Interpolatory Wavelet Packets †

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Abstract:

In this note, we present a construction of interpolatory wavelet packets. Interpolatory wavelet packets provide a finer decomposition of the 2^jth dilate cardinal interpolation space, hence, give a better localization for an adaptive interpolation. This can lead to a more efficient compression scheme which, in turn, provides an interpolation algorithm with a smaller set of data for use in applications.

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1. Introduction

We begin with a refinable continuous compactly supported or exponentially decaying function φ that satisfies

$$\varphi(\alpha) = \delta(\alpha), \qquad \alpha \in \mathbb{Z}. \tag{1.1}$$

The function φ satisfying (1.1) is called the fundamental function.

Since φ is fundamental, the values of the refinement mask m of φ are determined on the even integers:

$$m(2\beta) = \varphi(\beta) = \delta(\beta), \quad \beta \in \mathbb{Z}.$$

Compactly supported refinable fundamental functions with high order of smoothness were given in [3]. Exponentially decaying spline fundamental functions can be obtained from B-splines as follows: Let B_{2n} be the univariate B-spline of order $2n$ with integer knots. The function φ defined via its Fourier transform by

$$\widehat{\varphi} := \widehat{B}_{2n} / \sum_{\beta} \widehat{B}_{2n}(\,\cdot\, + 2\pi\beta)$$

is a refinable continuous fundamental function with exponential decay.

Define a function $p_0 = \varphi$ and functions p_n inductively as follows:

$$\begin{aligned} p_{2n} &:= \sum_{\alpha\in\mathbb{Z}} m(\alpha) p_n(2\,\cdot\, - \alpha) \\ p_{2n+1} &:= p_n(2\,\cdot\, - 1). \end{aligned} \tag{1.2}$$

We denote $\mathcal{P}_n$ to be the closed shift invariant subspace in $C(\mathbb{R})$ generated by p_n; that is,

$$\mathcal{P}_n := \overline{\{f : f = \sum_{\alpha\in\mathbb{Z}} a(\alpha) p_n(\,\cdot\, - \alpha), a \in \ell_0(\mathbb{Z})\}},$$

where $\ell_0(\mathbb{Z})$ is the space of finitely supported sequences. The space $\mathcal{P}_0$ is a cardinal interpolation space; that is, the space spanned by translates of a fundamental solution for cardinal interpolation. The collection

$$\big\{p_n(2^k\,\cdot\, - \alpha) : n \in \mathbb{Z}_+,\ k \in \mathbb{Z},\ \alpha \in \mathbb{Z}\big\}$$

are the wavelet packets from which we will draw bases for our interpolation spaces.

The dilation operator $\sigma : \mathcal{P}_n \mapsto \sigma\mathcal{P}_n$ is defined by $\sigma f := \sqrt{2} f(2\,\cdot)$. The space $\sigma^j \mathcal{P}_0$ is the 2^j dilate of the cardinal interpolation space. It is generated by $\varphi(2^j \cdot - \alpha),\ \alpha \in \mathbb{Z}$ which are the dyadic shifts of a fundamental solution for interpolation on the lattice $\mathbb{Z}/2^j$. Our goal is to decompose the spaces $\sigma\mathcal{P}_n$ and to find ℓ_∞-stable bases for the decomposition.

For a given space $\mathcal{P}_n$, it is easy to show that (see [6] and [7]) $\sigma\mathcal{P}_n = \mathcal{P}_{2n} \oplus \mathcal{P}_{2n+1}$, and $\big\{p_{2n}(\,\cdot\, - \alpha),\ p_{2n+1}(\,\cdot\, - \alpha) : \alpha \in \mathbb{Z}\big\}$ forms an ℓ_∞–stable basis for $\sigma\mathcal{P}_n$. Further for each $j \in \mathbb{Z}$, the functions $\big\{p_n(2^j\,\cdot\, - \alpha) : \alpha \in \mathbb{Z},\ 2^{k-1} \le n \le 2^k - 1\big\}$ form an ℓ_∞-stable basis for the space $\sigma^{k-1+j}\mathcal{P}_1$.

A collection $\mathcal{J}$ of pairs (ℓ, k), $\ell \in \mathbb{Z}$ and $0 < k \in \mathbb{Z}$ disjointly covers the integer interval $[j_1 \,..\, j_2]$ if $\cup\{k + \ell : (\ell, k) \in \mathcal{J}\} = [j_1 \,..\, j_2] \cap \mathbb{Z}$ and the representation $j = k + \ell$ is unique from $\mathcal{J}$. We have the following Proposition:

(1.3) Proposition. *For given $j_1 < j_2$ and any disjoint cover $\mathcal{J}$ of $[j_1 \,..\, j_2] \cap \mathbb{Z}$, the functions*

$$\{p_0(2^{j_1-1} \cdot -\alpha),\ p_n(2^\ell \cdot -\alpha) : 2^{k-1} \le n \le 2^k - 1,\ (\ell, k) \in \mathcal{J},\ \alpha \in \mathbb{Z}\}$$

form an ℓ_∞-basis for the space $\sigma^{j_2}\mathcal{P}_0$.

Univariate orthogonal wavelet packets were introduced in [2] and its multivariate counterpart can be found in [6]. Recently, interpolatory wavelets, wavelet packets and their applications were discussed in [5], [1] and [7]. The interpolatory wavelet packets provided here, together with their decomposition and reconstruction algorithm, give wide choices of the decomposition of the lattice which, in turn, provides a possible way to interpolate scattered data. Applications are discussed in [4] and [5].

2. Decomposition of interpolation operators

The cardinal interpolation operator at the 2^jth dyadic level can be written as

$$\mathcal{L}_{2^j} f := \sum_{\alpha \in \mathbb{Z}} f(\alpha/2^j)\varphi(2^j \cdot -\alpha).$$

This provides an approximation for the function f which interpolates f at $\mathbb{Z}/2^j$. Further, $\lim_j \mathcal{L}_j f = f$ for any compactly supported continuous function f. To obtain higher accuracy, more interpolation points are needed. The interpolation operator $\mathcal{L}_{2^j}$ gives each point the same priority regardless of the shape of the function. It is numerically more desirable to interpolate the function in a coarser grid where the function f is flat, and in a finer grid where it has large variation. However, to do this we need to know the shape of the function from a given set of data, because in most of the applications the function f itself is not available. Decomposing the interpolation operator at the 2^j dyadic level into lower dyadic levels is one way to analyze the data structure and hence the shape of the function from the given set of data. Then, a quantization scheme designed according to the given practical problem is applied and finally the reconstruction algorithm leads to an interpolation of f which approximates f with the same order of the accuracy but using a smaller set of data. Similarly, this process also can be applied to decompose, compress and reconstruct the sampled data at 2^jth dyadic levels. Further, for the interpolatory wavelet packets, a standard subdivision scheme can provide a predication of 2^{j+1}th dyadic level sample data from the sampled data at the 2^jth dyadic level.

The cardinal interpolation operator at the 2^jth dyadic level can be decomposed as as a sum of elements from $\sigma^{j-1}\mathcal{P}_0$ and $\sigma^{j-1}\mathcal{P}_1$ as was given by Donoho [5]. Clearly, this decomposition can be iterated down to the j_1st level.

The wavelet packets given in this section provide a finer decomposition of $\sigma^j\mathcal{P}_0$. These bases have descriptions as interpolation operators. This gives a finer decomposition of the interpolation operator and a more detailed analysis of the the structure of the given set of data. This leads to a more effective compression, hence a better adapted interpolation.

The crucial Proposition that follows uses a correspondence

$$r := \sum_{\ell=0}^{k-1} \eta_\ell 2^\ell \longleftrightarrow r^* := \frac{1}{2^{k+1}} + \frac{1}{2}\sum_{\ell=0}^{k-1} \eta_\ell 2^{-\ell}, \quad \forall \eta_\ell \in \{0,1\} \tag{2.1}$$

between the integers r in $[0\,..\,2^k - 1]$ and points r^* in the lattice $(2^{-k-1}\mathbb{Z} \backslash 2^{-k}\mathbb{Z})$ in $[0\,..\,1]$:

(2.2)Proposition. *For any $k \geq 0$, $\beta \in \mathbb{Z}$, $0 \leq s, r \leq 2^k - 1$, and s and s^* related by (2.1), we have*

$$p_{2^k+r}(\beta/2^k) = 0,$$
$$p_{2^k+r}(\beta + \frac{1}{2^{k+1}} + \frac{1}{2}s^*) = \begin{cases} 0, & \text{if } s < r \text{ and } \beta \text{ arbitrary;} \\ \delta(\beta), & \text{if } s = r \text{ and } \beta \in \mathbb{Z}. \end{cases}$$

Proof. The proof is by induction on k. For $k = 0$, the only r is $r = 0$. In that case, since $p_0 = \varphi$ is a fundamental function for cardinal interpolation from (1.2), we have

$$p_1(\beta) = p_0(2\beta - 1) = 0, \quad p_1(\beta + \frac{1}{2}) = p_0(2\beta + 1 - 1) = \delta(2\beta)$$

Now assume that the lemma holds for integers $< k$. Consider $2n = 2^k + 2r$. If $s = \sum_{\ell=0}^{k-1} \eta_\ell 2^\ell \leq 2r$, then for strict inequality we must have $\sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^\ell < r$ while equality implies equality in the last inequality as well since $\eta_0 = 0$ in that case. Hence, by (1.2)

$$p_{2^k+2r}(\beta/2^k) = \sum_{\alpha\in\mathbb{Z}} m(\alpha) p_{2^{k-1}+r}(\beta/2^{k-1} - \alpha) = 0,$$
$$p_{2^k+2r}(\beta + \frac{1}{2^{k+1}} + \frac{1}{2}s^*) = \sum_{\alpha\in\mathbb{Z}} m(\alpha) p_{2^{k-1}+r}(2\beta + \frac{1}{2^k} + \eta_0 - \alpha + \frac{1}{2}\sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^{-\ell})$$
$$= \begin{cases} 0, & \text{if } s < 2r \text{ and } \beta \in \mathbb{Z}; \\ m(2\beta) = \delta(\beta), & \text{if } s = 2r. \end{cases}$$

Similarly, for $2n + 1 = 2^k + 2r + 1$, $s = \sum_{\ell=0}^{k-1} \eta_\ell 2^\ell \leq 2r + 1$, then we must have $\sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^\ell < r$ while equality implies equality in the last inequality and $\eta_0 = 1$. Hence,

$$p_{2^k+2r+1}(\beta + \frac{1}{2^{k+1}} + \frac{1}{2}s^*) = p_{2^{k-1}+r}(2\beta + \eta_0 - 1 + \frac{1}{2^k} + \frac{1}{2}\sum_{\ell=0}^{k-2} \eta_{\ell+1} 2^{-\ell})$$
$$= \begin{cases} 0, & \text{if } s < 2r + 1 \text{ and } \beta \text{ arbitrary;} \\ \delta(\beta), & \text{if } s = 2r + 1. \end{cases}$$

□

We define interpolation operator $\mathcal{L}_{(2^k,r)} f$ by

$$\mathcal{L}_{(2^k,r)} f := \sum_{\beta\in\mathbb{Z}} f\big(\beta + \frac{1}{2^{k+1}} + \frac{r^*}{2}\big) p_{2^k+r}(\cdot - \beta).$$

Then for $s, r \in \{0, 1, \ldots, 2^k - 1\}$, $\mathcal{L}_{(2^k,r)} f(\frac{\beta}{2^k}) = 0$, $\forall\, \beta \in \mathbb{Z}$ and

$$\mathcal{L}_{(2^k,r)} f\big(\beta + \frac{1}{2^{k+1}} + \frac{1}{2}s^*\big) = \begin{cases} 0, & \text{if } s < r \text{ and } \beta \in \mathbb{Z}; \\ f\big(\beta + \frac{1}{2^{k+1}} + \frac{r^*}{2}\big), & \text{if } s = r \text{ and } \beta \in \mathbb{Z}. \end{cases}$$

The interpolation operator $\mathcal{L}_{2^k}$ at the 2^kth dyadic level can be decomposed between levels k and $k-1$ using the operators $\mathcal{L}_{(2^k,r)}f$. Here we give such a decomposition by the bases defined in Proposition 1.3 with $j_1 = j_2 = k$. Define

$$g_0 := (f - \mathcal{L}_{2^{k-1}}f), \quad \text{and} \quad g_r := g_{r-1} - \mathcal{L}_{(2^{k-1},r)}g_{r-1}, r = 1, \ldots, 2^{k-1} - 1.$$

Then from the uniqueness of the interpolation, it follows that

$$\mathcal{L}_{2^k}f = \mathcal{L}_{2^{k-1}}f + \sum_{r=0}^{2^{k-1}-1} \mathcal{L}_{(2^{k-1},r)}g_r,$$

since both sides are in $\sigma^k\mathcal{P}_0$ and interpolate f at the points $\{\beta/2^k\}_{\beta\in\mathbb{Z}}$.

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