Refinable Function Vectors

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Abstract: Refinable function vectors are usually given in the form of an infinite product of their refinement (matrix) masks in the frequency domain and approximated by a cascade algorithm in both time and frequency domains. We provide necessary and sufficient conditions for the convergence of the cascade algorithm. We also give necessary and sufficient conditions for the stability and orthonormality of refinable function vectors in terms of their refinement matrix masks. Regularity of function vectors gives smoothness orders in the time domain, and decay rates at infinity in the frequency domain. Regularity criteria are established in terms of the vanishing moment order of the matrix mask.

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1. Introduction

This paper presents a complete characterization of the convergence of the cascade algorithm and the stability and orthonormality of compactly supported refinable function vectors in terms of their refinement matrix masks. Regularity criteria for refinable function vectors are also established in terms of the vanishing moment order of the matrix mask.

We start with a finite set of compactly supported functions $\Phi \subset L_2(\mathbb{R}^s)$. The **FSI space** (finitely generated shift invariant; see [BDR]) $S(\Phi)$ generated by Φ is the smallest (closed) shift invariant subspace of $L_2(\mathbb{R}^s)$ containing Φ. Here we recall that a space is **shift invariant** if it is invariant under all **shifts**, i.e. invariant under all integer translations.

It is very convenient to discuss the shift invariant space in the frequency domain by using Gramian analysis. For a given set of functions Φ, the **pre-Gramian** matrix at $\omega \in \mathbb{T}^s$ is defined as a $\mathbb{Z}^s \times \Phi$ matrix by

$$J(\omega) := J_\Phi(\omega) := (\widehat{\phi}(\omega + 2\pi\alpha))_{\alpha,\phi},$$

where $\widehat{\phi}$ is the Fourier transform of the function ϕ. Its adjoint matrix

$$J^*(\omega) := J^*_\Phi(\omega) := (\overline{\widehat{\phi}(\omega + 2\pi\alpha)})_{\phi,\alpha}$$

is a $\Phi \times \mathbb{Z}^s$ matrix. The **Gramian matrix** of functions Φ is a $\Phi \times \Phi$ matrix defined as the product of J^* and J, i.e. $J^*_\Phi(\omega) J_\Phi(\omega)$. The pre-Gramian matrix was first introduced in [RS]; the basic properties of the pre-Gramian and its roles in the Gramian analysis for shift invariant spaces (not necessary a FSI space) can be found in [RS]. In this paper, we will often use the matrix $\overline{J^* J} =: G_\Phi =: G$. Since the properties of the Gramian matrix $J^* J$ which we are interested do not change when the conjugation is taken, we also call G_Φ the Gramian matrix of Φ.

This paper uses functions that are defined on $\mathbb{T}^s$, the s-dimensional torus. These can be viewed as 2π-periodic functions, via the standard transformation $\mathbb{R}^s \ni \omega \mapsto e^{i\omega} := (e^{i\omega_1}, ..., e^{i\omega_s}) \in \mathbb{T}^s$. Though we may refer to such functions as defined on $\mathbb{T}^s$, we always treat their arguments as *real*. Thus, "multiplying a function defined on $\mathbb{T}^s$ by a function defined on $\mathbb{R}^s$" simply means "multiplying a 2π-periodic function by ...". Following this slight abuse of language, we write "$\Omega \subset \mathbb{T}^s$" to mean "$\Omega \subset [-\pi, \pi]^s$".

The functions Φ used in this paper are solutions to functional equations of the type

$$\Phi = \sum_{\alpha \in \mathbb{Z}^s} P_\alpha \Phi(2 \cdot - \alpha), \tag{1.1}$$

where the "coefficients" P_α are $\Phi \times \Phi$ matrices and Φ is a $\#\Phi-$dimensional column refinable function vector. We assume throughout that the refinement matrix masks are supported in $[0,N]^s$. Here we use Φ to denote both the set of functions Φ and the column function vector Φ.

Define

$$\mathbf{P} := 2^{-s} \sum_{\alpha\in\mathbb{Z}^s} P_\alpha \exp(-i\alpha\cdot)\,.$$

Then, $\mathbf{P}$ is a $\Phi\times\Phi$ matrix, so that each entry is a trigonometric polynomial such that their Fourier coefficients are supported in $[0,N]^s$. The functional equations (1.1) can be written as

$$\widehat{\Phi} := \mathbf{P}(\cdot/2)\widehat{\Phi}(\cdot/2)\,. \tag{1.2}$$

Equations of the type (1.2) are called **vector refinement equations**; the matrix $\mathbf{P}$ is called the **refinement (matrix) mask** and Φ is a ($\mathbf{P}-$) **refinable function vector**.

Since each entry of $\mathbf{P}$ is a trigonometric polynomial, the function matrix $\mathbf{P}$ satisfies

$$\|\mathbf{P}(\cdot)-\mathbf{P}(0)\| \le \mathrm{const}\|\cdot\|,$$

where for any $d\times d$ matrix M, $\|M\| := \max_{\|\mathbf{v}\|=1}\|M\mathbf{v}\|/\|\mathbf{v}\|$, with $\|\mathbf{v}\|$ the Euclidean norm of the column vector $\mathbf{v}\in\mathbb{R}^d$.

If $\lim_{n\to\infty}\mathbf{P}(0)^n$ exists and is nontrivial, then the infinite product

$$\mathbf{P}^\infty := \prod_{k=1}^{\infty}\mathbf{P}(2^{-k}\cdot)$$

converges uniformly on compact sets. Further, $\widehat{\Phi} = \mathbf{P}^\infty\mathbf{a}$ is a solution of (1.2), where $\mathbf{a}$ is a right eigenvector of $\mathbf{P}(0)$ (see [HC], [H] for the univariate case and [LCY] for the multivariate one). The functions Φ are compactly supported distributions with $\mathrm{supp}(\Phi)\subset[0,N]^s$. We further remark that the existence of a solution $\widehat{\Phi}$ of (1.2) only requires the convergence of $\prod_{j=1}^n\mathbf{P}(2^{-j}\cdot)\mathbf{a}$, where $\mathbf{a}$ is a right eigenvector of $\mathbf{P}(0)$ corresponding to the eigenvalue 1 (see [HC] and [CDP]). It has been shown in [CDP] that $\prod_{j=1}^n\mathbf{P}(2^{-j}\cdot)\mathbf{a}$ converges if $\rho(\mathbf{P}(0))<2$ (see also [HC]).

We say that a matrix M (or linear operator) satisfies the Condition on Eigenvalues or, **Condition E** for short, if the spectral radius $\rho(M)\le 1$, 1 is required to be the only eigenvalue on the unit circle, and must be a simple eigenvalue. Condition E is a useful concept in the wavelet theory and applications (see [CD], [S1], [SN] and [LLS3]).

Assume that $\mathbf{P}(0)$ satisfies Condition E. Then, there is a nonsingular matrix U so that $U\mathbf{P}(0)U^{-1}$ has the form

$$\begin{pmatrix} 1 & \mathbf{0}\\ \mathbf{0} & \Lambda\end{pmatrix}, \tag{1.3}$$

where

$$\Lambda := \begin{pmatrix} \lambda_2 & \mu_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_3 & \mu_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_{\#\Phi} \end{pmatrix}$$

with $|\lambda_i| < 1$, and $\mu_i = 1$, or 0, $i = 2, \dots, \#\Phi$. Define $\mathbf{P}_1 = U\mathbf{P}U^{-1}$, then $\Phi^1 := U\Phi$ satisfies the refinement equations

$$\widehat{\Phi}^1 = \mathbf{P}_1(\cdot/2)\widehat{\Phi}^1(\cdot/2), \tag{1.4}$$

where Φ is a solution of (1.2).

The stability, regularity and the convergence of the cascade algorithm discussed in the paper do not change, even if we consider the refinement equation (1.4) instead of (1.1). Furthermore, as we will see in §3, the problem of checking the orthonormality of Φ can be reduced to that of checking the stability of Φ^1. Therefore, we can always assume that $\mathbf{P}(0)$ has the form given in (1.3), without losing anything.

In this case the vector $\mathbf{i}_1^T := (1, 0, \dots, 0)$ is a left eigenvector and $\mathbf{i}_1$ is a right eigenvector of $\mathbf{P}(0)$ corresponding to eigenvalue 1. We further require that $\mathbf{P}$ have vanishing moments of at least order 1, that is equivalent to the fact that

$$\mathbf{i}_1^T \mathbf{P}(\pi\nu) = \delta_\nu \mathbf{i}_1^T, \quad \nu \in \mathbb{Z}^s/2\mathbb{Z}^s.$$

This implies that

$$\mathbf{i}_1^T \widehat{\Phi}(2\pi\alpha) = \delta_\alpha, \; \alpha \in \mathbb{Z}^s.$$

Altogether, we assume, throughout this paper, that the mask $\mathbf{P}$ satisfies the following conditions:

Basic conditions 1.5. *We say that* $\mathbf{P}$ *satisfies the basic conditions, if*

(i) $\mathbf{P}(0)$ *has the form of (1.3), and*

(ii) $\mathbf{i}_1^T \mathbf{P}(\pi\nu) = \delta_\nu \mathbf{i}_1^T, \quad \nu \in \mathbb{Z}^s/2\mathbb{Z}^s.$

It has further been shown in [LCY] (see also [HC]) that if the basic conditions 1.5 hold for $\mathbf{P}$, then

$$\mathbf{P}^\infty = \prod_{j=1}^{\infty} \mathbf{P}(\cdot/2^j) = \begin{pmatrix} \widehat{\Phi} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}.$$

In particular, if $\widehat{\Phi}(0) \neq 0$, the solution Φ is determined uniquely up to a constant factor. In fact, $\widehat{\Phi} = c\mathbf{P}^\infty \mathbf{b}$, where $\mathbf{b}$ is an arbitrary vector satisfying $\mathbf{i}_1^T \mathbf{b} = 1$.

The functions Φ can be approximated by the following **cascade algorithm**: starting with a function vector Φ_0 which satisfies

$$\sum_{\alpha\in\mathbb{Z}^s} \mathbf{i}_1\Phi_0(\cdot-\alpha)=1,$$

the function vector Φ_n is defined inductively by

$$\Phi_n := \sum_{\alpha\in\mathbb{Z}^s} P_\alpha\Phi_{n-1}(2\cdot-\alpha). \tag{1.6}$$

The cascade algorithm can be iterated in the frequency domain by taking the Fourier transform of (1.6)

$$\widehat{\Phi}_n = \mathbf{P}(\cdot/2)\widehat{\Phi}_{n-1}(\cdot/2). \tag{1.7}$$

It is clear that the sequence $(\Phi_n)_n$ converges in the L_2-norm if and only if the sequence $(\widehat{\Phi}_n)_n$ does. (We say $(\Phi_n)_n$ converges to Φ in the L_2-norm, if each component of $(\Phi_n)_n$ converges to the corresponding component of Φ in the L_2-norm.) A sufficient condition for the convergence of the cascade algorithm is given in [CDP], under the assumption that Φ and its shifts are linearly independent; $s=1$ and $\widehat{\Phi}_0=\chi_{[-\pi,\pi]}\mathbf{a}$, where $\mathbf{a}$ is a right eigenvector of $\mathbf{P}(0)$ corresponding to eigenvalue 1. If the sequence $(\Phi_n)_n$ defined by (1.6) converges, then $\Phi\subset L_2(\mathbb{R}^s)$.

Define

$$S^k := S^k(\Phi) := \{f(2^k\cdot) : f\in S(\Phi)\}. \tag{1.8}$$

Then,

$$S^k\subset S^{k+1}. \tag{1.9}$$

It is proven in [JS] that if

$$\cup_{k\in\mathbb{Z}}\cup_{\phi\in\Phi}\operatorname{supp}\widehat{\phi}(2^k\cdot)=\mathbb{R}^s \tag{1.10}$$

holds up to a null set and (1.9) holds, then $\cup_{k\in\mathbb{Z}}S^k$ is dense in $L_2(\mathbb{R}^s)$. If the refinable function vector Φ is compactly supported, (1.10) is always true. It has further been shown in [JS] that if $\Phi\subset L_2(\mathbb{R}^s)$, then $\cap_{k\in\mathbb{Z}}S^k=\{0\}$. Altogether, we have the following result:

Result 1.11. *Let Φ be the compactly supported $\mathbf{P}-$ refinable function vector. If $\Phi\subset L_2$, then,*

$$\overline{\cup_{k\in\mathbb{Z}}S^k}=L_2(\mathbb{R}^s), \quad \textit{and} \quad \cap_{k\in\mathbb{Z}}S^k=\{0\}. \tag{1.12}$$

We say that a set of functions Φ is **stable**, if Φ and their shifts form a Riesz basis of $S(\Phi)$, and a set of functions Φ is **orthonormal**, if Φ and their shifts form an orthonormal basis of $S(\Phi)$.

A set of functions $\Phi \subset L_2(\mathbb{R}^s)$ is stable if and only if

$$0 < C_1 \leq \|\lambda\|_\infty \leq \|\Lambda\|_\infty \leq C_2 < \infty, \quad \text{a. e.} \quad \omega \in \mathbb{T}^s,$$

where $\lambda(\omega)$ and $\Lambda(\omega)$ are the smallest and largest eigenvalues of the Gramian matrix $G_\Phi(\omega)$. If the set of functions Φ is compactly supported, then Φ is stable if and only if $\det G(\omega) \neq 0$ for all $\omega \in \mathbb{T}^s$. The set of functions Φ is orthonormal if and only if $G(\omega) = I$, a.e. $\omega \in \mathbb{T}^s$, where I is the identity matrix. The proofs of these results can be found in many articles (see e.g. [JM], [BDR], [GL], [DGHM], [RS], [CL], and [LCY]).

Once the set of functions $\Phi \subset L_2(\mathbb{R}^s)$ is stable (or orthonormal), it would be advantageous to know the regularity of Φ in order to make better use of Φ. An estimation of the regularity of Φ ($s = 1$) in terms of $\mathbf{P}$ has been given in [CDP], under the assumption that the refinable function vector Φ and its shifts are linearly independent.

By the above discussion, if the refinable function vector $\Phi \subset L_2(\mathbb{R}^s)$ is stable, or orthonormal, the sequence of subspaces $(S^k)_k$, $k \in \mathbb{Z}$ of $L_2(\mathbb{R}^s)$ forms a multiresolution; recall that a sequence $(S^k)_k$ forms a **multiresolution** if the sequence $(S^k)_k$ satisfies (1.12) and is refinable ($S^k \subset S^{k+1}$, $k \in \mathbb{Z}$), and if the refinable function vector Φ is stable or orthonormal.

The multiresolution generated by several functions was first introduced by [GLT], [GL] (see also [AR] and [DGH]). Result 1.11 is due to [JS]. The first set of examples of orthonormal refinable function vectors Φ were given in [GHM], [DGHM] and [DGH]. Examples of stable refinable function vectors Φ were given in [GL]. Compactly supported wavelets and prewavelets from these examples were constructed in [DGHM], [GHM], [SS], and [LLS1] (see also [CL]).

It is of particular interest to construct compactly supported wavelets and prewavelets from compactly supported refinable function vectors and the refinement matrix masks. An algorithmic method in the construction of compactly supported wavelets and prewavelets from an arbitrarily given $\mathbf{P}$ refinable function vector Φ was obtained in [LLS1], where $s = 1$. The problem of wavelet constructions is much more challenging in higher dimensions even when $\#\Phi = 1$ (see [JM] and [JS]). However, in dimensions no greater than 3, a method for the case $\#\Phi = 1$ has been provided in [RiS1] and [RiS2], under a mild condition on refinement masks.

Since the solutions Φ of (1.1) are defined via their Fourier transform by the refinement matrix mask $\mathbf{P}$, and since in practice only the refinement matrix mask is available for checking, it is useful to transfer the characterization of the stability and orthonormality of Φ by the Gramian matrix of Φ to characterization in terms of the mask. Similarly, it is necessary to characterize the convergence of the cascade algorithm defined by (1.6) and set criteria for the regularity of refinable function vectors in terms of the refinable matrix mask $\mathbf{P}$.

For this, we introduce the transition operator defined on $\mathbb{H}$, the space of all $\Phi \times \Phi$ matrices whose entries are trigonometric polynomials such that their Fourier coefficients are supported in $[-N, N]^s$. Here, we recall that the refinement mask $(P_\alpha)_\alpha$ is supported in $[0, N]^s$. The **transition operator** $\mathbf{T}_\Phi := \mathbf{T}$ is defined by

$$\mathbf{T}H := \sum_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{P}(\cdot/2 + \pi\nu) H(\cdot/2 + \pi\nu) \mathbf{P}^*(\cdot/2 + \pi\nu), \quad H \in \mathbb{H}.$$

Then, $\mathbf{T}$ is a linear operator on $\mathbb{H}$.

Denote by $\mathbb{H}_M$ the space of all $\Phi \times \Phi$ matrices whose entries are trigonometric polynomials such that their Fourier coefficients are supported in $[-M, M]^s$. Then, if $M \geq N$, the transition operator $\mathbf{T}$ can be defined as an operator on $\mathbb{H}_M$. Further, since if $M > N$, $\mathbf{T}$ is a Fourier coefficient support reduced operator on $\mathbb{H}_M$, any eigenmatrix of nonzero eigenvalues of $\mathbf{T}_{|\mathbb{H}_M}$ is in $\mathbb{H}$. Therefore, all results of this paper can be stated in terms of the transition operator $\mathbf{T}$ on $\mathbb{H}_M$ for $M > N$, although they are stated in terms of the transition operator $\mathbf{T}$ on $\mathbb{H}$.

If the functions Φ are the solutions of refinable equations (1.1), and if one writes $J(\omega)$ as a column block matrix by

$$J(\omega) = (\widehat{\phi}(\omega + 2\pi\nu + 4\pi\alpha))_{(\nu,\alpha)\times\phi \in (\mathbb{Z}^s/2\mathbb{Z}^s \times \mathbb{Z}^s)\times\Phi},$$

then

$$\overline{J^*}(\omega) = (\mathbf{P}(\omega/2 + \pi\nu)\overline{J^*}(\omega/2 + \pi\nu))_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s}.$$

Hence

$$G(\omega) = \overline{J^*}(\omega)\overline{J}(\omega) = \sum_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{P}(\omega/2 + \pi\nu) G(\omega/2 + \pi\nu) \mathbf{P}^*(\omega/2 + \pi\nu). \tag{1.13}$$

Therefore, the Gramian matrix $G_\Phi \in \mathbb{H}$ is an eigenmatrix of eigenvalue 1 of the transition operator $\mathbf{T}$.

Equation (1.13) also leads to the following result, which was proven by [GLT],[GL], [GHM], [CL], and [LCY].

Result 1.14. *If the compactly supported refinable function vector Φ is orthonormal, then*

$$I = \sum_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{P}(\omega/2 + \pi\nu) \mathbf{P}^*(\omega/2 + \pi\nu), \omega \in \mathbb{T}^s. \tag{1.15}$$

If the refinable function vector Φ is stable, then the matrix

$$\sum_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{P}(\omega/2+\pi\nu)\mathbf{P}^*(\omega/2+\pi\nu)$$

is not singular for all $\omega \in \mathbb{T}^s$.

A mask $\mathbf{P}$ satisfying (1.15) is called a **conjugate quadrature filter**, or CQF.

Since $\mathbb{H}$ is a finite dimensional space, the operator $\mathbf{T}$ can be represented by a finite order matrix with respect to some fixed basis of $\mathbb{H}$. The matrix is also denoted by $\mathbf{T}$, and we will identify the operator $\mathbf{T}$ with the matrix $\mathbf{T}$.

We say that the cascade algorithm defined in (1.6) **converges**, if Φ_n defined by (1.6) converges to Φ with $\widehat{\Phi}(0) = \mathbf{i}_1$ for all Φ_0 which satisfy

$$\sum_{\alpha \in \mathbb{Z}^s} \mathbf{i}_1^T \Phi_0(\cdot - \alpha) = 1, \text{and} \quad G_{\Phi_0} \in \mathbb{H}. \tag{1.16}$$

We note that if $\mathbf{P}$ satisfies the basic conditions (1.5); and Φ_0 satisfies (1.16), then the Φ_n defined by the cascade algorithm (1.6) always converges to Φ with $\widehat{\Phi}(0) = \mathbf{i}_1$ in the distribution sense.

The rest of the paper is organized as follows. In Section 2, we will prove that the cascade algorithm converges if and only if the transition operator $\mathbf{T}$ satisfies Condition E. In Section 3, we will prove that Φ is stable if and only if the transition operator $\mathbf{T}$ satisfies Condition E, and the corresponding eigenmatrix non-singular on $\mathbb{T}^s$. Consequently, we show that if Φ is stable, then the cascade algorithm converges; and if $\mathbf{P}$ is a CQF mask, then Φ is orthonormal if and only if it is stable. Regularity criteria in terms of mask are established in Section 4. We also remark that most of the results in this paper can be generalized to a general dilation matrix easily.

Finally, we remark that the corresponding results for the case $\#\Phi = 1$ were obtained in [LLS3] (convergence of the cascade algorithm), [LLS2] (stability and orthonormality) and [RiS3] (regularity).

2. Convergence of cascade algorithms

In this section we present a complete characterization of the convergence of the cascade algorithm defined by (1.6).

In what follows, we will identify the matrix $H \in \mathbb{H}$ with the corresponding unique sequence $(h_B^H)_{B \in \mathbb{B}}$ for a fixed basis $\mathbb{B}$, where

$$H = \sum_{B \in \mathbb{B}} h_B^H B.$$

We use the standard basis

$$\begin{aligned}\mathbb{B}_{st} :=\{&B_{i,j}^{\alpha} = (b_{l,l'}^{\alpha})_{1\le l,l'\le \#\Phi} \in \mathbb{H} :\\ &b_{i,j}^{\alpha} = \exp(-i\alpha\cdot); \quad b_{l,l'}^{\alpha} = 0,\ (l,l') \neq (i,j); \quad \alpha \in [-N,N]^s\}\end{aligned}$$

in the proof of the sufficiency part of the next theorem.

We also note that a sequence of matrices $(\mathbf{T}^n)$ generated by a finite order matrix $\mathbf{T}$ converges to a nontrivial matrix if and only if the spectral radius $\rho(\mathbf{T}) \le 1$, and 1 is the only eigenvalue on the unit circle and is nondegenerate. Furthermore the sequence $(\mathbf{T}^n)$ converges if and only if for all $H \in \mathbb{H}$, the sequence $(\mathbf{T}^n H)$ converges. Here the convergence of the sequence $(\mathbf{T}^n H)$ is equivalent to the convergence of the sequence $(h_B^{\mathbf{T}^n H})_n$ for a fixed basis $\mathbb{B}$. Since $\mathbf{T}(\lim_n \mathbf{T}^{n-1}H) = \lim_n \mathbf{T}^n H$, the matrix $\lim_n \mathbf{T}^n H$ is an eigenmatrix of $\mathbf{T}$ corresponding to the eigenvalue 1. In particular, if $\mathbf{T}$ satisfies Condition E, then, for arbitrary $H \in \mathbb{H}$, $\lim_n \mathbf{T}^n H = \mathrm{const} G_\Phi$.

A special basis of $\mathbb{H}$ is needed in the proof of the necessity part of the next theorem. The basis $\mathbb{B}_{sp}$ chosen is the one such that for each $B \in \mathbb{B}_{sp}$, there are Φ_0 and Ψ_0 satisfying (1.16) and $B = \overline{J_{\Phi_0}^* J_{\Psi_0}}$.

Define $\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$, where

$$\begin{aligned}\mathbb{D}_1 := \{&D_{1,1}^{\alpha} = (d_{l,l'}^{\alpha})_{1\le l,l'\le \#\Phi} \in \mathbb{H} :\\ &d_{1,1}^{\alpha} = \exp(-i\alpha\cdot), d_{l,l'}^{\alpha} = 0, (l,l') \neq (1,1); \quad \alpha \in [-N,N]^s\};\end{aligned}$$

and

$$\begin{aligned}\mathbb{D}_2 :=\{&D_{i,i}^{\alpha} = (d_{l,l'}^{\alpha})_{1\le l,l'\le \#\Phi} \in \mathbb{H} : \quad d_{1,1}^{\alpha} = 1,\ d_{i,i}^{\alpha} = \exp(-i\alpha\cdot), 1 < i,\\ &d_{1,i}^{\alpha} = \exp(-i\alpha\cdot), \quad d_{i,1} = 1,\\ &d_{l,l'}^{\alpha} = 0,\ \mathrm{if}(l,l') \neq (i,i), (1,1), (i,1)\,\mathrm{and}\,(1,i);\ \alpha \in [-N,N]^s\}.\end{aligned}$$

Then define $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_3$, where

$$\begin{aligned}\mathbb{E}_1 :=\{&E_{i,j}^{\alpha} = (e_{l,l'}^{\alpha})_{1\le l,l'\le \#\Phi} \in \mathbb{H} :\\ &e_{1,1}^{\alpha} = 1, \quad e_{i,j}^{\alpha} = \exp(-i\alpha\cdot),\ 1 < i,j\ i \neq j,\\ &e_{i,1}^{\alpha} = 1, \quad e_{1,j}^{\alpha} = \exp(-i\alpha\cdot),\\ &e_{l,l'}^{\alpha} = 0,\ \mathrm{if}\,(l,l') \neq (1,1), (i,1), (1,j), \mathrm{and}\,(i,j); \quad \alpha \in [-N,N]^s\};\end{aligned}$$

$$\begin{aligned}\mathbb{E}_2 :=\{&E_{1,j}^{\alpha} = (e_{l,l'}^{\alpha})_{1\le l,l'\le \#\Phi} \in \mathbb{H} : \quad e_{1,1}^{\alpha} = 1, \quad e_{1,j}^{\alpha} = \exp(-i\alpha\cdot), 1 < j,\\ &e_{l,l'}^{\alpha} = 0,\ \text{otherwise}; \quad \alpha \in [-N,N]^s\};\end{aligned}$$

and

$$\mathbb{E}_3 := \{B^T : B \in \mathbb{E}_2\}.$$

Then, the set $\mathbb{B}_{sp} := \mathbb{D} \cup \mathbb{E}$ is a basis of $\mathbb{H}$.

For function vectors Φ_0 and Ψ_0 satisfying (1.16), define function vectors

$$\Phi_n = (\phi_n^i)_{1\le i\le \#\Phi}^T \quad \text{and} \quad \Psi_n = (\psi_n^i)_{1\le i\le \#\Phi}$$

via their Fourier transform as

$$\widehat{\Phi}_n := \mathbf{P}(\cdot/2)\widehat{\Phi}_{n-1}(\cdot/2), \quad \text{and} \quad \widehat{\Psi}_n := \mathbf{P}(\cdot/2)\widehat{\Psi}_{n-1}(\cdot/2).$$

Let $G_n = \overline{J_{\Phi_n}^* J_{\Psi_n}}$. Then

$$\mathbf{T}G_{n-1} = \sum_{\nu\in\mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{P}(\cdot/2+\pi\nu)\overline{J_{\Phi_{n-1}}^*}(\cdot/2+\pi\nu)\overline{J_{\Psi_{n-1}}}(\cdot/2+\pi\nu)\mathbf{P}^*(\cdot/2+\pi\nu) = G_n.$$

If the cascade algorithm converges in the L_2-norm, then each entry of G_n converges to the corresponding entry of G_Φ in the L_1-norm. This implies G_n converges to G_Φ in the $\|\cdot\|_1-$norm on $\mathbb{H}$, where for $H(\omega) := (h_{i,j}(\omega))_{1\le i,j\le\#\Phi}$,

$$\|H\|_1 := \sum_{1\le i,j\le \#\Phi} \|h_{i,j}(\cdot)\|_1.$$

For a fixed $\mathbb{B}$, let

$$G_n = \sum_{B\in\mathbb{B}} a_B^n B, \quad \text{and} \quad G_\Phi = \sum_{B\in\mathbb{B}} a_B B.$$

Then, the convergence of the cascade algorithm implies that the sequence of the sequences $(a^n)_n$ converges to the sequence a.

Finally, we note that for any $\Phi := (\phi^l)^T$ and $\Psi := (\psi^{l'})^T$, the $(\phi^l, \psi^{l'})$th entry of $\overline{J_\Phi^* J_\Psi}$ can be written as

$$\sum_{\beta\in\mathbb{Z}^s} \widehat{\phi}^l(\cdot+2\pi\beta)\overline{\widehat{\psi}^{l'}(\cdot+2\pi\beta)} = \sum_{\alpha\in\mathbb{Z}^s} (\phi^l * \overline{\psi^{l'}(-\cdot)})(\alpha)\exp(-i\alpha\cdot). \tag{2.1}$$

We are ready to prove the following theorem:

Theorem 2.2. *If the basic conditions 1.5 hold for* $\mathbf{P}$*, then the cascade algorithm converges if and only if the transition operator* $\mathbf{T}$ *satisfies Condition E.*

Proof. "$\Longrightarrow$" Since $\mathbf{T}$ satisfies Condition E, the sequence of linear operators $\mathbf{T}^n$ converges. Let Φ_n be a sequence of function vectors generated by the cascade algorithm (1.6) with Φ_0 satisfying (1.16). Then $G_{\Phi_n} = \mathbf{T}^n G_{\Phi_0}$ converges. Since for each fixed l, $\|\phi_n^l\|$ is the coefficient of $B_{l,l}^0 \in \mathbb{B}_{st}$, if we express G_{Φ_n} by $\mathbb{B}_{st}$, and since $\mathbb{H}$ is a finite dimensional space, Φ_n is bounded in the L_2-norm. Hence any subsequence of Φ_n contains a a weakly convergent subsubsequence of Φ_n. Since Φ_n converges to Φ in the distribution sense, and since weak convergence is stronger than convergence in the distribution sense,

Φ_n converges to Φ weakly. Therefore, to show that Φ_n converges strongly to Φ, it remains to show that the L_2-norm of Φ_n converges to that of Φ. Since G_Φ is an eigenmatrix of $\mathbf{T}$ corresponding to the eigenvalue 1 and since $\mathbf{T}$ satisfies Condition E, G_Φ is the unique eigenmatrix of $\mathbf{T}$. Therefore,

$$\lim_{n\to\infty} \mathbf{T}^n G_{\Phi_0} = \lim_{n\to\infty} G_{\Phi_n} = \mathrm{const} G_\Phi.$$

Since $\mathbf{i}_1^T G_{\Phi_0}(0)\mathbf{i}_1 = \mathbf{i}_1^T G_{\Phi_n}(0)\mathbf{i}_1 \neq 0$ for all n, we have

$$0 \neq \mathbf{i}_1^T G_{\Phi_0}(0)\mathbf{i}_1 = \lim_{n\to\infty} \mathbf{i}_1^T \mathbf{T}^n G_{\Phi_0}(0)\mathbf{i}_1 = \lim_{n\to\infty} \mathbf{i}_1^T G_{\Phi_n}(0)\mathbf{i}_1 = \mathbf{i}_1^T G_\Phi(0)\mathbf{i}_1 = \mathrm{const}\mathbf{i}_1^T G_\Phi(0)\mathbf{i}_1.$$

Hence const $= 1$, and

$$\lim_{n\to\infty} \mathbf{T}^n G_{\Phi_0} = \lim_{n\to\infty} G_{\Phi_n} = G_\Phi.$$

Since for each fixed l, $\|\phi_n^l\|$ is the coefficient of $B_{l,l}^0 \in \mathbb{B}_{st}$ and since $\mathbb{H}$ is a finite dimensional space, we must have $\lim_n \|\phi_n^l\| = \|\phi^l\|$. Hence, Φ_n converges to Φ strongly in the L_2-norm.

"$\Longleftarrow$" We first prove that for any element $B \in \mathbb{B}_{sp}$, there is a proper choice of $\Phi_0 := (\phi^{l'})^T$ and $\Psi_0 := (\psi^l)^T$ satisfying (1.16) so that $\overline{J_{\Phi_0}^* J_{\Psi_0}} = B$.

For this, let $f := \chi_{[-1/2,1/2]^s}$.

First, if $B = D_{1,1}^\alpha \in \mathbb{D}_1$, then define $\Phi_0 = (\phi_0^l)^T$ and $\Psi_0 = (\psi_0^{l'})^T$, so that $\phi_0^1 = f$, $\psi_0^1 = f(\cdot - \alpha)$ and $\phi_0^l = 0$ and $\psi_0^{l'} = 0$, if $l, l' \neq 1$. Then Φ_0 and Ψ_0 satisfy (1.16) and $\overline{J_{\Phi_0}^* J_{\Psi_0}} = D_{1,1}^\alpha \in \mathbb{D}_1$ by (2.1). For $E_{i,j}^\alpha \in \mathbb{E}_1$ (or $D_{i,i}^\alpha \in \mathbb{D}_2$), define $\Phi_0 = (\phi_0^l)^T$ and $\Psi_0 = (\psi_0^{l'})^T$ to be the function vectors such that $\phi_0^1 = \psi_0^1 = f$ and $\phi_0^i = f$ and $\psi_0^j = f(\cdot - \alpha)$, $\phi_0^l = 0$, $l \neq 1, i$ and $\psi_0^{l'} = 0$ $l' \neq 1, j$. Then the function vectors Φ_0 and Ψ_0 satisfy (1.16). Further, the matrix $\overline{J_{\Phi_0}^* J_{\Psi_0}} = E_{i,j}^\alpha \in \mathbb{E}_1$, if $i \neq j$ (and $D_{i,i} \in \mathbb{D}_2$, if $i = j$) by (2.1). For $E_{i,j}^\alpha \in \mathbb{E}_2$, define $\Phi_0 = (\phi_0^l)^T$ and $\Psi_0 = (\psi_0^{l'})^T$ so that $\phi_0^1 = \psi_0^1 = f$, and $\phi_0^j = 0$ and $\psi_0^j = f(\cdot - \alpha)$ and $\phi_0^l = \psi_0^{l'} = 0$ if $l, l' \neq 1, j$. Then Φ_0 and Ψ_0 satisfy (1.16) and $\overline{J_{\Phi_0}^* J_{\Psi_0}} = E_{i,j}^\alpha \in \mathbb{E}_2$ by (2.1).

Since the cascade algorithm converges, $\mathbf{T}^n B$ converges to G_Φ for all $B \in \mathbb{B}_{sp}$. Thus for any $H \in \mathbb{H}$, $\lim_{n\to\infty} \mathbf{T}^n H = \mathrm{const} G_\Phi$; consequently, the sequence of matrices $(\mathbf{T}^n)$ converges. Therefore, the spectral radius $\rho(\mathbf{T}) \leq 1$ and 1 is the only eigenvalue of $\mathbf{T}$ on the unit circle. Further, 1 is a nondegenerate eigenvalue of $\mathbf{T}$. To prove that $\mathbf{T}$ satisfies Condition E, it remains to show that G_Φ is the only eigenmatrix of eigenvalue 1. Let $E \in \mathbb{H}$, so that $\mathbf{T}E = E$, then

$$\lim_{n\to\infty} \mathbf{T}^n E = E = \mathrm{const} G_\Phi.$$

Hence, G_Φ is the only eigenmatrix (up to a constant multiple) of $\mathbf{T}$ corresponding to eigenvalue 1. $\square$

3. Stability, orthonormality and biorthonormality

In this section we will discuss the stability of refinable function vectors. We first give here a sufficient condition in terms of the eigenvalue of the transition operator $\mathbf{T}$ under which the function vector Φ is stable. Then, we will show that this condition is also necessary.

Proposition 3.1. *Suppose that the basic conditions (1.5) holds for* $\mathbf{P}$ *and* $\Phi \subset L_2(\mathbb{R}^s)$. *If* 1 *is a simple eigenvalue of the transition operator* $\mathbf{T}$ *on* $\mathbb{H}$ *and the corresponding eigenmatrix non-singular on* $\mathbb{T}^s$*, then the* $\mathbf{P}$ *refinable function vector* Φ *is stable.*

Proof. Since G_Φ is an eigenmatrix of $\mathbf{T}$ of the eigenvalue 1, the hypothesis of the theorem implies that the matrix $G_\Phi(\omega)$ non-singular on $\mathbb{T}^s$. Hence, the function vector Φ is stable. □

We note that if Φ is stable, the simplicity of the eigenvalue 1 of $\mathbf{T}$ implies that the corresponding eigenmatrix of the eigenvalue 1 of $\mathbf{T}$ non-singular on $\mathbb{T}^s$, since in this case G_Φ is the only eigenmatrix of $\mathbf{T}$. Therefore, to show that the condition in the above theorem is necessary, one only requires to show that if Φ is stable, then 1 is a simple eigenvalue of $\mathbf{T}$.

Define

$$V_1 := \{H \in \mathbb{H} : (\mathbf{i}_1^T H \mathbf{i}_1)(0) = 0\}.$$

Since for any $H \in \mathbb{H}$, $H = \sum_{B \in \mathbb{B}_{sp}} h_B B$, where the set $\mathbb{B}_{sp}$ is the basis defined in the previous section, a matrix $H \in V_1$, if and only if $\sum_{B \in \mathbb{B}_{sp}} h_B = 0$ by the structure of the element of $\mathbb{B}_{sp}$. Hence, the space V_1 has codimension 1. Since $(\mathbf{i}_1^T G_\Phi \mathbf{i}_1)(0) \neq 0$, $G_\Phi \notin V_1$. Since $\mathbf{P}$ satisfies the basic condition (1.5), for any $H \in V_1$,

$$(\mathbf{i}_1^T(\mathbf{T}H)\mathbf{i}_1)(0) = \sum_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{i}_1^T \mathbf{P}(\pi\nu) H(\pi\nu) \mathbf{P}^*(\pi\nu) \mathbf{i}_1 = 0.$$

Hence, V_1 is a $\mathbf{T}$ invariant subspace of $\mathbb{H}$.

The proofs of the following two propositions (Propositions 3.2 and 3.5) were originally in our earlier drafts. Before completing the paper, we received a preprint of [LCY], which contains the same results (Proposition 3.3 and Theorem 5.1 Theorem 5.2 in there) with similar proofs. Thus, we will only provide an outline of the proofs here.

Proposition 3.2. *Let* $H_1(\omega)$ *and* $H_2(\omega)$ *be matrices so that each entry is a continuous function on* $\mathbb{T}^s$*. Then,*

$$\int_{\mathbb{T}^s} H_1(\omega)(\mathbf{T}^n H_2)(\omega) d\omega = \int_{\mathbb{R}^s} H_1(\omega)\Pi_n(\omega) H_2(2^{-n}w)\Pi_n^*(\omega) d\omega, \tag{3.3}$$

where

$$\Pi_n(\omega) := \chi_{2^n \mathbb{T}^s}(\omega) \prod_{j=1}^{n} \mathbf{P}(\omega/2^j);\ n = 1, 2, \ldots. \tag{3.4}$$

Here we define the transition operator as an operator on the space of the all $\Phi \times \Phi$ matrices whose entries are continuous functions on $\mathbb{T}^s$.

Proof. One can easily show that for any such H

$$\mathbf{T}^n H = \sum_{\alpha \in \mathbb{Z}^s} \Pi_n(\cdot + 2\pi\alpha) H(2^{-n}(\cdot + 2\pi\alpha) \Pi_n^*(\cdot + 2\pi\alpha),$$

by induction. Replacing H by H_2, multiplying by H_1 and integrating both sides of the above identity, lead to the fact that for any H_1 and H_2, (3.3) holds. □

Proposition 3.5. *Assume that the* **P** *refinable function vector* Φ *is stable and its mask* **P** *satisfies basic conditions (1.5). Then,*

(i) For any $H_1 \in \mathbb{H}$ *and* $H_2 \in V_1$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^s} H_1(\omega)\Pi_n(\omega)H_2(2^{-n}\omega)\Pi_n^*(\omega)d\omega = \int_{\mathbb{R}^s} \lim_{n\to\infty} H_1(\omega)\Pi_n(\omega)H_2(2^{-n}\omega)\Pi_n^*(\omega)d\omega = \mathbf{0}.$$

(ii) The transition operator $\mathbf{T}$ restricted to V_1 has spectral radius < 1.

Proof. Since Φ is stable, $G_\Phi \geq cI$, with $c > 0$. This leads to the fact that the sequence $(\Pi_n \Pi_n^*)$ is uniformly integrable (details in the proof of Theorem 5.2 of [LCY]). Recall that the sequence

$$(\Pi_n(\omega)\Pi_n^*(\omega)), \quad n = 0, 1, \ldots,$$

is uniformly integrable, if for an arbitrary $\varepsilon > 0$ there exist a finite measure set F and $\delta > 0$ so that

$$\int_{\mathbb{R}^s \setminus F} \Pi_n(\omega)\Pi_n^*(\omega)d\omega \leq \varepsilon,$$

and

$$\int_D \Pi_n(\omega)\Pi_n^*(\omega)d\omega \leq \varepsilon,$$

hold for all n, for any measurable set D with the measure of $D \leq \delta$.

That the sequence $(\Pi_n(\omega)\Pi_n^*(\omega))_n$ is uniformly integrable implies that the sequence $(H_1(\omega)\Pi_n(\omega)H_2(2^{-n}\omega)\Pi_n^*(\omega))$ is uniformly integrable for any $H_1 \in \mathbb{H}$ and $H_2 \in V_1$. This implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^s} H_1(\omega)\Pi_n(\omega)H_2(2^{-n}\omega)\Pi_n^*(\omega)d\omega = \int_{\mathbb{R}^s} \lim_{n\to\infty} H_1(\omega)\Pi_n(\omega)H_2(2^{-n}\omega)\Pi_n^*(\omega)d\omega.$$

Since $(\mathbf{i}_1^T H_2 \mathbf{i}_1)(0) = 0$ and since $\mathbf{P}^\infty = \begin{pmatrix} \widehat{\Phi} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}$,

$$\lim_{n\to\infty} H_1(\omega)\Pi_n(\omega)H_2(2^{-n}\omega)\Pi_n^*(\omega) = H_1(\omega)\mathbf{P}^\infty(\omega)H_2(0)\mathbf{P}^{*\infty}(\omega) = \mathbf{0}.$$

Hence, the first statement holds.

For the second statement, assume that λ is an eigenvalue of $\mathbf{T}$ restricted to V_1 and $H \in V_1$ is the corresponding nontrivial eigenmatrix. Then

$$\lambda^n \int_{\mathbb{T}^s} H^*(\omega)H(\omega)d\omega = \int_{\mathbb{R}^s} H^*(\omega)\Pi_n(\omega)H(2^{-n}\omega)\Pi_n^*(\omega)d\omega.$$

Hence, (i) implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^s} H^*(\omega)\Pi_n(\omega)H(2^{-n}\omega)\Pi_n^*(\omega)d\omega = \int_{\mathbb{R}^s} \lim_{n\to\infty} H^*(\omega)\Pi_n(\omega)H(2^{-n}\omega)\Pi_n^*(\omega)d\omega = \mathbf{0}.$$

This gives

$$\lim_{n\to\infty} \lambda^n \int_{\mathbb{T}^s} H^*(\omega)H(\omega)d\omega = 0.$$

Therefore $|\lambda| < 1$ by the fact

$$\int_{\mathbb{T}^s} H^*(\omega)H(\omega)d\omega \neq 0.$$

□

From the two propositions above, we obtain the following result:

Lemma 3.6. *Assume that* $\mathbf{P}$ *satisfies basic conditions (1.5) and the corresponding refinable function vector* Φ *is stable, then the transition operator* $\mathbf{T}$ *satisfies Condition E. In particular, 1 is a simple eigenvalue of the transition operator* $\mathbf{T}$.

Proof. Let $\mathbb{B}_0$ be a basis of V_1. Since G_Φ is not in V_1 and V_1 has codimension 1, $G_\Phi \cup \mathbb{B}_0$ is a basis of $\mathbb{H}$. Therefore, an arbitrary $H \in \mathbb{H}$ can be written uniquely as

$$H = aG_\Phi + H_0, \quad H_0 \in V_1.$$

Let H be the eigenmatrix of the eigenvalue λ of $\mathbf{T}$,

$$\lambda a G_\Phi + \lambda H_0 = \mathbf{T}H = a\mathbf{T}G_\Phi + \mathbf{T}H_0 = aG_\Phi + \mathbf{T}H_0.$$

If $\lambda \neq 1$, then $a = 0$. Thus, $H = H_0 \in V_1$ is an eigenmatrix of λ. This implies $|\lambda| < 1$ by Proposition 3.5. If $\lambda = 1$, then $H_0 \in V_1$ is also the eigenmatrix of $\mathbf{T}$ corresponding to the eigenvalue 1, thus $H_0 = 0$ again by Proposition 3.5. Hence, $\rho(\mathbf{T}) \leq 1$ and 1 is the unique eigenvalue on the unit circle. Further G_Φ is the only eigenmatrix of eigenvalue 1 up to a constant multiple.

Finally we need to show that 1 is a simple eigenvalue of $\mathbf{T}$. If not, it must be a degenerate eigenvalue with only one eigenmatrix. In this case, there exists a matrix $H \in \mathbb{H}$ such that $\mathbf{T}H = G_\Phi + H$. Let $H_1 = cG_\Phi + H$ so that $H_1 \in V_1$. Then

$$\int_{\mathbb{T}^s} (\mathbf{T}^n H_1)(\omega)d\omega = \int_{\mathbb{R}^s} \Pi_n(\omega)H_1(2^{-n}\omega)\Pi_n^*(\omega)d\omega = \int_{\mathbb{T}^s} ((c+n)G_\Phi(\omega) + H(\omega))d\omega.$$

The left hand side tends to 0 by the stability of the vector function Φ and (i) of Proposition 3.5, while the right hand side tends to ∞, which is a contradiction. □

An immediate consequence of this lemma is:

Corollary 3.7. *Assume that* $\mathbf{P}$ *satisfies the conditions (1.5). If the refinable vector function* Φ *is stable, then the cascade algorithm converges.*

The next theorem, the main result of this section, follows directly from Proposition 3.1 and Lemma 3.6.

Theorem 3.8. *Assume that* $\mathbf{P}$ *satisfies the basic conditions (1.5). The* $\mathbf{P}$ *refinable function vector* Φ *is stable if and only if the corresponding transition operator* $\mathbf{T}$ *satisfies Condition E and the eigenmatrix of eigenvalue 1 is nonsingular on* $\mathbb{T}^s$*.*

Proof. If the transition operator satisfies Condition E, then the cascade algorithm converges and $\Phi \subset L_2(\mathbb{R}^s)$. Therefore, Φ is stable by Proposition 3.1. If Φ is stable, then Lemma 3.6 implies that the transition operator satisfies Condition E. G_Φ is the eigenmatrix of a simple eigenvalue 1 of $\mathbf{T}$ which is nonsingular on $\mathbb{T}^s$. □

Remark 3.9. *If the transition operator* $\mathbf{T}$ *satisfies Condition E and if eigenvalue 1 has an eigenmatrix which is non-singular on* $\mathbb{T}^s$*, then the compactly supported* $\mathbf{P}$ *refinable functions* $\Phi \subset L_2(\mathbb{R}^s)$ *by the fact that the corresponding cascade algorithm converges. Hence the sequence* $(S^k(\Phi))$ *forms a multiresolution of* $L_2(\mathbb{R}^s)$ *with the functions* Φ *and their shifts forming a Riesz basis of* $S(\Phi)$*, by Result 1.11 and Theorem 3.8.*

If $\mathbf{P}$ is CQF, the identity matrix I is an eigenmatrix of the transition operator $\mathbf{T}$ corresponding to eigenvalue 1. A consequence of Theorem 3.8 is:

Theorem 3.10. *Suppose that* $\mathbf{P}$ *is a CQF matrix mask which satisfies the basic conditions (1.5), then the following statements are equivalent:*
- *(i) the refinable function vector* Φ *is orthonormal,*
- *(ii) the transition operator* $\mathbf{T}$ *satisfies Condition E,*
- *(iii) the refinable function vector* Φ *is stable, and*
- *(iv) the corresponding cascade algorithm converges.*

Remark 3.9 gives the following corollary:

Corollary 3.11. *Suppose that* $\mathbf{P}$ *is a CQF matrix mask which satisfies the basic conditions (1.5). If the corresponding transition operator* $\mathbf{T}$ *satisfies Condition E, then the sequence of spaces* $(S^k(\Phi))_k$ *forms a multiresolution of* $L_2(\mathbb{R}^s)$ *with the functions* Φ *and their shifts forming an orthonormal basis of* $S(\Phi)$*.*

In the rest of this section, we discuss the biorthonormality of two refinable function vectors Φ and Ψ. Let $\mathbf{P}_\Phi$ and $\mathbf{P}_\Psi$ be the refinement masks of functions Φ and Ψ satisfying the basic conditions (1.5) and the condition

$$\sum_{\nu \in \mathbb{Z}^s/2\mathbb{Z}^s} \mathbf{P}_\Phi(\cdot/2 + \pi\nu)\mathbf{P}_\Psi^*(\cdot/2 + \pi\nu) = I. \tag{3.12}$$

We say that Φ and Ψ are **biorthonormal** if both function vectors Φ and Ψ are stable and

$$\overline{J_\Phi^*}(\omega)\overline{J_\Psi}(\omega) = I, \quad \in \mathbb{T}^s.$$

Here again we are interested in characterizing the biorthonormality in terms of the matrix masks $\mathbf{P}_\Phi$ and $\mathbf{P}_\Psi$. The following result was shown in [LCY] Theorem 5.3, which is the main result of [LCY].

Result 3.13. *Let $\mathbf{P}_\Phi$ and $\mathbf{P}_\Psi$ be the refinement masks of refinable function vectors Φ and Ψ satisfying the basic conditions (1.5) and (3.12). Assume that $G_\Phi(0) \geq \mathrm{const}I$ and $G_\Psi(0) \geq \mathrm{const}I$. Then Φ and Ψ are biorthonormal, if both $\mathbf{T}_\Phi$ and $\mathbf{T}_\Psi$ have the spectrum radius < 1 on V_1.*

We note that if the Φ and Ψ are stable, then by Proposition 3.5(ii) the conditions in the above result are satisfied.

Theorem 3.14. *Let $\mathbf{P}_\Phi$ and $\mathbf{P}_\Psi$ be the refinement matrix masks of Φ and Ψ which satisfy the basic conditions (1.5) and condition (3.12). Then the following statements are equivalent:*

(i) the refinable function vectors Φ and Ψ are biorthonormal,
(ii) both Φ and Ψ are stable, and
(iii) the transition operators $\mathbf{T}_\Phi$ and $\mathbf{T}_\Psi$ satisfy Condition E; and the corresponding eigenmatrices of eigenvalue 1 of $\mathbf{T}_\Phi$ and $\mathbf{T}_\Psi$ are nonsingular on $\mathbb{T}^s$.

Proof. The equivalence of (ii) and (iii) follows from Theorem 3.8. Since if Φ and Ψ are stable, the conditions in Result 3.13 are satisfied; (ii) implies (i) by that result. Finally, (i) implies (ii) by the definition of the biorthonormality of Φ and Ψ. □

4. Regularity of refinable function vectors

In this section, we establish some criteria for the regularity of refinable function vectors.

We say that the mask $\mathbf{P}$ satisfying the basic conditions (1.5) has **vanishing moment order** r, if conditions

$$D^\beta(A^*(2\omega)\mathbf{P}(\omega))_{|_{\omega=\pi\nu}} = i^{-|\beta|}(D^\beta A^*)(0)\delta_\nu, \quad \nu \in \mathbb{Z}^s/2\mathbb{Z}^s,\ |\beta| \leq r-1, \tag{4.1}$$

hold, for some

$$A = \sum_{|\beta|\leq r-1} \mathbf{a}_\beta^T \exp(-i\beta\cdot),$$

where $\mathbf{a}_\beta \in \mathbb{R}^{\#\Phi}$.

As we did in [RiS3] for the case $\#\Phi = 1$, we will connect this vanishing moment order to the regularity of Φ.

We say that $\Phi := (\phi^l)^T \in C^\gamma$, if each component $\phi^l \in C^\gamma$. Recall that a function $\phi \in C^\gamma$ for $n \le \gamma < n+1$ provided that $\phi \in C^n$ and

$$|D^\beta \phi(x+t) - D^\beta \phi(x)| \le \text{const}|t|^{\gamma-n}, \text{ for all } |\beta| = n \text{ and } |t| \le 1$$

for some constant independent of x. This number is related to

$$\kappa_2 := \sup\{\kappa : \int_{\mathbb{R}^s} (1+|w|^2)^\kappa |\widehat{\phi}(\omega)|^2 d\omega < \infty\}$$

by the inequality $\gamma \ge \kappa_2 - s/2$.

Define

$$V_r := \{H \in \mathbb{H} : (D^\beta(A^*(\omega)H(\omega))_{|\omega=0})^T = D^\beta(H(\omega)A(\omega))_{|\omega=0}, = \mathbf{0}, \quad |\beta| \le r-1\}.$$

In the case that $r - 1 > N$, we replace $\mathbb{H}$ by $\mathbb{H}_{r-1}$ in the above definition of V_r.

Since

$$D^\beta(A^*(2\omega)\mathbf{P}(\omega))_{|\omega=\pi\nu} = i^{-|\beta|}(D^\beta A^*)(0)\delta_\nu, \quad \nu \in \mathbb{Z}^s/2\mathbb{Z}^s,\ |\beta| \le r-1, \tag{4.2}$$

we have that

$$(D^\beta(A^*(\omega)\mathbf{T}H(\omega))_{|\omega=0})^T = D^\beta(\mathbf{T}H(\omega)A(\omega))_{|\omega=0} = \mathbf{0}, \text{ for all } H \in V_r, |\beta| \le r-1.$$

Hence, the space V_r is an invariant subspace of the transition operator $\mathbf{T}$.

In the case $r = 1$, the space V_1 defined here is an invariant subspace of V_1 defined in §3, since $A^*(0)$ is the left eigenvector of $\mathbf{P}$.

For each $H(\omega) := (h_{i,j}(\omega))_{1\le i,j\le \#\Phi} \in V_r$, define

$$\|H\|_F := \sum_{1\le i,j\le \#\Phi} \|h_{i,j}(\cdot)\|_\infty.$$

If H is a constant matrix, this norm is the sum of the modulus of all entries.

Then the operator norm $\|\mathbf{T}_{|V_r}\| := \sup_{H\in V_r\setminus\{0\}} \frac{\|\mathbf{T}H\|_F}{\|H\|_F}$ on V_r satisfies

$$\lim_{n\to\infty} \|\mathbf{T}^n_{|V_r}\|^{1/n} = \rho,$$

where ρ is the spectral radius of $\mathbf{T}_{|V_r}$. Hence, there exists $N_\mathbf{T}$, such that for any $H \in V_r$ and for all $n > N_\mathbf{T}$,

$$\|\mathbf{T}^n H\|_F \le \|\mathbf{T}^n\|\|H\|_F \le \text{const}(\rho+\varepsilon)^n\|H\|_F, \tag{4.3}$$

where ε is arbitrarily small.

The proof of the following proposition is carried out by modifying the proofs of Proposition 3.6 of [RiS3] and Theorem 5.2 of [LCY].

Proposition 4.4. *Suppose* $\mathbf{P}$ *satisfies the basic conditions (1.5) and conditions (4.1). Then for the* $\mathbf{P}$ *refinable function* $\Phi := (\phi^l)^T$, *there exists a constant* C *such that*

$$\int_{\mathbb{F}_n} \left|\widehat{\phi}^l(\omega)\right|^2 d\omega \le C(\rho+\varepsilon)^{n+1},$$

where $\mathbb{F}_n := 2^n\mathbb{T}^s \backslash 2^{n-1}\mathbb{T}^s$ *for all* $n > N_{\mathbf{T}}$ *and* ε *is arbitrarily small.*

Proof. It follows from (4.3) for any $H \in V_r$,

$$\| \int_{\mathbb{F}_n} \mathbf{T}^n H(\omega) d\omega \|_F \le \mathrm{const}(\rho+\varepsilon)^n \|H\|_F.$$

Since none of choices of the constants in this proof depend on n, for simplicity we denote all constants by "const" even though the value of this may change with each occurrence.

Let $H(\omega) := (\sum_{\ell=1}^s (1-\cos w(\ell))^{r-1})I$. Since

$$(D^\beta(A^*(\omega)H(\omega))_{|\omega=0})^T = D^\beta(H(\omega)A(\omega))_{|\omega=0} = \mathbf{0},\ |\beta| \le r-1,$$

we have $H \in V_r$ and $H \ge I$ for all $\omega \in \mathbb{T}^s \backslash (1/2\mathbb{T}^s)$. Since $\|\mathbf{P}(\omega) - \mathbf{P}(0)\| \le \mathrm{const}\|\omega\|$, the function $\widehat{\Phi}$ is bounded on $\mathbb{T}^s$.

We also note that

$$\widehat{\Phi}(\omega) = \Pi_n(\omega)\widehat{\Phi}(2^{-(n+1)}\omega).$$

Hence we have

$$\begin{aligned}
\int_{\mathbb{F}_n} |\widehat{\phi}^l(\omega)|^2 d\omega &= \int_{\mathbb{F}_n} \mathbf{i}_l^T \widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)\mathbf{i}_l d\omega \\
&= \int_{\mathbb{F}_n} \mathbf{i}_l^T \Pi_n(\omega)\widehat{\Phi}(2^{-(n+1)}\omega)\widehat{\Phi}^*(2^{-(n+1)}\omega)\Pi_n^*(\omega)\mathbf{i}_l d\omega \\
&\le \mathrm{const} \int_{\mathbb{F}_n} \mathbf{i}_l^T \Pi_n(\omega) H(2^{-(n+1)}\omega)\Pi_n^*(\omega)\mathbf{i}_l d\omega \\
&\le \mathrm{const} \int_{2^n\mathbb{T}^s} \mathbf{i}_l^T \Pi_n(\omega) H(2^{-(n+1)}\omega)\Pi_n^*(\omega)\mathbf{i}_l d\omega \\
&\le \mathrm{const} \| \int_{2^n\mathbb{T}^s} \Pi_n(\omega) H(2^{-(n+1)}\omega)\Pi_n^*(\omega) d\omega \|_F \\
&= \mathrm{const} \| \int_{\mathbb{T}^s} (\mathbf{T}^{(n+1)}H)(\omega) d\omega \|_F \\
&\le \mathrm{const}(\rho+\varepsilon)^{n+1},
\end{aligned}$$

where $\mathbf{i}_l$ is the $\Phi \times 1$ column vector whose lth entry is 1 and all others are 0. □

This Proposition together with the usual Littlewood-Paley technique leads to the following estimate of the regularity of the refinable function vector Φ.

Theorem 4.5. *Suppose* $\mathbf{P}$ *satisfies the basic conditions 1.5 and the conditions (4.1), and let* ρ *be the spectral radius of* $\mathbf{T}_{|V_r}$. *Then the function* $\Phi = (\phi^l)^T$ *is in* $C^{\gamma-\varepsilon}$ *for any* $\varepsilon > 0$ *and* $\gamma = -\log\rho/(2\log 2) - s/2$.

Proof. Since when $n > N_{\mathbf{T}}$,

$$\int_{\mathbb{F}_n} |\widehat{\phi}^l(\omega)|^2 d\omega \le \mathrm{const}\rho^{n+1},$$

and since the function $\widehat{\phi}^l$ is bounded on $2^{N_T}\mathbb{T}^s$,

$$\int_{\mathbb{R}^s} (1+|w|^2)^\kappa |\widehat{\phi}^l(\omega)|^2 d\omega \le \mathrm{const}(1+\sum_{n=1}^{\infty} 2^{2n\kappa}\rho^{n+1}).$$

Hence $\phi^l \in C^{\gamma-\varepsilon}$ where $\gamma = -\log\rho/(2\log 2) - s/2$. That is, $\Phi \in C^{\gamma-\varepsilon}$. □

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