A Pair of Orthogonal Frames *

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Abstract

We start with a characterization of a pair of frames to be orthogonal in a shift-invariant space and then give a simple construction of a pair of orthogonal shift-invariant frames. This is applied to obtain a construction of a pair of Gabor orthogonal frames as an example. This is also developed further to obtain constructions of a pair of orthogonal wavelet frames.

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1 Introduction

Let X be a (countable) Bessel system for a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. The *synthesis operator* $T_X : \ell_2(X) \to \mathcal{H}$, which is well-known to be bounded, is defined by

$$T_X a := \sum_{h \in X} a_h h$$

for $a = (a_h)_{h \in X}$. The adjoint operator T_X^* of T_X, called the *analysis operator*, is

$$T_X^* : \mathcal{H} \to \ell_2(X); \quad T_X^* f := (\langle f, h \rangle)_{h \in X}.$$

Recall that X is a frame for $\mathcal{H}$ if and only $S_X := T_X T_X^* : \mathcal{H} \to \mathcal{H}$, the *frame operator or dual Gramian*, is bounded and has a bounded inverse [4, 8] and it is a tight frame (with frame bound 1) if and only if S_X is the identity operator. The system X is a Riesz system (for $\overline{\operatorname{span}}\, X$) if and only if its *Gramian* $G_X := T_X^* T_X$ is bounded and has a bounded inverse; and it is an orthonormal system of $\mathcal{H}$ if and only if G_X is the identity operator.

Definition 1.1 *Let X and $Y = RX$, where $R : h \to Rh$ is a bijection between X and Y, be two frames for $\mathcal{H}$. We call X and Y a pair of orthogonal frames for $\mathcal{H}$ if $T_Y T_X^* = 0$, i.e., $\sum_{h \in X} \langle f, h \rangle Rh = 0$ for all $f \in \mathcal{H}$.*

Note that the definition is symmetric with respect to X and Y. Orthogonal frames have been studied in [13] and [1]. Various applications of orthogonal frames are also discussed in both papers. We use one of examples from [13] to illustrate some ideas of applications of orthogonal frames. Let X and $Y = RX$ be a given pair of orthogonal frames for $\mathcal{H}$ such that both X and Y are also tight frames with frame bound 1 for $\mathcal{H}$. Let $f, g \in \mathcal{H}$. Suppose that the data sequence is given as $(\langle f, h \rangle + \langle g, Rh \rangle)_{h \in X}$, i.e. the data sequence is given as the sum of samples of two different elements f and g of $\mathcal{H}$. Then, since

$$f = \sum_{h \in X} (\langle f, h \rangle + \langle g, Rh \rangle)\, h \quad \text{and} \quad g = \sum_{h \in X} (\langle f, h \rangle + \langle g, Rh \rangle)\, Rh$$

we can recover both f and g from a single sequence $(\langle f, h\rangle + \langle g, Rh\rangle)_{h\in X}$. This idea can be used in multiple access communication systems.

For a pair of frames X and $Y = RX$ in $\mathcal{H}$, we have the following simple characterization of orthogonal frames via their Gramians.

Proposition 1.2 *Let X and $Y = RX$ be frames for $\mathcal{H}$ with synthesis operators T_X and T_Y, respectively. Then, X and Y are a pair of orthogonal frames for $\mathcal{H}$ if and only if $G_Y G_X = 0$.*

Proof. Suppose that $T_Y T_X^* = 0$. Then $G_Y G_X = T_Y^* T_Y T_X^* T_X = T_Y^* 0 T_X = 0$. Suppose, on the other hand, that $T_Y^* T_Y T_X^* T_X = 0$. Then

$$0 = (T_Y T_Y^*)(T_Y T_X^*)(T_X T_X^*) = S_Y (T_Y T_X^*) S_X.$$

Since $S_Y, T_Y T_X^*$ and S_X are bounded operators from $\mathcal{H}$ to $\mathcal{H}$ and since S_X and S_Y are invertible, $0 = T_Y T_X^*$. □

The paper is organized as follows: in Section 2, we discuss orthogonal frames in a general shift-invariant subspace of $L_2(\mathbb{R}^d)$, and apply the results to construct Gabor orthogonal frames. Section 3 provides a construction of wavelet orthogonal frames.

2 Orthogonal frames in a shift-invariant space

This section is devoted to the orthogonal frames in shift-invariant systems. The major tool used here is the dual Gramian analysis of [9].

2.1 Characterizations of shift-invariant orthogonal frames

We consider orthogonal frames in a shift-invariant subspace of $L_2(\mathbb{R}^d)$. Let Φ be a countable subset of $L_2(\mathbb{R}^d)$, and $E(\Phi) := \{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$. Define

$$\mathcal{S}(\Phi) := \overline{\operatorname{span}}\, E(\Phi),$$

the smallest closed subspace that contains $E(\Phi)$. Throughout the rest of this article, we assume that $E(\Phi)$ is a Bessel sequence for $\mathcal{S}(\Phi)$. This assumption settles most of the convergence issues. The space $\mathcal{S}(\Phi)$ is called the *shift-invariant space generated by* Φ and Φ a *generating set* for $\mathcal{S}(\Phi)$. Shift-invariant spaces have been studied extensively in the literature, e.g., [2, 3, 7, 9].

For $\omega \in \mathbb{R}^d$ we define the *pre-Gramian* via

$$J_\Phi(\omega) = \left(\widehat{\phi}(\omega + \alpha)\right)_{\alpha \in 2\pi\mathbb{Z}^d, \phi \in \Phi},$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. Note that the domain of the pre-Gramian matrix as an operator is $\ell_2(\Phi)$ and its co-domain is $\ell_2(\mathbb{Z}^d)$. The pre-Gramian can be regarded as the synthesis operator represented in Fourier domain as it was extensively studied in [9]. In particular, we have (see, e.g., [9, 3])

Proposition 2.1 *The shift-invariant system $E(\Phi)$ is a frame for $\mathcal{S}(\Phi)$ if and only if $J_\Phi(\omega)J_\Phi^*(\omega)$ is uniformly bounded with uniformly bounded inverse on the range of $J_\Phi(\omega)$ for a.e. ω such that $\operatorname{ran} J_\Phi(\omega) \neq \{0\}$. In particularly, when $\mathcal{S}(\Phi) = L_2(\mathbb{R}^d)$, $E(\Phi)$ is a frame for $L_2(\mathbb{R}^d)$ if and only if there are $0 < A \leq B < \infty$, such that $AI_{\ell_2(\mathbb{Z}^d)} \leq J_\Phi(\omega)J_\Phi^*(\omega) \leq BI_{\ell_2(\mathbb{Z}^d)}$ for a.e. $\omega \in \mathbb{R}^d$; and it is a tight frame with frame bound 1 for $L_2(\mathbb{R}^d)$ if and only if $J_\Phi(\omega)J_\Phi^*(\omega) = I_{\ell_2(\mathbb{Z}^d)}$, for a.e. $\omega \in \mathbb{R}^d$.*

Let Φ and $\Psi = R\Phi$, where R is a bijection satisfying $R(\phi(\cdot - k)) = (R\phi)(\cdot - k)$, be countable subsets of $L_2(\mathbb{R}^d)$. Suppose that $\mathcal{S}(\Phi) = \mathcal{S}(\Psi)$ and that both $E(\Phi)$ and $E(\Psi)$ are frames for $\mathcal{S}(\Phi)$. Then, by definition, $E(\Phi)$ and $E(\Psi)$ are a pair of orthogonal frames for $\mathcal{S}(\Phi)$ if and only if for all $f \in \mathcal{S}(\Phi)$

$$Sf := T_{E(\Psi)}T_{E(\Phi)}^* f = 0.$$

We define the *mixed dual Gramian* (cf. [11]) as

$$\widetilde{G}(\omega) = J_\Psi(\omega)J_\Phi^*(\omega),$$

and *Gramians* as

$$G_\Phi(\omega) = J_\Phi^*(\omega)J_\Phi(\omega) \quad \text{and} \quad G_\Psi(\omega) = J_\Psi^*(\omega)J_\Psi(\omega).$$

Then, it is proven in [11] that for any $f \in L_2(\mathbb{R}^d)$

$$\widehat{(Sf)}_{|\omega+\alpha} = \widetilde{G}(\omega)\hat{f}_{|\omega+\alpha},$$

where $\hat{g}_{|\omega+\alpha}$ is the column vector $(\hat{g}(\omega+\gamma))^T_{\gamma\in 2\pi\mathbb{Z}^d}$. With this, one can prove easily that $Sf = 0$ for all $f \in L_2(\mathbb{R}^d)$ if and only if $\widetilde{G}(\omega) = 0$ for a.e. $\omega \in \mathbb{R}^d$. Putting everything together, we have:

Theorem 2.2 *Let Φ and $\Psi = R\Phi$ be defined as above. Suppose that $\mathcal{S}(\Phi) = \mathcal{S}(\Psi)$ and that $E(\Phi)$ and $E(\Psi)$ are frames for $\mathcal{S}(\Phi)$. Then, the following are equivalent:*

(1) *$E(\Phi)$ and $E(\Psi)$ are a pair of orthogonal frames for $\mathcal{S}(\Phi)$;*

(2) *$J_\Psi(\omega)J^*_\Phi(\omega)J_\Phi(\omega) = 0$ a.e. $\omega \in \mathbb{R}^d$;*

(3) *$G_\Psi(\omega)G_\Phi(\omega) = 0$ a.e. $\omega \in \mathbb{R}^d$.*

In particular, when $\mathcal{S}(\Phi) = L_2(\mathbb{R}^d)$, $E(\Phi)$ and $E(\Psi)$ are a pair of orthogonal frames if and only if $J_\Psi(\omega)J^*_\Phi(\omega) = 0$ for a.e. $\omega \in \mathbb{R}^d$.

Proof. For the equivalence of (1) and (2), one notes that $f \in \mathcal{S}(\Phi)$ if and only if the Fourier transform of f can be written as

$$\hat{f} = \sum_{\phi\in\Phi} \widehat{a}_\phi \hat{\phi},$$

for some $\widehat{a}_\phi \in L_2(\mathbb{T}^d)$. Moreover,

$$\hat{f}_{|\omega+\alpha} = J_\Phi(\omega)(\widehat{a}_\phi(\omega))^T_{\phi\in\Phi}.$$

Hence, Item (1) is equivalent to the statement that for any $f \in S(\Phi)$

$$\widehat{(Sf)}_{|\omega+\alpha} = \widetilde{G}(\omega)\hat{f}_{|\omega+\alpha} = J_\Psi(\omega)J^*_\Phi(\omega)J_\Phi(\omega)(\widehat{a}_\phi(\omega))^T_{\phi\in\Phi} = 0,$$

which is equivalent to Item (2), i.e. $J_\Psi(\omega)J^*_\Phi(\omega)J_\Phi(\omega) = 0$ a.e. $\omega \in \mathbb{R}^d$. Finally, the equivalence of Item (2) and Item (3) follows from the fact that $J^*_\Psi(\omega)$ has bounded inverse on the range of $J_\Psi(\omega)$ for a.e. $\omega \in \mathbb{R}^d$ if $E(\Psi)$ is a frame for $S(\Psi)$ by Proposition 2.1 (see [9]). □

2.2 Construction of a pair of orthogonal shift-invariant frames from a given shift-invariant frame

Theorem 2.2 can be applied to construct a pair of shift-invariant orthogonal frames from a given shift-invariant frame as stated below.

Theorem 2.3 *Suppose that* $\Phi := \{\phi_1, \phi_2, \cdots, \phi_r\} \subset L_2(\mathbb{R}^d)$ *where* r *can be* ∞*, and that* $E(\Phi)$ *is a frame for* $\mathcal{S}(\Phi)$*. Let* $U := (U_1; U_2)$ *be a* $2r \times 2r$ *matrix with* $L_2(\mathbb{T}^d)$ *entries satisfying* $U^*(\omega)U(\omega) = I_{2r}$ *for a.e.* $\omega \in \mathbb{R}^d$*, where* U_1 *is the submatrix of the first* r *columns and* U_2 *the remaining* r *columns. Define* $\widehat{\Phi}_1 := U_1\widehat{\Phi}$*, and* $\widehat{\Phi}_2 := U_2\widehat{\Phi}$*. Then* $E(\Phi_1)$ *and* $E(\Phi_2)$ *are a pair of orthogonal frames for* $\mathcal{S}(\Phi)$*.*

Proof. It is easy to check by the Bessel property of $E(\Phi)$ that $\mathcal{S}(\Phi) = \mathcal{S}(\Phi_1) = \mathcal{S}(\Phi_2)$ with each of Φ_1 and Φ_2 consists of $2r$ elements of $L_2(\mathbb{R}^d)$. Furthermore, it is direct to check that

$$J_{\Phi_1}(\omega) = J_\Phi(\omega)U_1^T(\omega) \quad \text{and} \quad J_{\Phi_2}(\omega) = J_\Phi(\omega)U_2^T(\omega).$$

Moreover, $\operatorname{ran} J_{\Phi_1}(\omega) = \operatorname{ran} J_\Phi(\omega)$ a.e., since $U_1^T(\omega) : \ell_2(\Phi_1) \to \ell_2(\Phi)$ is onto by $U^T(\omega)(U^T(\omega))^* = I_{2r}$ for a.e. $\omega \in \mathbb{T}^d$. Moreover,

$$\begin{aligned} J_{\Phi_1}(\omega)J_{\Phi_1}^*(\omega) &= J_\Phi(\omega)U_1^T(\omega)(J_\Phi(\omega)U_1^T(\omega))^* = J_\Phi(\omega)(U_1^*(\omega)U_1(\omega))^T J_\Phi^*(\omega) \\ &= J_\Phi(\omega)I_r J_\Phi^*(\omega) = J_\Phi(\omega)J_\Phi^*(\omega). \end{aligned}$$

Hence, $E(\Phi_1)$ is a frame for $\mathcal{S}(\Phi_1) = \mathcal{S}(\Phi)$ by Proposition 2.1. Similarly, $E(\Phi_2)$ forms a frame for $\mathcal{S}(\Phi_2) = \mathcal{S}(\Phi)$ as well. It remains to show that $E(\Phi_1)$ and $E(\Phi_2)$ form a pair of orthogonal frames for $\mathcal{S}(\Phi)$. Indeed, this follows from the fact that, for a.e.

$\omega \in \mathbb{R}^d$,

$$\begin{aligned} G_{\Phi_1}(\omega)G_{\Phi_2}(\omega) &= J^*_{\Phi_1}(\omega)J_{\Phi_1}(\omega)J^*_{\Phi_2}(\omega)J_{\Phi_2}(\omega) \\ &= J^*_{\Phi_1}(\omega)J_{\Phi}(\omega)U_1^T(\omega)(U_2^T(\omega))^*J^*_{\Phi}(\omega)J_{\Phi_2}(\omega) \\ &= J^*_{\Phi_1}(\omega)J_{\Phi}(\omega)U_1^T(\omega)(U_2^*(\omega))^T J^*_{\Phi}(\omega)J_{\Phi_2}(\omega) \\ &= J^*_{\Phi_1}(\omega)J_{\Phi}(\omega)(U_2^*(\omega)U_1(\omega))^T J^*_{\Phi}(\omega)J_{\Phi_2}(\omega) \\ &= J^*_{\Phi_1}(\omega)J_{\Phi}(\omega)0J^*_{\Phi}(\omega)J_{\Phi_2}(\omega) = 0 \end{aligned}$$

and Theorem 2.2. □

Finally, we note that there are many choices of U. One of the easiest choices of U is a constant $2r \times 2r$ unitary matrix.

2.3 Construction of a pair of Gabor orthogonal frames

The constructions given above can be applied to the Gabor system to obtain a pair of orthogonal Gabor frames, since it is shift-invariant. Let $G := \{g_1, g_2, \cdots, g_\gamma\} \subset L_2(\mathbb{R}^d)$, where γ is a positive integer, and

$$\Phi := \{M^l g_j : l \in \mathbb{Z}^d, 1 \le j \le \gamma\},$$

where $M^t f(x) := e^{it\cdot x} f(x)$ is the *modulation operator* for $t \in \mathbb{R}^d$. Then $E(\Phi)$ is equivalent to a Gabor system generated by G [12]. Note that, in general, the shift operator and modulation operator can be chosen to be any d-dimensional lattice instead of $\mathbb{Z}^d$. For simplicity, we assume that both the shift lattice and the modulation lattice are $\mathbb{Z}^d$. However, the discussion here can be carried out similarly for more general shift and modulation lattices.

Suppose that $E(\Phi)$ is a frame for its closed linear span. Let $V := (V_1; V_2)$ be a $2\gamma \times 2\gamma$ constant unitary matrix, where V_1 is the submatrix formed by the first γ columns of V and V_2 by the remaining γ columns of V. We show that the Gabor systems generated by $G_1 := V_1 G$ and $G_2 := V_2 G$ are orthogonal frames by Theorem 2.3.

Let U_1 be the block diagonal (infinite) matrix of size $(\mathbb{Z}^d \times \{1, 2 \cdots, 2\gamma\}) \times (\mathbb{Z}^d \times \{1, 2, \cdots, \gamma\})$ such that

$$\text{the } (l,j)(l',j')\text{-th entry of } U_1 = \begin{cases} 0 & \text{if } l \neq l', \\ (V_1)_{j,j'} & \text{if } l = l'. \end{cases}$$

Similarly, one can define a block diagonal matrix U_2 by V_2. Then, the matrix $U := (U_1; U_2)$ is unitary. Furthermore, the Gabor system generated by $V_1 G$ is $E(\Phi_1)$ satisfying $\Phi_1 := U_1 \Phi$ and the Gabor system generated by $V_2 G$ is $E(\Phi_2)$ satisfying $\Phi_2 := U_2 \Phi$. Since U is a constant matrix, $\widehat{\Phi}_i = U_i \widehat{\Phi}_i$ for $i = 1, 2$. Hence $E(\Phi_1)$ and $E(\Phi_2)$ are a pair of orthogonal Gabor frames by Theorem 2.3.

3 Orthogonal wavelet frames

This section is devoted to construction of a pair of orthogonal wavelet frames. Let $\Psi := \{\psi_1, \psi_2, \cdots, \psi_r\} \subset L_2(\mathbb{R}^d)$, where r is a positive integer, and s an integer-valued invertible $d \times d$ matrix such that s^{-1} is contractive. Define a unitary dilation operator D on $L_2(\mathbb{R}^d)$ via

$$D : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) : f \mapsto |\det s|^{1/2} f(s\cdot).$$

Then, the following collection is called a *wavelet (or affine)* system generated by Ψ:

$$X(\Psi) := \{D^j E^k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq l \leq r\}, \tag{3.1}$$

where $E^k f := f(\cdot - k)$.

The wavelet system is not shift-invariant. To apply Theorem 2.3, one needs to use the quasi-affine system $X^q(\Psi)$, i.e. the smallest shift-invariant system containing $X(\Psi)$. Then, applying an approach similar to that in [11], one can obtain that two wavelet frame systems $X(\Psi_1)$ and $X(\Psi_2)$ are a pair of orthogonal frames if and only if the mixed dual Gramian of the corresponding quasi-affine systems $X^q(\Psi)$ and $X^q(\Psi_2)$ are zero almost everywhere. This is exactly what has been obtained by Weber in [13], with a different approach, as given below:

Proposition 3.1 ([13]) *Let* $\Psi_1 := \{\psi_1^{(1)}, \psi_2^{(1)}, \ldots, \psi_r^{(1)}\}$ *and* $\Psi_2 := \{\psi_1^{(2)}, \psi_2^{(2)}, \ldots, \psi_r^{(2)}\}$. *Suppose that* $X(\Psi_1)$ *and* $X(\Psi_2)$ *are frames for* $L_2(\mathbb{R}^d)$. $X(\Psi_1)$ *and* $X(\Psi_2)$ *are a pair of orthogonal frames for* $L_2(\mathbb{R}^d)$ *if and only if the following two equations are satisfied a.e.:*

$$\sum_{i=1}^{r}\sum_{j\geq 0}\overline{\widehat{\psi_i^{(2)}}(s^{*j}\omega)}\widehat{\psi_i^{(1)}}\left(s^{*j}(\omega+q)\right) = 0, \quad q \in 2\pi\mathbb{Z}^d \setminus 2\pi s^*\mathbb{Z}^d; \tag{3.2}$$

$$\sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d}\overline{\widehat{\psi_i^{(2)}}(s^{*j}\omega)}\widehat{\psi_i^{(1)}}(s^{*j}\omega) = 0. \tag{3.3}$$

We remark here that the double sums in (3.2) and (3.3) are the entries of the mixed dual Gramian of the affine systems generated by Ψ_1 and Ψ_2 [11].

Applying the above result of Weber, one can construct a pair of orthogonal wavelet frames easily.

Theorem 3.2 *Let* $\Psi := \{\psi_1, \psi_2, \cdots, \psi_r\} \subset L_2(\mathbb{R}^d)$ *for some positive integer* r. *Suppose that* $X(\Psi)$ *is a frame for* $L_2(\mathbb{R}^d)$. *Let* $V := (V_1; V_2)$ *be a* $2r \times 2r$ *constant unitary matrix, where* V_1 *denotes the submatrix formed by the first* r *columns of* V *and* V_2 *the last* r *columns of* V. *Then* $X(\Psi_1)$ *and* $X(\Psi_2)$ *are a pair of orthogonal frames for* $L_2(\mathbb{R}^d)$, *where* $\Psi_1 := V_1\Psi$ *and* $\Psi_2 := V_2\Psi$.

Proof. Note that $\widehat{\Psi}_1 := V_1\widehat{\Psi}$ and $\widehat{\Psi}_2 := V_2\widehat{\Psi}$ since V is a constant matrix. Direct calculations of the dual Gramians of $X^q(\Psi_1)$ and $X^q(\Psi_2)$, similar to what we do in the remaining part of the proof, show that $X(\Psi_1)$ and $X(\Psi_2)$ are frames for $L_2(\mathbb{R}^d)$ by using the dual Gramian characterization of frames in [10, Corollary 5.7].

We now show that the wavelet systems generated by Ψ_1 and Ψ_2 are a pair of orthogonal frames for $L_2(\mathbb{R}^d)$. Since $X(\Psi)$ is assumed to be a frame, the double sums converge absolutely a.e. We apply Theorem 3.1 to $\Psi_1 := \{\psi_1^{(1)}, \psi_2^{(1)}, \cdots, \psi_{2r}^{(1)}\}$ and $\Psi_2 := \{\psi_1^{(2)}, \psi_2^{(2)}, \cdots, \psi_{2r}^{(2)}\}$. Let $V = (v_{ij})_{1\leq i,j\leq 2r}$. For a fixed $q \in 2\pi\mathbb{Z}^d \setminus 2\pi s^*\mathbb{Z}^d$, we have

$$
\begin{aligned}
\sum_{i=1}^{2r}\sum_{j\geq 0}\overline{\widehat{\psi_i^{(1)}}}(s^{*j}\omega)\widehat{\psi_i^{(2)}}\left(s^{*j}(\omega+q)\right) &= \sum_{i=1}^{2r}\sum_{j\geq 0}\sum_{l=1}^{r}\overline{v_{i,l}}\overline{\widehat{\psi}_l}(s^{*j}\omega)\sum_{l'=1}^{r}v_{i,r+l'}\widehat{\psi_{l'}}\left(s^{*j}(\omega+q)\right)\\
&= \sum_{j\geq 0}\sum_{l=1}^{r}\overline{\widehat{\psi}_l}(s^{*j}\omega)\sum_{l'=1}^{r}\widehat{\psi_{l'}}\left(s^{*j}(\omega+q)\right)\sum_{i=1}^{2r}\overline{v_{i,l}}v_{i,r+l'}\\
&= \sum_{j\geq 0}\sum_{l=1}^{r}\overline{\widehat{\psi}_l}(s^{*j}\omega)\sum_{l'=1}^{r}\widehat{\psi_{l'}}\left(s^{*j}(\omega+q)\right)0 = 0,
\end{aligned}
\tag{3.4}
$$

where we used the orthogonality of the columns of V. A similar calculation shows that (3.3) also holds. Hence Ψ_1 and Ψ_2 generate a pair of orthogonal frames by Proposition 3.1. □

When the wavelet tight frame system $X(\Psi)$ is constructed from a multiresolution analysis based on the *unitary extension principle* (UEP) of [10], one can construct a pair of orthogonal tight frames from the same multiresolution analysis as we describe below.

We first give a brief discussion here on the UEP for the one variable case with trigonometric polynomial masks, while the more general version and comprehensive discussions of the UEP can be found in [5] and [10].

Let $\phi \in L_2(\mathbb{R})$ be a refinable function, i.e., $\widehat{\phi}(2\xi) = \widehat{a}_0(\xi)\widehat{\phi}(\xi)$, where $\widehat{a}_0$ is a trigonometric polynomial called the *refinement mask* of $\phi \in L_2(\mathbb{R})$ satisfying $\widehat{a}_0(0) = 1$, and let $\widehat{a}_j$, $j = 1, 2, \ldots, r$, be a set of trigonometric polynomials called the *wavelet masks.* The column vector $\vec{\widehat{a}} = (\widehat{a}_0, \widehat{a}_1, \ldots, \widehat{a}_r)^T$ is called the *refinement-wavelet mask.* Let

$$
A(\omega) = \begin{pmatrix} \widehat{a}_0(\omega) & \widehat{a}_0(\omega+\pi) \\ \widehat{a}_1(\omega) & \widehat{a}_1(\omega+\pi) \\ \vdots & \vdots \\ \widehat{a}_r(\omega) & \widehat{a}_r(\omega+\pi) \end{pmatrix} = (\vec{\widehat{a}}(\omega), \vec{\widehat{a}}(\omega+\pi)).
$$

Suppose that

$$
A^*(\omega)A(\omega) = I.
$$

for a.e. $\omega \in [-\pi, \pi]$. If we define $\Psi := \{\psi_1, \psi_2, \ldots, \psi_r\} \subset L_2(\mathbb{R})$ by

$$\widehat{\psi_l}(2\xi) := \widehat{a}_j(\xi)\widehat{\phi}(\xi), \quad l = 1, 2, \cdots, r,$$

then the UEP asserts that $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$.

By using the UEP the construction of compactly supported tight wavelet frames becomes painless. For example, it is easy to obtain compactly supported symmetric spline tight wavelet frames as shown in [10] and [5].

Next, we briefly describe how to obtain a pair of compactly supported orthogonal tight frames from a given compactly supported tight frame system $X(\Psi)$ constructed via the UEP. The main idea of this construction is from a paper by Bhatt, Johnson and Weber [1] where orthogonal wavelet tight frames are constructed from orthogonal wavelets.

Let $V(\omega) := (V_1(\omega); V_2(\omega)) = (v_{i,j}(\omega))$ be a $2r \times 2r$ unitary matrix with π periodic trigonometric polynomial entries, where V_1 denotes the submatrix formed by the first r columns of V and V_2 the last r columns of V. Let

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix}; \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}.$$

Define two new sets of the refinement-wavelet masks from $\vec{\widehat{a}}$ by

$$\vec{\widehat{a}}_1 = U_1\vec{\widehat{a}}; \quad \vec{\widehat{a}}_2 = U_2\vec{\widehat{a}}.$$

The corresponding wavelets are defined via its Fourier transform as: $\widehat{\Psi}_1 := V_1\widehat{\Psi}$ and $\widehat{\Psi}_2 := V_2\widehat{\Psi}$ with their wavelet masks given above. It is easy to check that both entries in the column vectors Ψ_1 and Ψ_2 are compactly supported. Let

$$A_1(\omega) = (\vec{\widehat{a}}_1(\omega); \vec{\widehat{a}}_1(\omega + \pi)); \quad A_2(\omega) = (\vec{\widehat{a}}_2(\omega); \vec{\widehat{a}}_2(\omega + \pi)).$$

Then, it is easy to see

$$A_1 = U_1 A; \quad A_2 = U_2 A,$$

since each entry of U_1 and U_2 is π periodic. This leads to

$$A_1^*(\omega)A_1(\omega) = I; \quad A_2^*(\omega)A_2(\omega) = I,$$

for all $\omega \in [-\pi, \pi]$. Hence, both $X(\Psi_1)$ and $X(\Psi_2)$ are tight frames by the UEP (see also e.g. [6]).

Let B_1 and B_2 be the matrices generated by A_1 and A_2 respectively by removing the first rows of them. Then, it is clear that

$$B_1^*(\omega)B_2(\omega) = 0,$$

for all $\omega \in [-\pi, \pi]$. This asserts that $X(\Psi_1)$ and $X(\Psi_2)$ are a pair of orthogonal frames by Theorem 2.1.1 of [1] whose proof was obtained by a computation similar to (3.4). In fact, Theorem 2.1.1 of [1] can also be proved via a method similar to the proof of the mixed unitary extension principle in [11]. Finally, we remark that this construction can be modified to more general cases, e.g., one may start with two different tight frames instead of starting with one tight frame $X(\Psi)$.

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