deconvolutions are given. The operators Q_n are used there to approximate the underlying function from a given data set. The interested reader should consult [1, 2, 3, 4, 5] for details.

The operators Q_n are closely related to other operators. For a sequence $\{\phi_n\}_{n=0}^\infty$ of functions in $L_2(\mathbb{R})$, we define the linear operators $P_n(f), n \in \mathbb{N}_0$, by

$$P_n(f) := \sum_{k\in\mathbb{Z}} \langle f, \phi_{n;n,k}\rangle \phi_{n;n,k}, \qquad f \in L_2(\mathbb{R}) \quad \text{with} \quad \phi_{n;n,k} := 2^{n/2}\phi_n(2^n \cdot -k). \tag{3.1}$$

Similar to the stationary case, by calculation, it is easy to verify that (1.12) implies

$$\sum_{\ell=1}^{\jmath_j}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j-1;j-1,k}\rangle \psi^\ell_{j-1;j-1,k} = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j;j,k}\rangle \phi_{j;j,k} - \sum_{k\in\mathbb{Z}} \langle f, \phi_{j-1;j-1,k}\rangle \phi_{j-1;j-1,k} \tag{3.2}$$

for $f \in L_2(\mathbb{R})$. Consequently, by the definition of the linear operators Q_n in (1.9), it follows from the relation in (3.2) that

$$Q_n(f) = \sum_{k\in\mathbb{Z}} \langle f, \phi_0(\cdot - k)\rangle \phi_0(\cdot - k) + \sum_{j=0}^{n-1}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j;j,k}\rangle \psi^\ell_{j;j,k} = \sum_{k\in\mathbb{Z}} \langle f, \phi_{n;n,k}\rangle \phi_{n;n,k} = P_n(f).$$

That is, if (1.12) holds, then the linear operators Q_n in (1.9) and P_n in (3.1) are the same.

The approximation order of Q_n's for a stationary tight wavelet frame is investigated in [14] through that of P_n's, since $Q_n = P_n$ for a tight frame system constructed from the unitary extension principle. The relationship has been studied in [14] for stationary tight wavelet frames between the approximation order of P_n's and the (best) approximation order provided by the spaces $S_n(\phi_n)$, where $S_n(\phi_n)$ is the smallest closed subspace of $L_2(\mathbb{R})$ generated by the linear span of $\phi_n(2^n \cdot -k)$, $k \in \mathbb{Z}$, that is, $S_n(\phi_n)$ is the same as the smallest closed subspace of $L_2(\mathbb{R})$ containing the truncated tight frame system $\{\phi_0(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j,k} \ : \ k \in \mathbb{Z}, 0 \leqslant j < n, \ell = 1, \dots, \jmath_{j+1}\}$. It is well-known that approximation order provided by the spaces $S_n(\phi_n)$ is determined by the order of the Strang-Fix conditions satisfied by ϕ_n. However, the approximation order of P_n's is determined by the order of the zero at the origin of the function $1 - |\widehat{\phi}|^2$, in addition to the order of the Strang-Fix conditions satisfied by ϕ_n. Consequently, the frame approximation order can be (much) smaller than the approximation order provided by the spaces $S_n(\phi_n)$ (see [14] for details). This is also true for the nonstationary case. For example, let $\widehat{a_j}(\xi) := 2^{-j}(1+e^{-i\xi})^j$, $j \in \mathbb{N}$, be the masks for the up-functions. Then, Item (iii) of Theorem 1.3 of this paper says that it does not have any "strong" frame approximation order in the sense of (1.16), i.e., for any given $\nu > 0$, there does not exist a positive constant C such that (1.16) is satisfied. But one can check that the corresponding spaces $S_n(\phi_n)$ provide a spectral approximation order.

If the integer shifts of ϕ_n are orthonormal, then the linear operator P_n in (3.1) becomes an orthogonal projection from $L_2(\mathbb{R})$ to $S_n(\phi_n)$. That is, for this case, $P_n(f)$ is the best approximation of $f \in L_2(\mathbb{R})$ in the closed subspace $S_n(\phi_n)$ of $L_2(\mathbb{R})$. This is the reason why the approximation order of Q_n's is identified with the best approximation order provided by the spaces $S_n(\phi_n)$ in [10] which is simpler to understand, since only orthonormal wavelets are studied in [10].

To summarize our discussion here, the understanding of the approximation order of the approximation operators Q_n for a given tight wavelet frame is necessary, since it is simple and used in applications such as image inpainting, and since unlike orthonormal wavelets, the approximation order of a truncated tight frame series is not necessarily the same as the best approximation order provided by the underlying nonstationary multiresolution analysis.

The main result of this section is Theorem 3.2, which is interesting in its own right and is independent of its role in our proofs of some of major parts of Theorems 1.1, 1.2, 1.3 and 1.4.

The following result can be directly obtained by applying Jetter and Zhou [28] and [29, Theorem 2.1].

Proposition 3.1. *Let $\varphi \in L_2(\mathbb{R})$ and $\nu \geqslant 0$. Define a linear operator P by*

$$P(f) := \sum_{k\in\mathbb{Z}} \langle f, \varphi(\cdot - k)\rangle \varphi(\cdot - k), \qquad f \in L_2(\mathbb{R}).$$

Then $\|f - P(f)\|_{L_2(\mathbb{R})} \leqslant C_\varphi |f|_{W_2^\nu(\mathbb{R})}$ for all $f \in W_2^\nu(\mathbb{R})$ with a positive constant

$$C_\varphi := \pi^{-1/2}\sqrt{\max(c_1, c_3) + \max(2c_2, 2c_4 + 1)}, \tag{3.3}$$

provided that there exist positive constants c_1, c_2, c_3, c_4 such that for almost every $\xi \in [-\pi, \pi]$, the following inequalities hold

$$\big|1 - |\hat\varphi(\xi)|^2\big|^2 \leqslant c_1|\xi|^{2\nu}, \tag{3.4}$$

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |\hat\varphi(\xi)|^2 |\hat\varphi(\xi + 2\pi k)|^2 \leqslant c_2|\xi|^{2\nu}, \tag{3.5}$$

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |\xi + 2\pi k|^{-2\nu} |\hat\varphi(\xi)|^2 |\hat\varphi(\xi + 2\pi k)|^2 \leqslant c_3, \tag{3.6}$$

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |\xi + 2\pi k|^{-2\nu} \sum_{\ell\in\mathbb{Z}\backslash\{0\}} |\hat\varphi(\xi + 2\pi \ell)|^2 |\hat\varphi(\xi + 2\pi k)|^2 \leqslant c_4. \tag{3.7}$$

Next, we present the following result on the approximation properties of the operators P_n defined in (3.1).

Theorem 3.2. *Let $\widehat{a_j}, j \in \mathbb{N}$, be 2π-periodic measurable functions such that* (1.3) *holds for all $j \in \mathbb{N}$ and for every $n \in \mathbb{N}_0$, the function $\widehat{\phi_n}(\xi) := \lim_{J\to\infty} \prod_{j=1}^J \widehat{a_{j+n}}(2^{-j}\xi)$ is well-defined for almost every $\xi \in \mathbb{R}$. Let $\nu \geqslant 0$. If for $n \in \mathbb{N}$,*

$$\begin{aligned} \Big|1 - |\widehat{\phi_n}(\xi)|^2\Big|^2 &\leqslant C_{\phi_n}|\xi|^{2\nu}, \qquad a.e.\ \xi \in [-\pi, \pi],\\ \sum_{k\in\mathbb{Z}\backslash\{0\}} |\widehat{\phi_n}(\xi)|^2 |\widehat{\phi_n}(\xi + 2\pi k)|^2 &\leqslant C_{\phi_n}|\xi|^{2\nu}, \qquad a.e.\ \xi \in [-\pi, \pi], \end{aligned} \tag{3.8}$$

where C_{ϕ_n} is a constant depending only on ϕ_n, then for the linear operators P_n in (3.1)*,*

$$\|f - P_n(f)\|_{L_2(\mathbb{R})} \leqslant \max(2, \sqrt{C_{\phi_n}})\, 2^{-\nu n} |f|_{W_2^\nu(\mathbb{R})} \qquad \forall\, f \in W_2^\nu(\mathbb{R}) \quad and \quad n \in \mathbb{N}. \tag{3.9}$$

In particular, (3.8) *is satisfied if*

$$1 - |\widehat{\phi_n}(\xi)|^2 \leqslant C_{\phi_n}|\xi|^{2\nu}, \qquad a.e.\ \xi \in [-\pi, \pi]. \tag{3.10}$$

Proof. By (1.3) and Lemma 2.2, we have $\phi_n \in L_2(\mathbb{R})$ for all $n \in \mathbb{N}_0$. For each fixed $n \in \mathbb{N}_0$, we denote $P_{n,0}$ the following linear operator on $L_2(\mathbb{R})$:

$$P_{n,0}(f) := \sum_{k\in\mathbb{Z}} \langle f, \phi_n(\cdot - k)\rangle \phi_n(\cdot - k), \qquad f \in L_2(\mathbb{R}).$$

It is apparent ([28]) that the operators P_n and $P_{n,0}$ are linked through the relation $P_n(f) = [P_{n,0}(f(2^{-n}\cdot))](2^n\cdot)$ and

$$\|f - P_n(f)\|_{L_2(\mathbb{R})} = 2^{-n/2}\|f(2^{-n}\cdot) - P_{n,0}(f(2^{-n}\cdot))\|_{L_2(\mathbb{R})}. \tag{3.11}$$

Since (1.3) holds, by Lemma 2.2, we have

$$[\widehat{\phi_n}, \widehat{\phi_n}](\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\phi_n}(\xi + 2\pi k)|^2 \leqslant 1, \quad a.e.\ \xi \in \mathbb{R}.$$

In particular, we have $|\widehat{\phi_n}(\xi)| \leqslant 1$ for almost every $\xi \in \mathbb{R}$.

Since $\nu \geqslant 0$, for $\xi \in [-\pi, \pi]$, it is evident that $|\xi + 2\pi k|^{-2\nu} \leqslant 1$ for all $k \in \mathbb{Z}\backslash\{0\}$ and

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |\xi+2\pi k|^{-2\nu} \sum_{\ell\in\mathbb{Z}\backslash\{0\}} |\widehat{\phi_n}(\xi+2\pi\ell)|^2|\widehat{\phi_n}(\xi+2\pi k)|^2 \leqslant \sum_{k\in\mathbb{Z}} |\widehat{\phi_n}(\xi+2\pi k)|^2 \sum_{\ell\in\mathbb{Z}} |\widehat{\phi_n}(\xi+2\pi\ell)|^2 \leqslant 1.$$

Hence, (3.7) holds with $c_4 = 1$ and $\varphi = \phi_n$. Similarly, for $\xi \in [-\pi, \pi]$, we have

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |\xi + 2\pi k|^{-2\nu}|\widehat{\phi_n}(\xi)|^2|\widehat{\phi_n}(\xi + 2\pi k)|^2 \leqslant \sum_{k\in\mathbb{Z}} |\widehat{\phi_n}(\xi + 2\pi k)|^2 \leqslant 1.$$

Therefore, (3.6) holds with $c_3 = 1$ and $\varphi = \phi_n$.

By (3.8), we see that (3.4) and (3.5) hold with $c_1 = c_2 = C_{\phi_n}$ and $\varphi = \phi_n$.

Note that $|f(2^{-n}\cdot)|_{W_2^\nu(\mathbb{R})} = 2^{n/2}2^{-\nu n}|f|_{W_2^\nu(\mathbb{R})}$. Therefore, by Proposition 3.1 and (3.11), we conclude that for all $f \in W_2^\nu(\mathbb{R})$,

$$\|f - P_n(f)\|_{L_2(\mathbb{R})} = 2^{-n/2}\|f(2^{-n}\cdot) - P_{n,0}(f(2^{-n}\cdot))\|_{L_2(\mathbb{R})} \leqslant \max(2, \sqrt{C_{\phi_n}})2^{-\nu n}|f|_{W_2^\nu(\mathbb{R})}, \quad (3.12)$$

since

$$\max(c_1, c_3) + \max(2c_2, 2c_4 + 1) = \max(C_{\phi_n}, 1) + \max(2C_{\phi_n}, 3) \leqslant \max(3C_{\phi_n}, 6) \leqslant \pi \max(C_{\phi_n}, 4).$$

Therefore, (3.9) is verified.

If (3.10) holds, then

$$\left|1 - |\widehat{\phi_n}(\xi)|^2\right|^2 \leqslant C_{\phi_n}^2|\xi|^{2\nu}|\xi|^{2\nu} \leqslant C_{\phi_n}|\xi|^{2\nu}, \qquad |\xi| \leqslant C_{\phi_n}^{-\frac{1}{2\nu}}$$

and by $|\widehat{\phi_n}(\xi)| \leqslant 1$,

$$\left|1 - |\widehat{\phi_n}(\xi)|^2\right|^2 \leqslant 1 = C_{\phi_n}[C_{\phi_n}^{-\frac{1}{2\nu}}]^{2\nu} \leqslant C_{\phi_n}|\xi|^{2\nu}, \qquad |\xi| \geqslant C_{\phi_n}^{-\frac{1}{2\nu}}.$$

Also, by $|\widehat{\phi_n}(\xi)|^2 \leqslant [\widehat{\phi_n}, \widehat{\phi_n}](\xi) \leqslant 1$ and the above two inequalities, we have

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |\widehat{\phi_n}(\xi)|^2|\widehat{\phi_n}(\xi + 2\pi k)|^2 \leqslant \sum_{k\in\mathbb{Z}\backslash\{0\}} |\widehat{\phi_n}(\xi + 2\pi k)|^2 = [\widehat{\phi_n}, \widehat{\phi_n}](\xi) - |\widehat{\phi_n}(\xi)|^2 \leqslant 1 - |\widehat{\phi_n}(\xi)|^2.$$

Consequently, (3.10) implies (3.8). ∎

The behavior of $1 - |\widehat{\phi_n}(\xi)|^2$ near the origin $\xi = 0$ in (3.10) is closely related to that of the masks $1 - |\widehat{a_j}(\xi)|^2$ for all $j \in \mathbb{N}$ near the origin $\xi = 0$. As we will see in the proof of the next section, when only masks are available and the nonstationary refinable functions are not explicitly given, one can use the estimate of $1 - |\widehat{a_j}(\xi)|^2$ near $\xi = 0$ for all $j \in \mathbb{N}$ to obtain the estimate of $1 - |\widehat{\phi_n}(\xi)|^2$ near $\xi = 0$. The following result provides the estimate of $1 - |\widehat{a_j}(\xi)|^2$ near the origin for the masks of pseudo-splines, which will be needed later in our proof of the spectral frame approximation order in Theorems 1.2 and 1.4.

Lemma 3.3. *Let $\widehat{a^I_{m,l}}$ and $\widehat{a_{m,l}}$ be pseudo-spline masks of type I and II of order (m, l) defined in* (1.19) *and* (1.20)*, respectively. For any $0 < \rho \leqslant 1$ and $\nu \geqslant 0$, there exist a positive integer N and a positive constant C, (both of them depend only on ρ and ν,) such that for all $N \leqslant \rho m < l \leqslant m$,*

$$0 \leqslant 1 - |\widehat{a^I_{m,l}}(\xi)|^2 \leqslant 1 - |\widehat{a_{m,l}}(\xi)|^2 \leqslant C|\xi|^{2\nu} \qquad \forall\, \xi \in [-\pi, \pi]. \quad (3.13)$$

Proof. Since $|\widehat{a^I_{m,l}}(\xi)|^2 = \widehat{a_{m,l}}(\xi) \leqslant 1$, we have

$$0 \leqslant 1 - |\widehat{a^I_{m,l}}(\xi)|^2 \leqslant 1 - |\widehat{a_{m,l}}(\xi)|^2 = [1 + \widehat{a_{m,l}}(\xi)][1 - \widehat{a_{m,l}}(\xi)] \leqslant 2[1 - \widehat{a_{m,l}}(\xi)].$$

Setting $x = \sin^2(\xi/2)$, by the definition of $\widehat{a_{m,l}}(\xi)$ in (1.19), we have $\widehat{a_{m,l}}(\xi) = (1 - x)^m P_{m,l}(x)$, where the polynomial $P_{m,l}$ is defined in (1.18). In order to prove (3.13), now it is easy to see that it is equivalent to proving that for any $0 < \rho \leqslant 1$ and any positive integer ν, there exist a positive integer N and a positive constant C, all depending only on ρ and ν, such that

$$1-(1-x)^m P_{m,l}(x) \leqslant C x^\nu, \qquad \forall\, x \in [0,1] \quad \text{and} \quad N \leqslant \rho m < l \leqslant m. \tag{3.14}$$

Since $(1-x)^m P_{m,m}(x) + x^m P_{m,m}(1-x) = 1$, we deduce that

$$\begin{aligned} 1-(1-x)^m P_{m,l}(x) &= (1-x)^m P_{m,m}(x) + x^m P_{m,m}(1-x) - (1-x)^m P_{m,l}(x) \\ &= x^m P_{m,m}(1-x) + (1-x)^m \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^j. \\ &= x^\nu \left[x^{m-\nu} P_{m,m}(1-x) + (1-x)^m \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^{j-\nu} \right]. \end{aligned}$$

By Lemma 2.7, (2.24) holds. In particular, replacing x by $1-x$ in (2.24), we conclude that

$$x^{m-\nu} P_{m,m}(1-x) \leqslant x^{N-\nu-1} P_{N,N}(1-x) \qquad \forall\, x \in [0,1] \quad \text{and} \quad m \geqslant N \geqslant \nu+1.$$

Therefore, on the one hand, we have

$$x^{m-\nu} P_{m,m}(1-x) \leqslant C_N := \max_{x\in[0,1]} x^{N-\nu-1} P_{N,N}(1-x) < \infty \qquad \forall\, x \in [0,1],\ m \geqslant N \geqslant \nu+1. \tag{3.15}$$

On the other hand, for $x \in [0,1]$, we have

$$\begin{aligned} (1-x)^m \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^{j-\nu} &= (1-x)^m x^{l-\nu} \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^{j-l} \\ &\leqslant (1-x)^m x^{l-\nu} \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} \\ &= (1-x)^m x^{l-\nu} \sum_{j=l}^{m-1} \left[\frac{(m+j)!}{j!m!} - \frac{(m+j-1)!}{(j-1)!m!} \right] \\ &\leqslant (1-x)^m x^{l-\nu} \frac{(2m-1)!}{(m-1)!m!}. \end{aligned}$$

By the Stirling's formula, we have $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} = 1$. So, when m is large enough and for all $x \in [0,1]$, we have

$$\frac{(2m-1)!}{(m-1)!m!} = \frac{1}{2} \frac{(2m)!}{m!m!} \leqslant \frac{1}{\sqrt{2\pi}} \frac{(2m)^{2m+1/2} e^{-2m}}{m^{2m+1} e^{-2m}} \leqslant 4^m m^{-1/2}.$$

So, for large enough m, we have

$$(1-x)^m \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^{j-\nu} \leqslant (1-x)^m x^{l-\nu} 4^m m^{-1/2} = [4(1-x)x^{(l-\nu)/m}]^m m^{-1/2}.$$

Since $\rho m < l$, we have $l/m > \rho$. Since ν is fixed, when m is large enough, we have $(l-\nu)/m \geqslant \rho$ and hence, $x^{(l-\nu)/m} \leqslant x^\rho$ for all $x \in [0,1]$. Therefore, we have

$$(1-x)^m \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^{j-\nu} \leqslant [4(1-x)x^\rho]^m m^{-1/2} \qquad \forall\, x \in [0,1].$$

Note that $\rho > 0$. The continuous function $(1-x)x^\rho$ has only one critical point $x = \frac{\rho}{1+\rho}$ on the interval $(0,1)$ and it takes value zero at $x = 0$. So, we can choose $0 < \tau \leqslant 1$ such that $4(1-x)x^\rho \leqslant 1$ for all $x \in [0,\tau]$; one may prefer to choose such τ as large as possible, in particular, τ may be obtained by solving $4(1-\tau)\tau^\rho = 1$. Therefore, there exists a positive integer N such that for $N \leqslant \rho m < l \leqslant m$,

$$(1-x)^m \sum_{j=l}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} x^{j-\nu} \leqslant [4(1-x)x^\rho]^m m^{-1/2} \leqslant m^{-1/2} \leqslant 1 \qquad \forall\, x \in [0,\tau]. \tag{3.16}$$

Combining (3.15) and (3.16), we conclude that

$$1-(1-x)^m P_{m,l}(x) \leqslant x^\nu [C_N + 1] \qquad \forall\, x \in [0,\tau],\ N \leqslant \rho m < l \leqslant m.$$

Observe that $0 \leqslant (1-x)^m P_{m,l}(x) \leqslant 1$ for all $x \in [0,1]$, we deduce that

$$1-(1-x)^m P_{m,l}(x) \leqslant 1 = x^{-\nu} x^\nu \leqslant \tau^{-\nu} x^\nu \qquad \forall\, x \in [\tau,1],\ 1 \leqslant l \leqslant m, m \in \mathbb{N}.$$

Therefore, (3.14) is verified with $C := \max(C_N + 1, \tau^{-\nu})$. This completes the proof. ■

4. Proofs of Theorems 1.1, 1.2, 1.3 and 1.4

In this section, we shall prove Theorems 1.1, 1.2, 1.3 and 1.4. We start with a proof to Theorem 1.1.

Proof of Theorem 1.1. By our assumption in Theorem 1.1, Item (i) follows from Lemmas 2.1 and 2.2. Since (1.3) holds, by Lemma 2.2, we have $[\widehat{\phi_n}, \widehat{\phi_n}] \leqslant 1$ and $\phi_n \in L_2(\mathbb{R})$. Therefore, the linear operators Q_n in (1.9) and P_n in (3.1) are well-defined, bounded and the same (see Section 3).

Let us first prove (1.15) in Item (iii). In order to do so, in the following, we estimate the constants C_{ϕ_n} in (3.10). Denote $\widehat{d_j}(\xi) := |\widehat{a_j}(\xi)|^2$ for $j \in \mathbb{N}$. Since $\widehat{a_j}(0) = 1$, we have $\widehat{d_j}(0) = 1$ for all $j \in \mathbb{N}$. By our assumption and Lemma 2.1, we have $|\widehat{\phi_n}(\xi)|^2 = \prod_{j=1}^\infty \widehat{d_{j+n}}(2^{-j}\xi)$ for all $\xi \in \mathbb{R}$. Therefore, we have

$$\begin{aligned} 1 - |\widehat{\phi_n}(\xi)|^2 &= \prod_{j=1}^\infty \widehat{d_{j+n}}(0) - \prod_{j=1}^\infty \widehat{d_{j+n}}(2^{-j}\xi) \\ &= \sum_{\ell=1}^\infty \Big[\prod_{j=1}^{\ell-1} \widehat{d_{j+n}}(0)\Big] \big[\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\xi)\big] \Big[\prod_{j=\ell+1}^\infty \widehat{d_{j+n}}(2^{-j}\xi)\Big]. \end{aligned}$$

Since $\widehat{d_{j+n}}(0) = 1$ and $0 \leqslant \widehat{d_{j+n}}(\xi) \leqslant 1$ by (1.3), we conclude that

$$0 \leqslant 1 - |\widehat{\phi_n}(\xi)|^2 \leqslant \sum_{\ell=1}^\infty |\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\xi)|, \qquad \xi \in \mathbb{R}. \tag{4.1}$$

Since $\widehat{a_j}$ is a 2π-periodic trigonometric polynomial, by the definition of $\widehat{d_j}$, we see that $\widehat{d_j}$ is a real-valued C^∞ function. Note that by our assumption, 2ν is a positive integer. Therefore, for $\xi \in [-\pi,\pi]$, there exists $\zeta_{\xi,j} \in [-\pi,\pi]$ such that

$$\widehat{d_j}(\xi) = \widehat{d_j}(0) + \frac{\widehat{d_j}^{(1)}(0)}{1!}\xi + \cdots + \frac{\widehat{d_j}^{(2\nu-1)}(0)}{(2\nu-1)!}\xi^{2\nu-1} + \frac{\widehat{d_j}^{(2\nu)}(\zeta_{\xi,j})}{(2\nu)!}\xi^{2\nu}. \tag{4.2}$$

By Bernstein's inequality and $0 \leqslant \widehat{d_j}(\xi) \leqslant 1$, we have

$$\|\widehat{d_j}^{(2\nu)}\|_{L_\infty(\mathbb{R})} \leqslant [\deg(\widehat{d_j})]^{2\nu} \|\widehat{d_j}\|_{L_\infty(\mathbb{R})} \leqslant [\deg(\widehat{d_j})]^{2\nu}.$$

By assumption in (1.14), for $j \geqslant N$, we have $\widehat{d_j}^{(\ell)}(0) = 0$ for all $\ell = 1, \ldots, 2\nu - 1$. Therefore, it follows from (4.2) that for $n \geqslant N$, $\ell \in \mathbb{N}$ and $\xi \in [-\pi,\pi]$,

$$|\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\xi)| \leqslant \frac{[\deg(\widehat{d_{\ell+n}})]^{2\nu}}{(2\nu)!} |2^{-\ell}\xi|^{2\nu} \leqslant \frac{1}{(2\nu)!} |\xi|^{2\nu} 2^{-2\nu\ell} [\deg(\widehat{d_{\ell+n}})]^{2\nu}.$$

Therefore, we have

$$\sum_{\ell=1}^{\infty} |\widehat{d_{\ell+n}}(0)-\widehat{d_{\ell+n}}(2^{-\ell}\xi)| \leqslant \frac{1}{(2\nu)!}|\xi|^{2\nu}\sum_{\ell=1}^{\infty} 2^{-2\nu\ell}[\deg(\widehat{d_{\ell+n}})]^{2\nu}.$$

That is, by (4.1), we see that for every $n \geqslant N$, (3.10) holds with

$$C_{\phi_n} := \frac{1}{(2\nu)!}\sum_{\ell=1}^{\infty} 2^{-2\nu\ell}[\deg(\widehat{d_{\ell+n}})]^{2\nu}. \tag{4.3}$$

Now we estimate C_{ϕ_n} using the condition in (1.13). By (1.13), there exists a positive constant C_1 such that

$$\deg(\widehat{a_j}) \leqslant C_1 j^\alpha 2^{\beta j} \qquad \forall\, j \in \mathbb{N}. \tag{4.4}$$

By the definition of $\widehat{d_j}$, we have $\deg(\widehat{d_j}) \leqslant 2\deg(\widehat{a_j})$ for all $j \in \mathbb{N}$. Therefore, from (4.4), we deduce that

$$\begin{aligned}2^{-2\nu\ell}[\deg(\widehat{d_{\ell+n}})]^{2\nu} &\leqslant 2^{2\nu}C_1^{2\nu}(\ell+n)^{2\nu\alpha}2^{2\nu\beta(\ell+n)}2^{-2\nu\ell}\\ &= 2^{2\nu}C_1^{2\nu}n^{2\nu\alpha}2^{2\nu\beta n}(1+\ell/n)^{2\nu\alpha}2^{-2\nu(1-\beta)\ell}\\ &\leqslant 2^{2\nu}C_1^{2\nu}n^{2\nu\alpha}2^{2\nu\beta n}(1+\ell)^{2\nu\alpha}2^{-2\nu(1-\beta)\ell}.\end{aligned}$$

Consequently, we have the following estimate for the constant C_{ϕ_n}:

$$C_{\phi_n} = \frac{1}{(2\nu)!}\sum_{\ell=1}^{\infty} 2^{-2\nu\ell}[\deg(\widehat{d_{\ell+n}})]^{2\nu} \leqslant \frac{2^{2\nu}C_1^{2\nu}}{(2\nu)!}n^{2\nu\alpha}2^{2\nu\beta n}\sum_{\ell=1}^{\infty}(1+\ell)^{2\nu\alpha}2^{-2\nu(1-\beta)\ell} = C_2 n^{2\nu\alpha}2^{2\nu\beta n},$$

where $C_2 := 2^{2\nu}C_1^{2\nu}[(2\nu)!]^{-1}\sum_{\ell=1}^{\infty}(1+\ell)^{2\nu\alpha}2^{-2\nu(1-\beta)\ell} < \infty$, since $1-\beta > 0$ and $\nu > 0$. Since $Q_n = P_n$, by Theorem 3.2, we conclude that

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant \max(2, \sqrt{C_2})n^{\nu\alpha}2^{-\nu(1-\beta)n}|f|_{W_2^\nu(\mathbb{R})} \qquad \forall\, f \in W_2^\nu(\mathbb{R}) \quad \text{and} \quad n \geqslant N.$$

That is, (1.15) holds with $C := \max(2, \sqrt{C_2}) < \infty$, which is independent of f and n.

Now we prove Item (ii). In order to show that $\{\phi_0(\cdot - k) \,:\, k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j,k} \,:\, j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, \dots, \mathfrak{J}_{j+1}\}$ is a tight frame of $L_2(\mathbb{R})$, since $Q_n = P_n$, now it suffices to show that

$$\lim_{n\to\infty}\|f - Q_n(f)\|_{L_2(\mathbb{R})} = \lim_{n\to\infty}\|f - P_n(f)\|_{L_2(\mathbb{R})} = 0 \qquad \forall\, f \in L_2(\mathbb{R}).$$

Since $\widehat{a_j}(0) = 1$ and $\widehat{d_j}(\xi) = |\widehat{a_j}(\xi)|^2$, it is evident that $\widehat{d_j}(0) = 1$. Since $\widehat{d_j}$ is a 2π-periodic trigonometric polynomial, the condition in (1.14) is automatically satisfied with $\nu = 1/2$.

Now from the above proof of Item (iii) and by Theorem 3.2, we see that (3.9) holds with the constant C_{ϕ_n} defined in (4.3) and $\nu = 1/2$. More precisely, by Theorem 3.2, for $\nu = 1/2$, we have

$$\|f - P_n(f)\|^2_{L_2(\mathbb{R})} \leqslant C_n|f|^2_{W_2^{1/2}(\mathbb{R})} \qquad \forall\, f \in W_2^{1/2}(\mathbb{R}) \tag{4.5}$$

with

$$C_n := \max(4, C_{\phi_n})2^{-n} \quad \text{and} \quad C_{\phi_n} := \sum_{\ell=1}^{\infty} 2^{-\ell}\deg(\widehat{d_{\ell+n}}). \tag{4.6}$$

Now we prove that $\lim_{n\to\infty} C_n = 0$ by showing $\lim_{n\to\infty} 2^{-n}C_{\phi_n} = 0$. Note that

$$2^{-n}C_{\phi_n} = \sum_{\ell=1}^{\infty} 2^{-(\ell+n)}\deg(\widehat{d_{\ell+n}}) = \sum_{j=n+1}^{\infty} 2^{-j}\deg(\widehat{d_j}). \tag{4.7}$$

Since $\deg(\widehat{d_j}) \leqslant 2\deg(\widehat{a_j})$, by our assumption in (1.4), we have

$$\sum_{j=1}^{\infty} 2^{-j}\deg(\widehat{d_j}) \leqslant 2\sum_{j=1}^{\infty} 2^{-j}\deg(\widehat{a_j}) < \infty.$$

Consequently, by (4.7), we conclude

$$0 \leqslant \lim_{n\to\infty} 2^{-n} C_{\phi_n} \leqslant \lim_{n\to\infty} \sum_{j=n+1}^{\infty} 2^{-j} \deg(\widehat{d_j}) = 0.$$

That is, $\lim_{n\to\infty} 2^{-n} C_{\phi_n} = 0$. Thus, we have $\lim_{n\to\infty} C_n = \lim_{n\to\infty} \max(4, C_{\phi_n}) 2^{-n} = 0$. Now from (4.5), we see that

$$\lim_{n\to\infty} \|f - P_n(f)\|^2_{L_2(\mathbb{R})} = \lim_{n\to\infty} C_n |f|^2_{W_2^{1/2}(\mathbb{R})} = 0 \qquad \forall\, f \in W_2^{1/2}(\mathbb{R}).$$

Since $P_n = Q_n$ in (3.1), we conclude that

$$\|f\|^2_{L_2(\mathbb{R})} = \lim_{n\to\infty} \langle Q_n(f), f\rangle = \sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\mathfrak{J}_{j+1}} \sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{j;j,k}\rangle|^2 \qquad \forall\, f \in W_2^{1/2}(\mathbb{R}).$$

Since $W_2^{1/2}(\mathbb{R})$ is dense in $L_2(\mathbb{R})$, (1.2) must hold for all $f \in L_2(\mathbb{R})$. Therefore, $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, \dots, \mathfrak{J}_{j+1}\}$ is a tight frame of $L_2(\mathbb{R})$. ■

Next, we prove Theorem 1.2:

Proof of Theorem 1.2. For Item (1), applying Theorem 2.8, we conclude that all $\phi_j, j \in \mathbb{N}_0$, are compactly supported functions in $C^\infty(\mathbb{R})$. Since all the masks $\widehat{a_j}$ are 2π-periodic trigonometric polynomials with real coefficients and are symmetric about the origin: $\overline{\widehat{a_j}(\xi)} = \widehat{a_j}(\xi)$, by the definition of $\widehat{\phi_j}$ in (1.1), it is straightforward to see that all $\phi_j, j \in \mathbb{N}_0$, are real-valued and $\overline{\widehat{\phi_j}(\xi)} = \widehat{\phi_j}(\xi)$; that is, all $\phi_j, j \in \mathbb{N}_0$, are symmetric about the origin.

By the definition of $\widehat{a^I_{m,l}}$ and $\widehat{a_{m,l}}$, we have

$$1 - |\widehat{a^I_{m,l}}(\xi)|^2 = 1 + O(|\xi|^{2l}) \quad \text{and} \quad 1 - |\widehat{a_{m,l}}(\xi)|^2 = 1 + O(|\xi|^{2l}), \qquad \xi \to 0. \tag{4.8}$$

By (1.12) and (4.8), we see that $\widehat{b^\ell_j}(\xi) = O(|\xi|^{l_j})$ as $\xi \to 0$. Therefore, ψ^ℓ_j has l_{j+1} vanishing moments. So, Item (2) holds.

For Item (3), by the definition of $\widehat{b^\ell_j}$ in (1.17), it is straightforward to check that (1.12) holds with $\mathfrak{J}_j = 3$ for all $j \in \mathbb{N}$. Now by Theorem 1.1, we see that $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi^\ell_{j;j;k} : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, \dots, \mathfrak{J}_{j+1}\}$ is a tight wavelet frame in $L_2(\mathbb{R})$.

Now we prove Item (4) by using Theorem 3.2 and Lemma 3.3. Let ν be an arbitrary positive integer. Since $\liminf_{j\to\infty} l_j/m_j > 0$, there exist a positive integer N and $0 < \rho < \liminf_{j\to\infty} l_j/m_j$ such that $2\nu < N < \rho m_j < l_j \leqslant m_j$ for all $j \geqslant N$. Denote $\widehat{d_j}(\xi) := |\widehat{a_j}(\xi)|^2$. By Lemma 3.3, we see that (3.13) holds. That is, there exists a positive constant C, independent of j, such that

$$0 \leqslant 1 - \widehat{d_j}(\xi) \leqslant C|\xi|^{2\nu}, \qquad \xi \in [-\pi, \pi] \quad \text{and} \quad j \geqslant N. \tag{4.9}$$

We now use (4.9) to estimate the constants C_{ϕ_n} in (3.10) of Theorem 3.2. For $n \geqslant N$ and $\ell \in \mathbb{N}$, since $\widehat{d_{\ell+n}}(0) = 1$, it follows from (4.9) that

$$|\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\xi)| = |1 - \widehat{d_{\ell+n}}(2^{-\ell}\xi)| \leqslant C 2^{-2\nu\ell} |\xi|^{2\nu} \qquad \forall\, \xi \in [-\pi, \pi].$$

Now by (4.1), we conclude that

$$1 - |\widehat{\phi_n}(\xi)|^2 \leqslant C|\xi|^{2\nu} \sum_{\ell=1}^{\infty} 2^{-2\nu\ell}, \qquad \xi \in [-\pi, \pi].$$

Therefore, (3.10) holds with

$$C_{\phi_n} := C \sum_{\ell=1}^{\infty} 2^{-2\nu\ell} = \frac{C}{1 - 2^{-2\nu}} < \infty.$$

Consequently, by $Q_n = P_n$ and Theorem 3.2, we conclude that

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant C_1 2^{-\nu n} |f|_{W_2^\nu(\mathbb{R})} \qquad \forall\, f \in W_2^\nu(\mathbb{R}) \quad \text{and} \quad n \geqslant N,$$

where $C_1 := \max(2, \sqrt{C/(1-2^{-2\nu})}) < \infty$ is independent of f and n. Since ν is arbitrary, the tight wavelet frame has the desired spectral frame approximation order. ∎

Now we prove Theorem 1.3:

Proof of Theorem 1.3. Item (i) has been proved in Theorem 1.2. It is evident that

$$\deg(\widehat{a_j}) = j \quad \text{and} \quad |\widehat{a_j}(\xi)|^2 = \cos^{2j}(\xi/2) = 1 + O(|\xi|^2), \qquad \xi \to 0.$$

Now it follows from the proof of Item (iii) of Theorem 1.1 that there exists a positive constant C_1 such that

$$1 - |\widehat{\phi_n}(\xi)|^2 \leqslant C_1 n^2 |\xi|^2 \qquad \forall\, \xi \in [-\pi, \pi] \quad \text{and} \quad n \in \mathbb{N}.$$

That is, we conclude that

$$\Big| 1 - |\widehat{\phi_n}(\xi)|^2 \Big|^2 \leqslant C_1^2 n^4 |\xi|^4 \qquad \forall\, \xi \in [-\pi, \pi] \quad \text{and} \quad n \in \mathbb{N}. \tag{4.10}$$

Let B_2 be the B-spline of order 2. Then

$$|\widehat{B_2}(\xi)|^2 = \frac{\sin^4(\xi/2)}{(\xi/2)^4} \quad \text{and} \quad |\widehat{B_2}(2\xi)|^2 = \cos^4(\xi/2) |\widehat{B_2}(\xi)|^2.$$

Since $|\widehat{a_j}(\xi)|^2 = \cos^{2j}(\xi/2) \leqslant \cos^4(\xi/2)$ for all $\xi \in \mathbb{R}$ and $j \geqslant 2$, it is evident that $|\widehat{\phi_n}(\xi)|^2 \leqslant |\widehat{B_2}(\xi)|^2$ for all $\xi \in \mathbb{R}$ and $n \geqslant 2$. In particular, for $\xi \in [-\pi, \pi]$, we deduce that

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} |\widehat{\phi_n}(\xi)|^2 |\widehat{\phi_n}(\xi + 2\pi k)|^2 \leqslant \sum_{k \in \mathbb{Z}\setminus\{0\}} |\widehat{B_2}(\xi + 2\pi k)|^2 = |\xi|^4 \frac{\sin^4(\xi/2)}{(\xi/2)^4} \sum_{k \in \mathbb{Z}\setminus\{0\}} |\xi + 2\pi k|^{-4}.$$

Setting $C_2 := \sup_{\xi \in [-\pi,\pi]} \sum_{k \in \mathbb{Z}\setminus\{0\}} |\xi + 2\pi k|^{-4} < \infty$, we conclude that

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} |\widehat{\phi_n}(\xi)|^2 |\widehat{\phi_n}(\xi + 2\pi k)|^2 \leqslant C_2 |\xi|^4, \qquad \xi \in [-\pi, \pi] \quad \text{and} \quad n \geqslant 2.$$

Now taking into account (4.10), we see that the two inequalities in (3.8) holds with $\nu = 2$ and $C_{\phi_n} := \max(C_1, C_2) n^4$. So, by Theorem 3.2, we see that Item (ii) holds.

Now we prove Item (iii) using proof by contradiction. Suppose that (1.16) holds for some $\nu > 0$. By [29, Theorem 2.2], we have $\big|1 - |\widehat{\phi_n}(\xi)|^2\big|^2 \leqslant \pi C^2 |\xi|^{2\nu}$ for almost every $\xi \in [-\pi, \pi]$ and $n \geqslant N$, where C is the positive constant in (1.16). That is, we must have

$$1 - |\widehat{\phi_j}(\xi)|^2 \leqslant C_3 |\xi|^\nu \qquad \forall\, \xi \in [-\pi, \pi] \quad \text{and} \quad j \geqslant N, \tag{4.11}$$

where $C_3 := \sqrt{\pi} C$ is a positive constant independent of j.

By the definition of $\widehat{\phi_j}$ in (1.1) and $\widehat{a_j}(\xi) = 2^{-j}(1 + e^{-i\xi})^j$, it is evident that $\widehat{\phi_j}(\xi) = \widehat{B_j}(\xi)\widehat{\phi_0}(\xi)$, where B_j is the B-spline of order j. Since $|\widehat{\phi_0}(\xi)| \leqslant 1$, by $\widehat{\phi_j}(\xi) = \widehat{B_j}(\xi)\widehat{\phi_0}(\xi)$, we have $|\widehat{\phi_j}(\xi)| \leqslant |\widehat{B_j}(\xi)|$ and therefore, It follows from (4.11) that

$$1 - |\widehat{B_j}(\xi)|^2 \leqslant 1 - |\widehat{\phi_n}(\xi)|^2 \leqslant C_3 |\xi|^\nu \qquad \forall\, \xi \in [-\pi, \pi] \quad \text{and} \quad j \geqslant N. \tag{4.12}$$

Since $|\widehat{B_j}(\xi)|^2 = \cos^{2j}(\xi/4) |\widehat{B_j}(\xi/2)|^2$ and $|\widehat{B_j}(\xi)| \leqslant 1$, we have

$$1 - |\widehat{B_j}(4\xi)|^2 = 1 - \cos^{2j}(\xi) + \cos^{2j}(\xi)(1 - |\widehat{B_j}(2\xi)|^2) \geqslant 1 - \cos^{2j}(\xi).$$

Consequently, (4.12) implies

$$\frac{1 - \cos^{2j}(\xi)}{|\xi|^\nu} \leqslant \frac{1 - |\widehat{B_j}(4\xi)|^2}{|\xi|^\nu} \leqslant C_4 \qquad \forall\, \xi \in [-\pi/4, \pi/4] \quad \text{and} \quad j \geqslant N, \tag{4.13}$$

where $C_4 := 4^\nu C_3 < \infty$. Noting $1 = [\cos^2(\xi) + \sin^2(\xi)]^j \geqslant \cos^{2j}(\xi) + j\cos^{2j-2}(\xi)\sin^2(\xi)$, by $4\pi^{-2}\xi^2 \leqslant \sin^2(\xi) \leqslant |\xi|^2$ for all $\xi \in [-\pi/2, \pi/2]$, we deduce that for $\xi \in [-\pi/4, \pi/4]$,

$$\frac{1-\cos^{2j}(\xi)}{|\xi|^\nu} \geqslant j\frac{\sin^2(\xi)\cos^{2j-2}(\xi)}{|\xi|^\nu} = j\frac{\sin^2(\xi)(1-\sin^2(\xi))^{j-1}}{|\xi|^\nu} \geqslant 4\pi^{-2}j(\xi^2)^{1-\nu/2}(1-\xi^2)^{j-1}.$$

Taking $\xi_j := \sqrt{\frac{1-\nu/2}{j-\nu/2}}$, we observe that $\lim_{j\to\infty}\xi_j = 0$ and it follows from the above inequalities and (4.13) that for $\xi \in [-\pi/4, \pi/4]$ and sufficiently large j,

$$C_4 \geqslant \frac{1-\cos^{2j}(\xi_j)}{|\xi_j|^\nu} \geqslant 4\pi^{-2}j(\xi_j^2)^{1-\nu/2}(1-\xi_j^2)^{j-1} = 4\pi^{-2}j\frac{(1-\nu/2)^{1-\nu/2}(j-1)^{j-1}}{(j-\nu/2)^{j-\nu/2}} := C_5 c_j,$$

where $0 < C_5 := 4\pi^{-2}(1-\nu/2)^{1-\nu/2} < \infty$ and $c_j := j\frac{(j-1)^{j-1}}{(j-\nu/2)^{j-\nu/2}}$. That is, we must have

$$c_j := j\frac{(j-1)^{j-1}}{(j-\nu/2)^{j-\nu/2}} \leqslant C_4/C_5 \qquad \text{for all sufficiently large integers } j. \tag{4.14}$$

By calculation, we have

$$c_j = j\frac{(j-1)^{j-1}}{(j-\nu/2)^{j-\nu/2}} = j\frac{j^{j-1}(1-\frac{1}{j})^{j-1}}{j^{j-\nu/2}(1-\frac{\nu}{2j})^{j-\nu/2}} = j^{\nu/2}\Big(1-\frac{1}{j}\Big)^{j-1}\Big(1-\frac{\nu}{2j}\Big)^{-j+\nu/2}.$$

We note that and $\lim_{j\to\infty}\left(1-\frac{\nu}{2j}\right)^{-j+\nu/2} = e^{\nu/2}$. Hence, by $\nu > 0$, we conclude that $\lim_{j\to\infty} c_j = \lim_{j\to\infty} j^{\nu/2}e^{\nu/2-1} = \infty$, which is a contradiction to (4.14). So, the tight wavelet frame does not have any frame approximation order. Now Item (iii) is verified. ■

We finish this paper by proving Theorem 1.4.

Proof of Theorem 1.4. For Item (1), by a direct calculation ([23, Lemma 3]), we observe that

$$\begin{aligned}|\widehat{a_j}(\xi)|^2 &= \cos^{2m_j}(\xi/2)([P^r_{m_j}(\sin^2(\xi/2))]^2 + [P^i_{m_j}(\sin^2(\xi/2))]^2)\\ &= \cos^{2m_j}(\xi/2)P_{m_j,m_j}(\sin^2(\xi/2)) = |\widehat{a^I_{m_j,m_j}}(\xi)|^2.\end{aligned}$$

Hence $|\widehat{a_j}(\xi)|^2 + |\widehat{a_j}(\xi+\pi)|^2 = 1$. By Theorem 2.8 and Proposition 2.6, we see that the non-stationary cascade algorithm associated with $\{\widehat{a_j}\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$ for any $\nu \geqslant 0$ and therefore, all $\phi_j, j \in \mathbb{N}_0$, are well-defined compactly supported functions in $C^\infty(\mathbb{R})$. By Theorem 2.10, (2.28) holds and $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k} : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ is an orthonormal basis of $L_2(\mathbb{R})$. By the same proof as in Theorem 1.2 and Lemma 3.3 with $l_j = m_j$, this orthonormal wavelet basis has the spectral approximation order. So, Item (3) is verified.

The symmetry $\phi_j(1-\cdot) = \phi_j$ follows ([23, Lemma 2]) from the definition of ϕ_j in (1.1) and the symmetry of the masks $\widehat{a_j}$: $\widehat{a_j}(\xi) = e^{-i\xi}\widehat{a_j}(-\xi)$. Item (2) can be easily verified by (4.8). ■

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