WAVELETS WITH SHORT SUPPORT

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Abstract. This paper is to construct Riesz wavelets with short support. Riesz wavelets with short support are of interests in both theory and application. In theory, it is known that a B-spline of order m has the shortest support among all compactly supported refinable functions with the same regularity. However, it remained open whether a Riesz wavelet with the shortest support and m vanishing moments can be constructed from the multiresolution analysis generated by the B-spline of order m. In various applications, a Riesz wavelet with a short support, a high order of regularity and vanishing moments is often desirable in signal and image processing, since they have a good time frequency localization and approximation property, as well as fast algorithms. This paper is to present a theory for the construction of Riesz wavelets with short support and to give various examples. In particular, from the multiresolution analysis whose underlying refinable function is the B-spline of order m, we are able to construct the shortest supported Riesz wavelet with m vanishing moments. The support of the wavelet functions can be made even much shorter by reducing their orders of vanishing moments. The study here also provides a new insight of the structures of the spline tight frame systems constructed in [20, 9, 12] and bi-frame systems in [9, 8].

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1. Introduction

This paper is to design Riesz wavelets with short support from a multiresolution analysis. We start with some basic notions and definitions. A function ϕ is *refinable* if it satisfies the refinement equation

$$\phi = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2\cdot -k), \tag{1.1}$$

where $a:\mathbb{Z}\mapsto\mathbb{C}$ is a sequence on $\mathbb{Z}$, called the *refinement mask* for ϕ.

By $\ell_2(\mathbb{Z})$ we denote the set of all sequences $u:\mathbb{Z}\mapsto\mathbb{C}$ such that

$$\|u\|_{\ell_2(\mathbb{Z})} := \Big(\sum_{k\in\mathbb{Z}} |u(k)|^2\Big)^{1/2} < \infty.$$

The *Fourier series* $\hat{u}$ of a sequence u in $\ell_2(\mathbb{Z})$ is defined as

$$\hat{u}(\xi) := \sum_{k\in\mathbb{Z}} u(k)e^{-ik\xi}, \qquad \xi\in\mathbb{R}, \tag{1.2}$$

where i is the imaginary unit such that $i^2=-1$. Similarly, the Fourier transform of a function $f\in L_1(\mathbb{R})$ is defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-i\xi t}\,dt, \qquad \xi\in\mathbb{R},$$

which can be naturally extended to functions in $L_2(\mathbb{R})$ and tempered distributions. With these, refinement equation (1.1) can be rewritten as, in terms of the Fourier transform,

$$\hat{\phi}(\xi) = \hat{a}(\xi/2)\hat{\phi}(\xi/2), \qquad a.e.\,\xi\in\mathbb{R}.$$

Using the above definition, one extends the concept of refinable functions to that of refinable distributions. Throughout the paper we assume that $\hat{\phi}(0)\ne 0$ and $\hat{a}(0)=1$. We also call $\hat{a}$ a refinement mask for convenience.

For a given compactly supported refinable function $\phi\in L_2(\mathbb{R})$, define V_0 to be the smallest closed subspace of $L_2(\mathbb{R})$ generated by $\phi(\cdot-k)$, $k\in\mathbb{Z}$. Then, V_0 is a shift (integer translate) invariant subspace of $L_2(\mathbb{R})$. Let $V_j := \{f(2^j\cdot) : f\in V_0\}$, for $j\in\mathbb{Z}$. Then the sequence of subspaces V_j, $j\in\mathbb{Z}$, forms a multiresolution analysis (MRA) in $L_2(\mathbb{R})$, which is generated by ϕ, i.e., (i) $V_j\subset V_{j+1}$, $j\in\mathbb{Z}$; (ii) $\overline{\cup_{j\in\mathbb{Z}}V_j} = L_2(\mathbb{R})$ and $\cap_{j\in\mathbb{Z}}V_j=\{0\}$ (see e.g. [1] and [15]). In this paper, a function $\psi\in V_1$ (or more precisely, $\psi\in V_1\backslash V_0$) is called an MRA-based wavelet function, or simply a wavelet, derived from the multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$.

In this paper, we are interested in finding a function ψ such that the wavelet system

$$X(\psi) := \{\psi_{j,k} := 2^{j/2}\psi(2^j\cdot -k) \;:\; j,k\in\mathbb{Z}\}$$

forms a Riesz basis for $L_2(\mathbb{R})$. The set $X(\psi)$ is called the wavelet system generated by ψ. Recall that the system $X(\psi)$ is a *Riesz basis* of $L_2(\mathbb{R})$ if the linear span of $X(\psi)$ is dense in $L_2(\mathbb{R})$ and $X(\psi)$ is a Riesz sequence, that is, there exist two positive constants C_1 and C_2 such that

$$C_1\|\{c_{j,k}\}\|_{\ell_2(\mathbb{Z}^2)} \leqslant \Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}\Big\|_{L_2(\mathbb{R})} \leqslant C_2\|\{c_{j,k}\}\|_{\ell_2(\mathbb{Z}^2)} \qquad \forall\,\{c_{j,k}\}_{j,k\in\mathbb{Z}}\in\ell_2(\mathbb{Z}^2). \tag{1.3}$$

If $X(\psi)$ is a Riesz basis of $L_2(\mathbb{R})$, then ψ is called a *Riesz wavelet*. To construct a compactly supported MRA-based Riesz wavelet ψ, one starts with a compactly supported refinable function ϕ with stable shifts. Recall that the shifts of a function ϕ are *stable* if $\{\phi(\cdot-k) \;:\; k\in\mathbb{Z}\}$ is a Riesz sequence, that is, there exist two positive constants C_1 and C_2 such that $C_1\|\{c_k\}\|_{\ell_2(\mathbb{Z})} \leqslant \|\sum_{k\in\mathbb{Z}} c_k\phi(\cdot-k)\|_{L_2(\mathbb{R})} \leqslant C_2\|\{c_k\}\|_{\ell_2(\mathbb{Z})}$ for all $\{c_k\}_{k\in\mathbb{Z}}\in\ell_2(\mathbb{Z})$. Then a compactly supported Riesz wavelet ψ is obtained by selecting some desirable finitely supported sequence b that is called a *wavelet mask* or a high-pass filter in the language of engineering. With the wavelet mask b, the wavelet function ψ is obtained from b and the refinable function ϕ via

$$\psi = 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot -k), \quad \text{or equivalently,} \quad \hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2). \tag{1.4}$$

When $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal system, a natural choice of b is

$$b(k) = (-1)^{k-1}\overline{a(1-k)}, \qquad k \in \mathbb{Z}.$$

Its Fourier series can be written as

$$\hat{b}(\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)}. \tag{1.5}$$

Then it is well known that the wavelet system $X(\psi)$ forms an orthonormal basis of $L_2(\mathbb{R})$. The wavelet function ψ has the same length of support as that of the corresponding refinable function ϕ. Thus, once a compactly supported refinable function ϕ whose shifts form an orthonormal system is given, the corresponding orthonormal wavelet ψ can be obtained easily. Hence, the major task and difficulty in constructing compactly supported dyadic orthonormal wavelets in dimension one are to design refinement masks whose corresponding refinable functions have the required smoothness order and whose shifts form an orthonormal system. This was discussed in detail in [6].

On the other hand, compactly supported refinable functions whose shifts form a Riesz sequence are much easier to get. One class of such refinable functions are B-splines.

The B-spline function of order m ($m \in \mathbb{N}$), denoted by B_m, can be obtained via the following recursive formula: $B_1 = \chi_{[0,1]}$, the characteristic function of the interval $[0,1]$, and

$$B_m(x) := \int_0^1 B_{m-1}(x-t)\,dt, \qquad x \in \mathbb{R},\ m \in \mathbb{N}. \tag{1.6}$$

The B-spline function $B_m \in C^{m-2}(\mathbb{R})$ is a function of piecewise polynomials of degree less than m, vanishes outside the interval $[0,m]$ and is symmetric about the point $m/2$, that is, $B_m(m-x) = B_m(x)$ for all $x \in \mathbb{R}$. It is well known that the B-spline function B_m is a refinable function satisfying the refinement equation

$$\widehat{B_m}(2\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^m \widehat{B_m}(\xi), \qquad \xi \in \mathbb{R}. \tag{1.7}$$

When $m = 1$, the shifts of B_1 form an orthonormal basis of the shift invariant space V_0 generated by B_1. The shifts of B_m, $m > 1$, form a Riesz, but not an orthonormal, basis of the shift invariant space V_0 generated by B_m. When m is even, $B_m(\cdot - m/2)$ is symmetric about the origin with the refinement mask

$$\hat{a}(\xi) = \cos^{2m}(\xi/2).$$

While compactly supported refinable functions with stable shifts are easier to obtain, the construction of compactly supported Riesz wavelets from an MRA generated by a B-spline of order m is not straightforward. A compactly supported Riesz wavelet ψ from the MRA generated by B_m was first constructed in [3]. While $X(\psi)$ forms a Riesz basis of $L_2(\mathbb{R})$ and the system keeps the orthogonality between different dilation levels, the support of the pre-wavelet ψ in [3] is $[0, 2m-1]$ (therefore, almost two times that of B_m) and ψ has m vanishing moments. Recently, [16] derived a Riesz wavelet ψ from B_m such that the Riesz wavelet system $X(\psi)$ forms a Riesz basis of $L_2(\mathbb{R})$. When ψ is required to have m vanishing moments, the construction of [16] gives the pre-wavelet of [3] and hence its support is $[0, 2m-1]$; efforts are made in [16] to shorten the support of the Riesz wavelets at the cost of a reduced order of vanishing moments of the Riesz wavelets. For example, the support of the Riesz wavelet ψ

can be reduced to $[0, m]$, when m is odd (or $[0, m+1]$, when m is even) with 1 or 2 vanishing moments. It is a bit surprising to us that there are no discussions in the literature whether the following natural choice of a function

$$\psi := 2\sum_{k\in\mathbb{Z}} b(k)B_m(2\cdot -k) \tag{1.8}$$

with

$$b(k) = (-1)^{k-1}\overline{a(1-k)}, \qquad k \in \mathbb{Z}, \tag{1.9}$$

where $\hat{a}(\xi) = 2^{-m}(1+e^{-i\xi})^m$ is the refinement mask of B_m, is a Riesz wavelet. There are several other motivations that lead to the discussions here. First, the mask defined in (1.9) works for the case $m = 1$. It is clear that when $m = 1$, the corresponding wavelet function ψ is the Haar wavelet. Hence, $X(\psi)$ is an orthonormal basis of $L_2(\mathbb{R})$. In fact, it works for an arbitrary compactly supported refinable function whose shifts form an orthonormal system. It is natural to ask whether the function ψ in (1.8) with the wavelet mask defined in (1.9) is a Riesz wavelet. Secondly, ψ in (1.8) has the same length of the support as that of B_m. Further, as it will be shown in Section 2, in some sense, ψ is the shortest supported Riesz wavelet of regularity $m - 1/2$ with m vanishing moments. Recall that a function ψ has m *vanishing moments* if

$$\hat{\psi}^{(j)}(0) = 0 \qquad \forall\, j = 0, \ldots, m-1,$$

where $\hat{\psi}^{(j)}$ denotes the jth derivative of $\hat{\psi}$. This means that it has good time frequency localization and it can lead to efficient algorithms in applications. One of the main objectives of this paper is to prove that the system $X(\psi)$, where ψ is defined in (1.8), forms a Riesz basis of $L_2(\mathbb{R})$, which will be established in Section 2. The spline-based Riesz wavelets in this paper with short support and high vanishing moments may be of interest in wavelet-based numerical algorithms, since the wavelet functions are piecewise polynomials, while it is well known that most other wavelets do not have explicit analytic forms.

In many applications, it is not only important to have Riesz wavelets with short support, but also desirable to have short supported Riesz wavelets with a small condition number, namely, a small ratio of the upper and lower Riesz bounds in (1.3). The condition number of the spline Riesz wavelet suggested here cannot be smaller than that of the system $\{B_m(\cdot - k) : k \in \mathbb{Z}\}$. However, it is well known that the condition number of the system $\{B_m(\cdot - k) : k \in \mathbb{Z}\}$ increases as m goes to ∞. In this regard, it is of interest to construct Riesz wavelets with short support, which are as close as possible to some orthonormal wavelets or tight frame wavelets, for a given order of regularity or vanishing moments. Such wavelet systems will then have small condition numbers. This is of interest and importance in various applications, although it is not a topic addressed in this paper.

The general theory needed for this paper is given in Section 3. The theory also provides a new insight of the systematic constructions of the tight frame systems from the B-spline of order m by using the unitary extension principle of [20] and the oblique extension principle of [9]. In those constructions, for a given B_m, a set $\Psi := \{\psi^1, \ldots, \psi^L\}$ of functions is obtained so that the system

$$X(\Psi) := \{\psi^\ell_{j,k} := 2^{j/2}\psi^\ell(2^j \cdot -k) \;:\; \ell = 1, \ldots, L \;\text{ and }\; j, k \in \mathbb{Z}\}$$

forms a tight frame for $L_2(\mathbb{R})$. That is,

$$f = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j,k}\rangle \psi^\ell_{j,k} \qquad \forall\, f \in L_2(\mathbb{R}).$$

The system $X(\Psi)$ is a redundant system. However, we discover from the study here that in all spline constructions, there exists a function $\psi \in \Psi$, such that $X(\psi)$ (in some cases $X(\psi(\cdot - 1/2))$

forms a Riesz basis for $L_2(\mathbb{R})$. This finding roughly says that one of the functions in the set Ψ already can generate a Riesz basis for $L_2(\mathbb{R})$, the other functions in Ψ are there just to either improve the condition number determined by the upper and lower frame bounds or provide a better dual system.

More generally, a system $X(\psi)$ is *Bessel* in $L_2(\mathbb{R})$ if there exists a positive constant C such that

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \leqslant C\|f\|^2 \qquad \forall\, f \in L_2(\mathbb{R}). \tag{1.10}$$

A system $X(\psi)$ is Bessel if both functions $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\cdot + 2\pi k)|$ and $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\cdot)|$ are in $L_\infty(\mathbb{R})$ (see [22, Corollary 15]). This will hold whenever ψ has a sufficient smoothness order. For example, it is known ([11, Propositions 2.6 and 3.5]) that if for some $\varepsilon > 0$, there exists a positive constant C such that $|\hat{\psi}(\xi)| \leqslant C(1+|\xi|)^{-1/2-\varepsilon}$ and $|\hat{\psi}(\xi)| \leqslant C|\xi|^{\varepsilon}$ for all $\xi \in \mathbb{R}$, then the corresponding system $X(\psi)$ is Bessel in $L_2(\mathbb{R})$. It is clear that for a finite set Ψ, $X(\Psi)$ is Bessel if and only if $X(\psi)$ is Bessel for each $\psi \in \Psi$.

A system $X(\Psi)$ with $\Psi = \{\psi^1, \dots, \psi^L\}$ is a *frame* for $L_2(\mathbb{R})$ if there exist positive constants C_1 and C_2 such that

$$C_1\|f\|^2 \leqslant \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{j,k}\rangle|^2 \leqslant C_2\|f\|^2 \qquad \forall\, f \in L_2(\mathbb{R}). \tag{1.11}$$

Let $\Psi := \{\psi^1, \dots, \psi^L\}$ and $\tilde{\Psi} := \{\tilde{\psi}^1, \dots, \tilde{\psi}^L\}$ be two sets of functions in $L_2(\mathbb{R})$. We say that $(X(\Psi), X(\tilde{\Psi}))$ is a pair of *bi-frames* in $L_2(\mathbb{R})$ if each of $X(\Psi)$ and $X(\tilde{\Psi})$ is Bessel in $L_2(\mathbb{R})$, and if $X(\Psi)$ and $X(\tilde{\Psi})$ satisfy

$$\langle f, g\rangle = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}^\ell_{j,k}\rangle\langle \psi^\ell_{j,k}, g\rangle \qquad \forall\, f, g \in L_2(\mathbb{R}), \tag{1.12}$$

where $\langle f, g\rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}\,dt$. If (1.12) holds with $\tilde{\psi}^\ell = \psi^\ell$ for all $\ell = 1, \dots, L$, then $X(\Psi)$ is a tight wavelet frame in $L_2(\mathbb{R})$.

A similar phenomenon also can be found for pairs of bi-frames constructed from B-splines in [9, 8]. All these will be discussed in details in Section 4. More examples of Riesz wavelets with short support will be given in the last section. Some of our examples are even from non-spline refinable functions.

2. Riesz Spline Wavelet Bases with Short Support

In order to show that the system $X(\psi)$ with ψ defined in (1.8) forms a Riesz basis for $L_2(\mathbb{R})$, we need the following Lemma that is a special case of Corollary 3.3.

Lemma 2.1. *Let a be a finitely supported refinement mask for a compactly supported refinable function $\phi \in L_2(\mathbb{R})$ with $\hat{a}(0) = 1$ and $\hat{a}(\pi) = 0$ such that $\hat{\phi}(0) \neq 0$ and $\hat{a}$ can be factorized into the form*

$$\hat{a}(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^m \hat{A}(\xi),$$

where $\hat{A}$ is the Fourier series of a finitely supported sequence A with $\hat{A}(\pi) \neq 0$. Suppose that

$$|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 \neq 0 \quad \textit{for all} \quad \xi \in \mathbb{R}.$$

Define

$$\hat{\psi}(2\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}\hat{\phi}(\xi)$$

and

$$\hat{\tilde{A}}(\xi) := \frac{\hat{A}(\xi)}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2}.$$

Assume that

$$\rho_A := \inf_{n\in\mathbb{N}} \|\widehat{A_n}\|^{1/n}_{L_\infty(\mathbb{R})} < 2^{m-1/2} \qquad \text{and} \qquad \rho_{\tilde{A}} := \inf_{n\in\mathbb{N}} \|\widehat{\tilde{A}_n}\|^{1/n}_{L_\infty(\mathbb{R})} < 2^{m-1/2},$$

where $\widehat{A_n}(\xi) := \hat{A}(2^{n-1}\xi)\cdots\hat{A}(2\xi)\hat{A}(\xi)$ *and* $\widehat{\tilde{A}_n}(\xi) := \hat{\tilde{A}}(2^{n-1}\xi)\cdots\hat{\tilde{A}}(2\xi)\hat{\tilde{A}}(\xi)$. *Then* $X(\psi)$ *is a Riesz basis for* $L_2(\mathbb{R})$.

Recall that a function f is in the Sobolev space $W^\beta(\mathbb{R})$ if

$$\int |\hat{f}(\xi)|^2(1+|\xi|^2)^\beta d\xi < \infty.$$

We say that f has the *regularity* α if $f \in W^\beta(\mathbb{R})$ for all $\beta < \alpha$. It is well known that the B-spline B_m of order m has the regularity $m - 1/2$. A compactly supported function ϕ satisfies the Strang-Fix condition of order m if

$$\hat{\phi}(0) \neq 0 \quad \text{and} \qquad \hat{\phi}^{(j)}(2\pi k) = 0 \qquad \forall\, j = 0, 1, \dots, m-1, \quad k \in \mathbb{Z}\backslash\{0\}.$$

Applying Lemma 2.1 to the B-spline functions, we obtain the following result.

Theorem 2.2. *Let* B_m *be the B-spline function of order* m *with the refinement mask*

$$\hat{a}(\xi) = 2^{-m}(1+e^{-i\xi})^m.$$

Define

$$\hat{\psi}(2\xi) = 2^{-m}e^{-i\xi}(1-e^{i\xi})^m\widehat{B_m}(\xi). \tag{2.1}$$

Then

(i) *The function* ψ *has the regularity* $m - 1/2$ *and has* m *vanishing moments. It is either symmetric or antisymmetric satisfying* $\psi = (-1)^m\psi(1-\cdot)$ *and it is supported on the interval* $[1/2 - m/2, 1/2 + m/2]$*;*

(ii) $X(\psi)$ *forms a Riesz basis for* $L_2(\mathbb{R})$*;*

(iii) *Among all wavelets (redundant or nonredundant), which have* m *vanishing moments and are based on an MRA whose underlying refinable function has the regularity* $m - 1/2$*, the Riesz wavelet* ψ *has the shortest support.*

Proof. Conclusion (i) follows directly from the properties of B-splines and the definition of ψ.

For (ii), we apply Lemma 2.1. First, it is easy to check that $|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 \neq 0$ for all $\xi \in \mathbb{R}$. Since $\hat{A}$ in Lemma 2.1 is 1, clearly, $\rho_A = 1 < 2^{m-1/2}$ for all $m \in \mathbb{N}$. The corresponding $\hat{\tilde{A}}$ in Lemma 2.1 is

$$\hat{\tilde{A}}(\xi) = \frac{1}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2} = \frac{1}{\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)}.$$

To prove that $\rho_{\tilde{A}} < 2^{m-1/2}$, we observe that the function $f_m(x) := x^m + (1-x)^m$ decreases on $[0, 1/2]$ and increases on $[1/2, 1]$ by $f'_m(x) = m[x^{m-1} - (1-x)^{m-1}]$. Consequently, we have $f_m(x) \geqslant f(1/2) = 2^{1-m}$ for all $x \in [0,1]$ and $m \in \mathbb{N}$. Since $\hat{\tilde{A}}(\xi) = [f(\cos^2(\xi/2))]^{-1}$, we conclude that $\rho_{\tilde{A}} \leqslant \|\hat{\tilde{A}}\|_{L_\infty(\mathbb{R})} \leqslant [f(1/2)]^{-1} = 2^{m-1} < 2^{m-1/2}$. By Lemma 2.1, $X(\psi)$ is a Riesz wavelet basis for $L_2(\mathbb{R})$.

For (iii), since the corresponding refinable function ϕ has the regularity $m - 1/2$, ϕ must satisfy the Strang-Fix condition of order m (see [19, 14]). Hence, ϕ must be the convolution of B_m with some function/distribution (see [18, Theorem 3.7]). Hence, B_m has the shortest support among all refinable functions of the regularity $m - 1/2$.

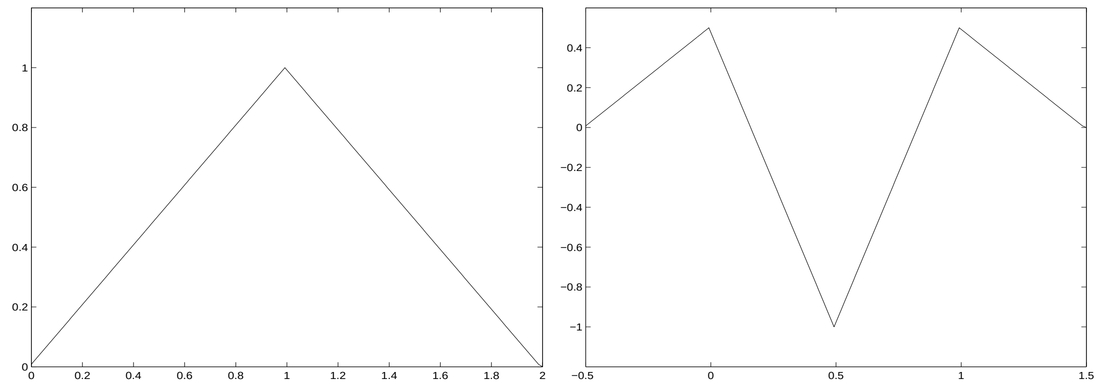

FIGURE 1. The graph of the B-spline B_2 (left) and the graph of the wavelet function ψ (right) in Example 2.3. The Riesz wavelet ψ has 2 vanishing moments and the regularity $3/2$. The wavelet system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

For any given MRA-based wavelet with m vanishing moments, since the refinable function ϕ satisfies $\hat{\phi}(0) \neq 0$, the wavelet mask must have the factor

$$\Big(\frac{1-e^{i\xi}}{2}\Big)^m.$$

This says that in order to have m vanishing moments, the wavelet mask cannot be shorter than $(\frac{1-e^{i\xi}}{2})^m$. Altogether, we conclude that ψ defined in (2.1) has the shortest support among all wavelets (redundant or nonredundant) which have m vanishing moments and are based on an MRA whose underlying refinable function has the regularity $m-1/2$. ∎

Since it is rare to be able to derive a wavelet of regularity $m-1/2$ from a multiresolution whose underlying refinable function has the regularity smaller than $m-1/2$, (iii) in the above theorem essentially says that ψ defined in (2.1) is the shortest supported MRA-based wavelet having the regularity $m-1/2$ and m vanishing moments.

Example 2.3. Let $m=2$. Then by (2.1), $\psi = \frac{1}{2}B_2(2\cdot -1) - B_2(2\cdot) + \frac{1}{2}B_2(2\cdot +1)$. By Theorem 2.2, $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$. The Riesz wavelet ψ has 2 vanishing moments and the regularity $3/2$. See Figure 1 for the graphs of the functions B_2 and ψ.

Example 2.4. Let $m=3$. Then by (2.1), $\psi = \frac{1}{4}B_3(2\cdot -1) - \frac{3}{4}B_3(2\cdot) + \frac{3}{4}B_3(2\cdot +1) - \frac{1}{4}B_3(2\cdot +2)$. By Theorem 2.2, $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$. The Riesz wavelet ψ has 3 vanishing moments and the regularity $5/2$. See Figure 2 for the graphs of the functions B_3 and ψ.

3. Biorthogonal Wavelets with Infinite Masks

In this section, we give a general form of Lemma 2.1. This not only leads to a proof of Lemma 2.1, but also leads to a result in a more general setting. This further allows us to connect the discussions here to the spline tight frame wavelet systems given in [20, 9, 12] and bi-frame systems in [9, 8], as we will discuss in Section 4.

We start with some basic notions. Recall that a function f on $\mathbb{R}$ has *polynomial decay* if

$$(1+|\cdot|)^j f \in L_\infty(\mathbb{R}) \qquad \forall\, j \in \mathbb{N},$$

and has *exponential decay* if there exists a positive number β such that

$$e^{\beta|\cdot|} f \in L_\infty(\mathbb{R}).$$

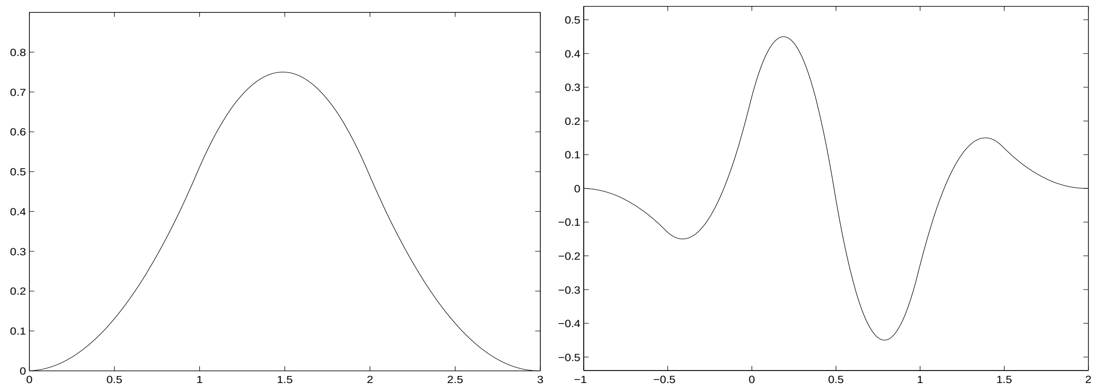

FIGURE 2. The graph of the B-spline B_3 (left) and the graph of the function ψ (right) in Example 2.4. The function ψ has 3 vanishing moments and the regularity 5/2. The wavelet system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

If a function f has polynomial decay or exponential decay, then clearly $f \in L_p(\mathbb{R})$ for all $1 \leqslant p \leqslant \infty$.

Similarly, a sequence a on $\mathbb{Z}$ has polynomial decay if

$$\sum_{k\in\mathbb{Z}} (1+|k|)^j |a(k)| < \infty \qquad \forall\, j \in \mathbb{N},$$

or equivalently, $\sup_{k\in\mathbb{Z}}(1+|k|)^j|a(k)| < \infty$ for all $j \in \mathbb{N}$. It is easy to see that a sequence a has polynomial decay if and only if $\hat{a} \in C^\infty(\mathbb{R})$.

For a sequence u on $\mathbb{Z}$ and a function f on $\mathbb{R}$, we define

$$\nabla u := u - u(\cdot - 1) \quad \text{and} \quad \nabla f := f - f(\cdot - 1). \tag{3.1}$$

In general, $\nabla^m u = \sum_{k=0}^m (-1)^k \frac{m!}{k!(m-k)!} u(\cdot - k)$.

The following Lemma is similar to [10, Theorem 3.6] and will be needed later.

Lemma 3.1. *Let $f \in L_2(\mathbb{R})$ be a function with polynomial decay and m be an arbitrary given positive integer. Then, the following statements are equivalent:*

(i) *$\hat{f}^{(j)}(2\pi k) = 0$ for all $k \in \mathbb{Z}$ and $j = 0, \ldots, m-1$;*

(ii) *The identity $f = \nabla^m h$ holds where*

$$h := \sum_{k=0}^{\infty} \frac{(k+m-1)!}{k!(m-1)!} f(\cdot - k) \tag{3.2}$$

has polynomial decay. In particular, h is in $L_2(\mathbb{R})$.

Proof. Assume that (i) holds. It is easy to see that (i) is equivalent to $\sum_{k\in\mathbb{Z}} k^j f(\cdot - k) = 0$ for all $j = 0, \ldots, m-1$. Let h be the function as given in (3.2). Since f has polynomial decay, it is easy to see that h is well defined on $\mathbb{R}$. Note that

$$\begin{aligned}\nabla h = h - h(\cdot - 1) &= \sum_{k=0}^{\infty} \frac{(k+m-1)!}{k!(m-1)!} f(\cdot - k) - \sum_{k=1}^{\infty} \frac{(k+m-2)!}{(k-1)!(m-1)!} f(\cdot - k) \\ &= f + \sum_{k=1}^{\infty} \left[\frac{(k+m-1)!}{k!(m-1)!} - \frac{(k+m-2)!}{(k-1)!(m-1)!} \right] f(\cdot - k) \\ &= f + \sum_{k=1}^{\infty} \frac{(k+m-2)!}{k!(m-2)!} f(\cdot - k) \\ &= \sum_{k=0}^{\infty} \frac{(k+m-2)!}{k!(m-2)!} f(\cdot - k).\end{aligned}$$

Hence, $\nabla^m h = f$ by induction on m. Next, we show that h has polynomial decay. Since f has polynomial decay, there exist positive constants $C_\ell, \ell \in \mathbb{N}$, such that

$$|f(t)| \leqslant C_\ell (1+|t|)^{-\ell} \qquad \forall\, t \in \mathbb{R}.$$

Let $t \leqslant 0$ and $j \in \mathbb{N}$. Then

$$\begin{aligned}(1+|t|)^j |h(t)| &\leqslant \sum_{k=0}^{\infty} \left| \frac{(k+m-1)\cdots(k+1)}{(m-1)!} \right| |f(t-k)|(1+|t|)^j \\ &\leqslant C_\ell \sum_{k=0}^{\infty} \frac{(k+m-1)^m}{(m-1)!} (1+|t-k|)^{-\ell} (1+|t|)^j \\ &\leqslant C_\ell \sum_{k=0}^{\infty} \frac{m^m (k+1)^m}{(m-1)!} (1+|t|+k)^{-\ell} (1+|t|)^j \\ &\leqslant C_\ell \frac{m^m}{(m-1)!} \sum_{k=0}^{\infty} (1+|t|+k)^{m+j-\ell} \\ &\leqslant C_\ell \frac{m^m}{(m-1)!} \sum_{k=0}^{\infty} (1+k)^{m+j-\ell}\end{aligned}$$

which is finite whenever $\ell > m+j+1$. (Note here that one always can choose such an ℓ by the definition of the polynomial decay of f.) For $t > 0$, we first note that

$$\frac{(k+m-1)!}{k!} = (k+m-1)\cdots(k+1)$$

(when $m = 1$, by convention it takes value 1) is a polynomial of degree $m-1$ for the variable k. Then (i) asserts that

$$0 = \sum_{k=-\infty}^{-1} \frac{(k+m-1)\cdots(k+1)}{(m-1)!} f(\cdot - k) + \sum_{k=0}^{\infty} \frac{(k+m-1)\cdots(k+1)}{(m-1)!} f(\cdot - k).$$

Hence,

$$h = -\sum_{k=-\infty}^{-1} \frac{(k+m-1)\cdots(k+1)}{(m-1)!} f(\cdot - k).$$

By a similar argument applying to the above identity, we conclude that h must have polynomial decay for $t > 0$.

Assume that (ii) holds. This implies that $\hat{f}(\xi) = (1 - e^{-i\xi})^m \hat{h}(\xi)$ which gives (i). ∎

By δ we denote the *Dirac sequence* on $\mathbb{Z}$ such that $\delta(0)=1$ and $\delta(k)=0$ for all $k\in\mathbb{Z}\backslash\{0\}$. The *bracket product* of two functions f and g in $L_2(\mathbb{R})$ is defined to be (see [13])

$$[f,g](\xi):=\sum_{k\in\mathbb{Z}} f(\xi+2\pi k)\overline{g(\xi+2\pi k)},\qquad \xi\in\mathbb{R}. \tag{3.3}$$

It is well known that $\langle f(\cdot-k),g\rangle=\delta(k)$ for all $k\in\mathbb{Z}$ if and only if $[\hat f,\hat g]=1$.

Assume that $(X(\Psi),X(\tilde\Psi))$ is a pair of bi-frames. If the system $(X(\Psi),X(\tilde\Psi))$ further satisfies $\langle\psi^\ell_{j,k},\psi^{\ell'}_{j',k'}\rangle=\delta(\ell-\ell')\delta(j-j')\delta(k-k')$ for all $\ell,\ell'=1,\dots,L$ and $j,j',k,k'\in\mathbb{Z}$, then $(X(\Psi),X(\tilde\Psi))$ forms a pair of *biorthogonal wavelet bases* in $L_2(\mathbb{R})$. Clearly, if $(X(\Psi),X(\tilde\Psi))$ forms a pair of biorthogonal wavelet bases in $L_2(\mathbb{R})$, then both systems $X(\Psi)$ and $X(\tilde\Psi)$ form a Riesz wavelet basis in $L_2(\mathbb{R})$ (see e.g. [21]).

With Lemma 3.1, we prove the following result on biorthogonal wavelets with infinite masks.

Theorem 3.2. *Let a and b be two sequences on $\mathbb{Z}$ satisfying the following two conditions:*

(i) *There are positive integers m and $\tilde m$ such that $\hat a(\xi)=(\frac{1+e^{-i\xi}}{2})^m\hat A(\xi)$ and $\hat b(\xi)=(\frac{1-e^{i\xi}}{2})^{\tilde m}\hat B(\xi)$, where A and B are sequences on $\mathbb{Z}$ with polynomial decay satisfying $\hat A(0)=1$ and $\hat B(\pi)\ne 0$.*

(ii) *The function $\hat d(\xi):=\hat a(\xi)\hat b(\xi+\pi)-\hat a(\xi+\pi)\hat b(\xi)$ does not vanish for all $\xi\in\mathbb{R}$.*

Let

$$\hat{\tilde a}(\xi):=\left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde m}\hat{\tilde A}(\xi),\quad \textit{where}\quad \hat{\tilde A}(\xi):=\frac{\overline{\hat B(\xi+\pi)}}{\overline{\hat d(\xi)}}\quad \textit{and}\quad \hat{\tilde b}(\xi):=-\frac{\overline{\hat a(\xi+\pi)}}{\overline{\hat d(\xi)}}. \tag{3.4}$$

Define

$$\hat\phi(\xi):=\prod_{j=1}^\infty \hat a(2^{-j}\xi)\quad \textit{and}\quad \hat{\tilde\phi}(\xi):=\prod_{j=1}^\infty \hat{\tilde a}(2^{-j}\xi),$$

$$\hat\psi(\xi):=\hat b(\xi/2)\hat\phi(\xi/2)\quad \textit{and}\quad \hat{\tilde\psi}(\xi):=\hat{\tilde b}(\xi/2)\hat{\tilde\phi}(\xi/2).$$

Assume that

$$\limsup_{n\to\infty}\|A_n\|^{1/n}_{\ell_2(\mathbb{Z})}<2^{m-1/2}\quad \textit{and}\quad \limsup_{n\to\infty}\|\tilde A_n\|^{1/n}_{\ell_2(\mathbb{Z})}<2^{\tilde m-1/2}, \tag{3.5}$$

where

$$\widehat{A_n}(\xi):=\hat A(2^{n-1}\xi)\cdots\hat A(2\xi)\hat A(\xi)\quad \textit{and}\quad \widehat{\tilde A_n}(\xi):=\hat{\tilde A}(2^{n-1}\xi)\cdots\hat{\tilde A}(2\xi)\hat{\tilde A}(\xi). \tag{3.6}$$

Then all the functions $\phi,\tilde\phi,\psi,\tilde\psi$ belong to $L_2(\mathbb{R})$ satisfying

$$\langle\phi,\tilde\phi(\cdot-k)\rangle=\langle\psi,\tilde\psi(\cdot-k)\rangle=\delta(k),\qquad \langle\phi,\tilde\psi(\cdot-k)\rangle=\langle\psi,\tilde\phi(\cdot-k)\rangle=0\qquad \forall\, k\in\mathbb{Z}. \tag{3.7}$$

If we further assume that

(iii) *$[\hat\phi,\hat\phi]\in L_\infty(\mathbb{R})$ and $[\hat{\tilde\phi},\hat{\tilde\phi}]\in L_\infty(\mathbb{R})$,*

(iv) *$X(\psi)$ and $X(\tilde\psi)$ are Bessel in $L_2(\mathbb{R})$,*

then $(X(\psi),X(\tilde\psi))$ forms a pair of biorthogonal wavelet bases in $L_2(\mathbb{R})$. In particular, $X(\psi)$ is a Riesz basis of $L_2(\mathbb{R})$.

Proof. The essential part of the proof is to show that the corresponding cascade algorithms to obtain ϕ and $\tilde\phi$ as defined below converge in $L_2(\mathbb{R})$. We shall use some ideas here from the proof of [10, Theorem 4.3] which deals with convergence of vector cascade algorithms in Sobolev spaces. We use the compactly supported orthonormal refinable function η that has a support $[0, 2\max(m,\tilde{m})+1]$ (see [7]) satisfying

$$\hat{\eta}(0)=1,\ \hat{\eta}^{(j)}(2\pi k)=0 \quad \forall\, k\in\mathbb{Z}\backslash\{0\} \quad \text{and} \quad j=0,\dots,\max(m,\tilde{m}) \tag{3.8}$$

to obtain the initial seed in the cascade algorithm. Since η and its shifts form an orthonormal system, $[\hat{\eta},\hat{\eta}]=1$. It can be easily verified that $(\hat{\eta}\overline{\hat{\eta}})^{(j)}(0)=\delta(j)$ for all $j=0,\dots,2\max(m,\tilde{m})$. Since $\hat{\tilde{a}}(0)=\hat{\eta}(0)=1$, by [10, Lemmas 2.2 and 3.4], there exists a finitely supported sequence c on $\mathbb{Z}$ such that

$$\hat{c}(0)=1, \qquad 2^{-j}(\hat{\tilde{a}}\hat{c}\hat{\eta})^{(j)}(0)=(\hat{c}\hat{\eta})^{(j)}(0) \qquad \forall\, j=0,\dots,\max(m,\tilde{m}) \tag{3.9}$$

and $\hat{c}(\xi)\neq 0$ for all $\xi\in\mathbb{R}$. In fact, as shown in [10, Lemmas 2.2 and 3.4], the values $\hat{c}^{(j)}(0), j=1,\dots,\max(m,\tilde{m})$ are uniquely determined by the system of linear equations given in (3.9).

Now, pick the initial seeds ϕ_0 and $\tilde{\phi}_0$ by $\widehat{\phi_0}(\xi):=\hat{\eta}(\xi)/\overline{\hat{c}(\xi)}$ and $\widehat{\tilde{\phi}_0}(\xi):=\hat{c}(\xi)\hat{\eta}(\xi)$. Since the sequence c is finitely supported and $\hat{c}(\xi)\neq 0$ for all $\xi\in\mathbb{R}$, we see that $\phi_0,\tilde{\phi}_0\in L_2(\mathbb{R})$ have exponential decay. Moreover, it is easy to check that $[\widehat{\phi_0},\widehat{\tilde{\phi}_0}]=[\hat{\eta}/\overline{\hat{c}},\hat{\eta}\hat{c}]=[\hat{\eta},\hat{\eta}]=1$. The corresponding cascade operators Q_a and $Q_{\tilde{a}}$ defined by a and $\tilde{a}$ are

$$Q_a f := 2\sum_{k\in\mathbb{Z}} a(k)f(2\cdot -k) \quad \text{and} \quad Q_{\tilde{a}} f := 2\sum_{k\in\mathbb{Z}} \tilde{a}(k)f(2\cdot -k), \qquad f\in L_2(\mathbb{R}). \tag{3.10}$$

Let $g:=Q_a\phi_0-\phi_0$ and $\tilde{g}=Q_{\tilde{a}}\tilde{\phi}_0-\tilde{\phi}_0$. Then

$$\hat{g}(\xi)=\hat{a}(\xi/2)\widehat{\phi_0}(\xi/2)-\widehat{\phi_0}(\xi)=\hat{a}(\xi/2)\hat{\eta}(\xi/2)/\overline{\hat{c}(\xi/2)}-\hat{\eta}(\xi)/\overline{\hat{c}(\xi)} \tag{3.11}$$

and

$$\hat{\tilde{g}}(\xi)=\hat{\tilde{a}}(\xi/2)\widehat{\tilde{\phi}_0}(\xi/2)-\widehat{\tilde{\phi}_0}(\xi)=\hat{\tilde{a}}(\xi/2)\hat{c}(\xi/2)\hat{\eta}(\xi/2)-\hat{c}(\xi)\hat{\eta}(\xi). \tag{3.12}$$

Since a and $\tilde{a}$ have polynomial decay, by the fact that the function η is compactly supported, it follows from (3.11) and (3.12) that both g and $\tilde{g}$ have polynomial decay.

Next, we prove that

$$\hat{g}^{(j)}(2\pi k)=0 \quad \forall\, j=0,\dots,m-1 \quad \text{and} \quad k\in\mathbb{Z} \tag{3.13}$$

and

$$\hat{\tilde{g}}^{(j)}(2\pi k)=0 \quad \forall\, j=0,\dots,\tilde{m}-1 \quad \text{and} \quad k\in\mathbb{Z}. \tag{3.14}$$

First, when $k\in 2\mathbb{Z}\backslash\{0\}$, applying (3.8), we obtain $\hat{g}^{(j)}(2\pi k)=\hat{\tilde{g}}^{(j)}(2\pi k)=0$ for all $j=0,\dots,\max(m,\tilde{m})$. Secondly, when $k\in 2\mathbb{Z}+1$, since $\hat{a}(\xi)=2^{-m}(1+e^{-i\xi})^m\hat{A}(\xi)$ and $\hat{\tilde{a}}(\xi)=2^{-\tilde{m}}(1+e^{-i\xi})^{\tilde{m}}\hat{\tilde{A}}(\xi)$, it is easy to see that $\hat{g}^{(j)}(2\pi k)=0$ for all $j=0,\dots,m-1$ and $\hat{\tilde{g}}^{(j)}(2\pi k)=0$ for all $j=0,\dots,\tilde{m}-1$.

Hence, in order to prove (3.13) and (3.14), it suffices to prove the case $k=0$. For (3.14), applying (3.9), one obtains that

$$\begin{aligned}\hat{\tilde{g}}^{(j)}(0)&=[\hat{\tilde{a}}(\cdot/2)\hat{c}(\cdot/2)\hat{\eta}(\cdot/2)-\hat{c}\hat{\eta}]^{(j)}(0)\\&=2^{-j}[\hat{\tilde{a}}\hat{c}\hat{\eta}]^{(j)}(0)-[\hat{c}\hat{\eta}]^{(j)}(0)=0 \qquad \forall\, j=0,\dots,\max(m,\tilde{m}).\end{aligned} \tag{3.15}$$

This gives (3.14).

Next, we prove (3.13). It follows from the definition of $\tilde{a}$ that

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi)+\overline{\hat{a}(\xi+\pi)}\hat{\tilde{a}}(\xi+\pi)=1.$$

This, together with

$$\hat{a}(\xi)=2^{-m}(1+e^{-i\xi})^m\hat{A}(\xi), \quad \text{and} \quad \hat{\tilde{a}}(\xi)=2^{-\tilde{m}}(1+e^{-i\xi})^{\tilde{m}}\hat{\tilde{A}}(\xi),$$

leads to that

$$\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) = 1 + O(|\xi|^{m+\tilde{m}-1}), \quad \text{as} \quad \xi \to 0,$$

and

$$\hat{a}(\xi) = (\overline{\hat{\tilde{a}}(\xi)})^{-1} + O(|\xi|^{m+\tilde{m}-1}).$$

Hence, as $\xi \to 0$,

$$\begin{aligned}
\hat{g}(\xi) &= \hat{a}(\xi/2)\hat{\eta}(\xi/2)/\overline{\hat{c}(\xi/2)} - \hat{\eta}(\xi)/\overline{\hat{c}(\xi)} \\
&= \hat{\eta}(\xi/2)\overline{\hat{\eta}(\xi/2)}\,(\overline{\hat{\tilde{a}}(\xi/2)\hat{\eta}(\xi/2)\hat{c}(\xi/2)})^{-1} - \hat{\eta}(\xi)\overline{\hat{\eta}(\xi)}\,(\overline{\hat{c}(\xi)\hat{\eta}(\xi)})^{-1} + O(|\xi|^{m+\tilde{m}-1}) \\
&= (\overline{\hat{\tilde{a}}(\xi/2)\hat{\eta}(\xi/2)\hat{c}(\xi/2)})^{-1} - (\overline{\hat{c}(\xi)\hat{\eta}(\xi)})^{-1} + O(|\xi|^{\max(m,\tilde{m})}) \\
&= (\overline{\hat{\tilde{a}}(\xi/2)\hat{\eta}(\xi/2)\hat{c}(\xi/2)})^{-1}(\overline{\hat{c}(\xi)\hat{\eta}(\xi)})^{-1}\overline{\Big[\hat{c}(\xi)\hat{\eta}(\xi) - \hat{\tilde{a}}(\xi/2)\hat{c}(\xi/2)\hat{\eta}(\xi/2)\Big]} + O(|\xi|^{\max(m,\tilde{m})}) \\
&= (\overline{\hat{\tilde{a}}(\xi/2)\hat{\eta}(\xi/2)\hat{c}(\xi/2)})^{-1}(\overline{\hat{c}(\xi)\hat{\eta}(\xi)})^{-1}\overline{\hat{\tilde{g}}(\xi)} + O(|\xi|^{\max(m,\tilde{m})}) \\
&= O(|\xi|^{\max(m,\tilde{m})}).
\end{aligned}$$

The third equality follows from that $|\hat{\eta}(\xi)|^2 = 1 + O(|\xi|^{\max(m,\tilde{m})})$, and the last equality holds by (3.15). Therefore, (3.13) holds.

With (3.13) and (3.14), Lemma 3.1 says that there exist two functions $h, \tilde{h} \in L_2(\mathbb{R})$ with polynomial decay such that $g = \nabla^m h$ and $\tilde{g} = \nabla^{\tilde{m}}\tilde{h}$.

Let $f_n := Q_a^n \phi_0$ and $\tilde{f}_n := Q_{\tilde{a}}^n \tilde{\phi}_0$. Then their Fourier transforms are

$$\widehat{f_n}(\xi) = \widehat{\phi_0}(2^{-n}\xi)\prod_{j=1}^{n}\hat{a}(2^{-j}\xi) \quad \text{and} \quad \widehat{\tilde{f}_n}(\xi) = \widehat{\tilde{\phi}_0}(2^{-n}\xi)\prod_{j=1}^{n}\hat{\tilde{a}}(2^{-j}\xi).$$

One can prove inductively that

$$f_{n+1} - f_n = Q_a^n g = 2^n \sum_{k\in\mathbb{Z}} a_n(k)[\nabla^m h](2^n \cdot -k) = 2^n \sum_{k\in\mathbb{Z}}[\nabla^m a_n](k)h(2^n \cdot -k) \tag{3.16}$$

and

$$\tilde{f}_{n+1} - \tilde{f}_n = Q_{\tilde{a}}^n \tilde{g} = 2^n \sum_{k\in\mathbb{Z}} \tilde{a}_n(k)[\nabla^{\tilde{m}} \tilde{h}](2^n \cdot -k) = 2^n \sum_{k\in\mathbb{Z}}[\nabla^{\tilde{m}} \tilde{a}_n](k)\tilde{h}(2^n \cdot -k), \tag{3.17}$$

where $\widehat{a_n}(\xi) := \hat{a}(2^{n-1}\xi)\cdots\hat{a}(2\xi)\hat{a}(\xi)$ and $\widehat{\tilde{a}_n}(\xi) := \hat{\tilde{a}}(2^{n-1}\xi)\cdots\hat{\tilde{a}}(2\xi)\hat{\tilde{a}}(\xi)$.

Since both h and $\tilde{h}$ are $L_2(\mathbb{R})$ functions with polynomial decay, we conclude that both $[\hat{h}, \hat{h}]$ and $[\hat{\tilde{h}}, \hat{\tilde{h}}]$ are in $L_\infty(\mathbb{R})$. Hence, identities (3.16) and (3.17) imply that there exists a positive constant C such that for all $n \in \mathbb{N}$,

$$\|f_{n+1} - f_n\|_{L_2(\mathbb{R})} \leqslant C2^{n/2}\|\nabla^m a_n\|_{\ell_2(\mathbb{Z})} \quad \text{and} \quad \|\tilde{f}_{n+1} - \tilde{f}_n\|_{L_2(\mathbb{R})} \leqslant C2^{n/2}\|\nabla^{\tilde{m}} \tilde{a}_n\|_{\ell_2(\mathbb{Z})}. \tag{3.18}$$

Since

$$\widehat{\nabla^m a_n}(\xi) = (1 - e^{-i\xi})^m \widehat{a_n}(\xi) = 2^{-mn}(1 - e^{-i2^n\xi})^m \widehat{A_n}(\xi)$$

and

$$\widehat{\nabla^{\tilde{m}} \tilde{a}_n}(\xi) = (1 - e^{-i\xi})^{\tilde{m}} \widehat{\tilde{a}_n}(\xi) = 2^{-\tilde{m}n}(1 - e^{-i2^n\xi})^{\tilde{m}} \widehat{\tilde{A}_n}(\xi),$$

we have

$$\|\nabla^m a_n\|_{\ell_2(\mathbb{Z})} \leqslant 2^{m-mn}\|A_n\|_{\ell_2(\mathbb{Z})}, \quad \text{and} \quad \|\nabla^{\tilde{m}} \tilde{a}_n\|_{\ell_2(\mathbb{Z})} \leqslant 2^{\tilde{m}-\tilde{m}n}\|\tilde{A}_n\|_{\ell_2(\mathbb{Z})}.$$

But, (3.5) says that there exist two positive constants ρ with $0 < \rho < 1$ and C_1 such that

$$\|A_n\|_{\ell_2(\mathbb{Z})} \leqslant C_1 \rho^n 2^{(m-1/2)n} \quad \text{and} \quad \|\tilde{A}_n\|_{\ell_2(\mathbb{Z})} \leqslant C_1 \rho^n 2^{(\tilde{m}-1/2)n} \qquad \forall\, n \in \mathbb{N}. \tag{3.19}$$

Therefore, we deduce from (3.18) that

$$\|f_{n+1} - f_n\|_{L_2(\mathbb{R})} \leqslant 2^m C C_1 \rho^n \quad \text{and} \quad \|\tilde{f}_{n+1} - \tilde{f}_n\|_{L_2(\mathbb{R})} \leqslant 2^{\tilde{m}} C C_1 \rho^n \qquad \forall\, n \in \mathbb{N}.$$

Since $0 < \rho < 1$, both $\{f_n\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in $L_2(\mathbb{R})$.

Since $\hat{a}(0) = 1$ and $\hat{b}(\pi) \neq 0$, we have $\hat{\tilde{a}}(0) = 1$. Hence, $\lim_{n\to\infty} \widehat{f_n}(\xi) = \hat{\phi}(\xi)$ and $\lim_{n\to\infty} \widehat{\tilde{f}_n}(\xi) = \hat{\tilde{\phi}}(\xi)$ for $\xi \in \mathbb{R}$. This leads to $\lim_{n\to\infty} \|f_n - \phi\|_{L_2(\mathbb{R})} = 0$ and $\lim_{n\to\infty} \|\tilde{f}_n - \tilde{\phi}\|_{L_2(\mathbb{R})} = 0$, since both $\{f_n\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in $L_2(\mathbb{R})$.

Next, we prove that $[\hat{\phi}, \hat{\tilde{\phi}}] = 1$. It is clear that $[\widehat{f_0}, \widehat{\tilde{f}_0}] = [\widehat{\phi_0}, \widehat{\tilde{\phi}_0}] = 1$. Using $\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) + \overline{\hat{a}(\xi+\pi)}\hat{\tilde{a}}(\xi+\pi) = 1$, one can prove that $[\widehat{f_n}, \widehat{\tilde{f}_n}] = 1$ for all $n \in \mathbb{N}$ inductively. Finally, since $\lim_{n\to\infty} \|f_n - \phi\|_{L_2(\mathbb{R})} = 0$ and $\lim_{n\to\infty} \|\tilde{f}_n - \tilde{\phi}\|_{L_2(\mathbb{R})} = 0$, we must have $[\hat{\phi}, \hat{\tilde{\phi}}] = 1$. Therefore, $(\phi, \tilde{\phi})$ is a pair of refinable functions whose shifts form a pair of biorthogonal systems.

We further note that by the definition of $\tilde{a}$ and $\tilde{b}$, it is easy to verify that

$$\begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}(\xi) & \hat{b}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \hat{\tilde{a}}(\xi) & \hat{\tilde{a}}(\xi+\pi) \\ \hat{\tilde{b}}(\xi) & \hat{\tilde{b}}(\xi+\pi) \end{bmatrix}}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.20}$$

With all these, (3.7) and the rest of the conclusions of this theorem follow directly from a standard argument in wavelet analysis on biorthogonal wavelets (see [5, 7] and [21]). ∎

In the above proof, the assumption that both the sequences A and B have polynomial decay is only used to apply Lemma 3.1 and to show that $[\hat{h}, \hat{h}] \in L_\infty(\mathbb{R})$ and $[\hat{\tilde{h}}, \hat{\tilde{h}}] \in L_\infty(\mathbb{R})$. Checking the proof of Lemma 3.1, we see that such polynomial decay condition on both A and B can be further weakened.

As a direct consequence of Theorem 3.2, we have the following result.

Corollary 3.3. *Let sequences a and b be given in Theorem 3.2; and sequences $\tilde{a}, \tilde{b}, A,$ and $\tilde{A}$ and functions $\phi, \tilde{\phi}, \psi, \tilde{\psi}$ be defined as in Theorem 3.2. Define*

$$\rho_A := \inf_{n\in\mathbb{N}} \|\widehat{A_n}\|_{L_\infty(\mathbb{R})}^{1/n} \quad \textit{and} \quad \rho_{\tilde{A}} := \inf_{n\in\mathbb{N}} \|\widehat{\tilde{A}_n}\|_{L_\infty(\mathbb{R})}^{1/n}, \tag{3.21}$$

where A_n and $\tilde{A}_n$ are defined in (3.6). *Then for any $\varepsilon > 0$, there exists a positive constant C such that*

$$\max(|\hat{\phi}(\xi)|, |\hat{\psi}(\xi)|) \leqslant C(1+|\xi|)^{-m+\varepsilon+\log_2 \rho_A} \qquad \forall\, \xi \in \mathbb{R} \tag{3.22}$$

and

$$\max(|\hat{\tilde{\phi}}(\xi)|, |\hat{\tilde{\psi}}(\xi)|) \leqslant C(1+|\xi|)^{-\tilde{m}+\varepsilon+\log_2 \rho_{\tilde{A}}} \qquad \forall\, \xi \in \mathbb{R}. \tag{3.23}$$

Consequently, if $\rho_A < 2^{m-1/2}$ and $\rho_{\tilde{A}} < 2^{\tilde{m}-1/2}$, then $(X(\psi), X(\tilde{\psi}))$ forms a pair of biorthogonal wavelet bases in $L_2(\mathbb{R})$. In particular, $X(\psi)$ is a Riesz basis of $L_2(\mathbb{R})$.

Proof. The proof of (3.22) and (3.23) follows from the proof of [7, Lemmas 7.1.1 and 7.1.2]. Note that

$$\limsup_{n\to\infty} \|A_n\|_{\ell_2(\mathbb{Z})}^{1/n} \leqslant \limsup_{n\to\infty} \|\widehat{A_n}\|_{L_\infty(\mathbb{R})}^{1/n} = \inf_{n\in\mathbb{N}} \|\widehat{A_n}\|_{L_\infty(\mathbb{R})}^{1/n},$$

by $\|A_n\|_{\ell_2(\mathbb{Z})} \leqslant \|\widehat{A_n}\|_{L_\infty(\mathbb{R})}$. Therefore, if $\rho_A < 2^{m-1/2}$ and $\rho_{\tilde{A}} < 2^{\tilde{m}-1/2}$, then it follows from (3.22) and (3.23) that there exist $\varepsilon > 0$ and $C > 0$ such that

$$\max(|\hat{\phi}(\xi)|, |\hat{\psi}(\xi)|, |\hat{\tilde{\phi}}(\xi)|, |\hat{\tilde{\psi}}(\xi)|) \leqslant C(1+|\xi|)^{-1/2-\varepsilon}$$

and $\max(|\hat{\psi}(\xi)|, |\hat{\tilde{\psi}}(\xi)|) \leqslant C|\xi|^\varepsilon$ for all $\xi \in \mathbb{R}$. Therefore, by [11, Propositions 2.6 and 3.5], both $X(\psi)$ and $X(\tilde{\psi})$ are Bessel. Moreover, it is evident that $[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R})$ and $[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}] \in L_\infty(\mathbb{R})$. The proof is completed by Theorem 3.2. ∎

For a refinement mask a, define b as

$$\hat{b}(\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)}.$$

Then

$$\hat{d}(\xi) := \hat{a}(\xi)\hat{b}(\xi+\pi) - \hat{a}(\xi+\pi)\hat{b}(\xi) = e^{-i(\xi+\pi)}(|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2). \tag{3.24}$$

Further, the masks $\tilde{a}$ and $\tilde{b}$ defined in (3.4) take the following form:

$$\hat{\tilde{a}}(\xi) = \frac{\hat{a}(\xi)}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2} \quad \text{and} \quad \hat{\tilde{b}}(\xi) = \frac{e^{-i\xi}\overline{\hat{a}(\xi+\pi)}}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2}. \tag{3.25}$$

Now, it is clear that Lemma 2.1 becomes a special case of Corollary 3.3. The mask for the B-spline of order m is $\hat{a}(\xi) = 2^{-m}(1+e^{-i\xi})^m$. It follows from (3.24) that $\hat{d}(\xi) = e^{-i(\xi+\pi)}[\cos^{2m}(\xi/2) + \sin^{2m}(\xi/2)] \neq 0$ for all $\xi \in \mathbb{R}$.

The following result is a direct consequence and a slight modification of Theorem 3.2.

Corollary 3.4. *Under the same notations and conditions as in Theorem 3.2, for any finitely supported sequence c on $\mathbb{Z}$ such that $\hat{c}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, we redefine the functions ψ and $\tilde{\psi}$ in Theorem 3.2 by*

$$\hat{\psi}(\xi) := \hat{c}(\xi)\hat{b}(\xi/2)\hat{\phi}(\xi/2) \quad \text{and} \quad \hat{\tilde{\psi}}(\xi) := \hat{\tilde{b}}(\xi/2)\hat{\tilde{\phi}}(\xi/2)/\overline{\hat{c}(\xi)}.$$

Then all the claims in Theorem 3.2 hold.

Proof. Denote $\hat{b}^{new} := \hat{c}(2\cdot)\hat{b}$ and $\hat{\tilde{b}}^{new} = \hat{\tilde{b}}/\overline{\hat{c}(2\cdot)}$. To apply Theorem 3.2, one observes that

$$\hat{d}^{new}(\xi) := \hat{a}(\xi)\hat{b}^{new}(\xi+\pi) - \hat{a}(\xi+\pi)\hat{b}^{new}(\xi) = \hat{c}(2\xi)[\hat{a}(\xi)\hat{b}(\xi+\pi) - \hat{a}(\xi+\pi)\hat{b}(\xi)] \neq 0 \qquad \forall\, \xi \in \mathbb{R}.$$

It is easy to see that (3.20) holds with $\hat{b}$ and $\hat{\tilde{b}}$ being replaced by $\hat{b}^{new}$ and $\hat{\tilde{b}}^{new}$, respectively. The rest of the proof follows directly from Theorem 3.2. ∎

Remark. As shown in Section 2, Riesz spline wavelets ψ, constructed using Theorem 3.2 and its corollaries, have, in some sense, the shortest support for a given order of smoothness and vanishing moments. However, its dual wavelet systems normally are not compactly supported and have low smoothness orders. Hence, Riesz wavelets constructed here may only be used in applications that either the reconstruction algorithm or the decomposition algorithm is not required. For example, it only needs the decomposition algorithm in some applications of signal analysis and classification. On the other hand, in other applications such as image compression, while the decomposition can be done off-line and reconstruction has to be done online. The short supported reconstruction filter is essential, for example, in computer graphics and numerical algorithms. Furthermore, a fast reconstruction algorithm can be derived by adding a system $X(\psi(\cdot - 1/2))$ to $X(\psi)$ to generate a frame system and by using compactly supported smooth dual wavelet system of the frame system $X(\psi) \cup X(\psi(\cdot - 1/2))$. The detailed discussion is given at the end of the next section.

4. Connections of Riesz Wavelets to Spline Frame Systems

The study here on Riesz wavelets also provides a better understanding of the structure of spline tight frame wavelet systems in [20] and [9, 12] by using the *unitary extension principle* of [20] and *the oblique extension principle* of [9] as we shall discuss in this section.

We first briefly describe the constructions of MRA-based tight frames by using the oblique extension principle in [9] and the details can be found there. For a given refinable function ϕ, with a refinement mask a, one first chooses a 2π-periodic trigonometric polynomial Θ with $\Theta(0) = 1$, called a fundamental function of the tight frame system, according to the approximation order of the refinable function ϕ and the required approximation order of the tight frame expansion (this is directly related to the order of the vanishing moments of frame wavelets). Suppose that a fundamental function Θ can be chosen so that it satisfies (i) $\Theta(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$; (ii)

$$H(\xi) := \Theta(\xi) - \Theta(2\xi)[|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2] \geqslant 0 \qquad \forall\, \xi \in \mathbb{R}. \tag{4.1}$$

The three wavelet masks are then

$$\widehat{b^1}(\xi) := e^{i\xi}\theta(2\xi)\overline{\hat{a}(\xi+\pi)}, \qquad \widehat{b^2}(\xi) := \frac{\sqrt{2}}{2}h(\xi) \quad \text{and} \quad \widehat{b^3}(\xi) := \frac{\sqrt{2}}{2}e^{i\xi}h(\xi),$$

where θ and h are the square roots of Θ and H, respectively; that is, $\theta(\xi)\overline{\theta(\xi)} = \Theta(\xi)$ and $h(\xi)\overline{h(\xi)} = H(\xi)$.

Define the frame wavelet set $\Psi := \{\psi^1, \psi^2, \psi^3\}$ by

$$\widehat{\psi^\ell}(\xi) := \widehat{b^\ell}(\xi/2)\hat{\phi}(\xi/2), \qquad \ell = 1,2,3. \tag{4.2}$$

Then, it was proven in [9] that the system $X(\Psi)$ forms a tight frame system of $L_2(\mathbb{R})$ by using the oblique extension principle (see [9, Proposition 1.11]).

One can reduce the number of frame wavelets to be two by defining

$$\widehat{b^1}(\xi) := e^{i\xi}\theta(2\xi)\overline{\hat{a}(\xi+\pi)}, \quad \widehat{b^2}(\xi) := \hat{a}(\xi)\, h(2\xi).$$

Then, it was proven in [9] that the $X(\Psi)$ with $\Psi := \{\psi^1, \psi^2\}$ defined by

$$\widehat{\psi^\ell}(\xi) = \widehat{b^\ell}(\xi/2)\hat{\phi}(\xi/2), \qquad \ell = 1,2 \tag{4.3}$$

is again a tight frame wavelet system in $L_2(\mathbb{R})$. Note that ψ^1 in (4.2) is the same function as ψ^1 in (4.3).

The construction of spline tight frame systems given in [9] starts with the MRA generated by a B-spline B_m with refinement mask $\hat{a}(\xi) = 2^{-m}(1+e^{-i\xi})^m$. The fundamental function Θ can be chosen according to the needs of the approximation order of the truncated wavelet system; it satisfies $\Theta(\xi) > 0$ for all $\xi \in \mathbb{R}$, and (4.1). An explicit form of Θ is given in [9, Lemma 3.4]. With Θ, $\hat{a}$, and H, it is easy to obtain the functions defined in (4.2) and (4.3) whose corresponding wavelet systems form spline tight frame systems of $L_2(\mathbb{R})$. In this case, setting $\hat{c}(\xi) = e^{i\xi}\theta(\xi)$ in Corollary 3.4 and applying Theorem 2.2, one can show easily that $X(\psi^1)$ is a Riesz basis for $L_2(\mathbb{R})$, where ψ^1 is defined in either (4.2) or (4.3).

In both spline function sets defined in (4.2) and (4.3), the first function ψ^1 is already able to generate a Riesz basis of $L_2(\mathbb{R})$, i.e., $X(\psi^1)$ is a Riesz basis of $L_2(\mathbb{R})$, which is an even stronger statement than that $X(\psi^1)$ is a frame of $L_2(\mathbb{R})$. The first role of $X(\psi^2)$ and $X(\psi^3)$ in (4.2) and $X(\psi^2)$ in (4.3) is to reduce the condition number determined by the upper and lower frame bounds of $X(\Psi)$ to be one, so that the whole system $X(\Psi)$ becomes a tight frame. The second role of other functions in both (4.2) and (4.3) is to give better dual systems. As we remarked before, the (unique) function corresponding to the biorthogonal dual of the system $X(\psi^1)$ normally has low order of smoothness with infinite support. With the help of ψ^2 and ψ^3 in system (4.2) and ψ^2 in the system (4.3), the whole system $X(\Psi)$ in both cases becomes a tight frame that is self-dual system. All these are obtained at the cost of changing a nonredundant system $X(\psi^1)$ to a redundant system $X(\Psi)$.

We further note that since the wavelet mask b^2 in (4.3) vanishes at both 0 and π, therefore, $[\widehat{\psi^2}, \widehat{\psi^2}](0) = 0$. Since ψ^2 in (4.3) is compactly supported, ψ^2 and its shifts cannot form a Riesz system. This concludes that $X(\psi^2)$ is not a Riesz basis of $L_2(\mathbb{R})$.

The spline tight frame systems constructed above may not be symmetric since the square root of Θ may not be symmetric. This weak point was overcome in [12] by an elegant and careful choice of a symmetric θ so that $\theta^2 = \Theta$ is the required fundamental function satisfying (4.1). Let h be a square root of H and set

$$\widehat{b^1}(\xi) := e^{-i\xi}\theta(2\xi)\overline{\hat a(\xi+\pi)}, \quad \widehat{b^2}(\xi) := \hat a(\xi)[h(2\xi)+\overline{h(2\xi)}]/2, \quad \widehat{b^3}(\xi) := \hat a(\xi)[h(2\xi)-\overline{h(2\xi)}]/2.$$

Define

$$\widehat{\psi^\ell}(\xi) := \widehat{b^\ell}(\xi/2)\hat\phi(\xi/2), \qquad \ell = 1,2,3. \tag{4.4}$$

Then $X(\Psi)$ with $\Psi := \{\psi^1,\psi^2,\psi^3\}$ being defined in (4.4) forms a tight frame system in $L_2(\mathbb{R})$ and all the functions ψ^1,ψ^2,ψ^3 are either symmetric or antisymmetric (see [12] for details). Similarly, $X(\psi^1)$ is a Riesz basis for $L_2(\mathbb{R})$, but both $X(\psi^2)$ and $X(\psi^3)$ cannot be a Riesz basis of $L_2(\mathbb{R})$.

Next, we discuss the spline wavelet system given in [20] via the unitary extension principle of [20]. We discuss the construction from B-splines with an even order. The other case can be discussed similarly. Let ϕ be the centered B-spline of order $2m$. Then its refinement mask is $\hat a(\xi) := \cos^{2m}(\xi/2)$. We define $2m$ wavelet masks by

$$\widehat{b^\ell}(\xi) := i^{2m+\ell}\sqrt{\frac{(2m)!}{\ell!(2m-\ell)!}}\sin^\ell(\xi/2)\cos^{2m-\ell}(\xi/2), \quad \ell = 1,\dots,2m.$$

Then, it was shown in [20] that the $2m$ functions, $\Psi = \{\psi^1,\dots,\psi^{2m}\}$, defined by

$$\widehat{\psi^\ell}(\xi) := \widehat{b^\ell}(\xi/2)\hat\phi(\xi/2), \qquad \ell = 1,\dots,2m, \tag{4.5}$$

forms a tight frame system of $L_2(\mathbb{R})$ by using the unitary extension principle ([20, Corollary 6.7]).

Consider the function $\psi := \psi^{2m}(\cdot - 1/2)$. Then ψ is the function derived from an MRA generated by the centered B-spline ϕ with the refinement mask $\hat a(\xi) := \cos^{2m}(\xi/2)$. The corresponding wavelet mask of ψ is

$$\hat b(\xi) = e^{-i\xi}\sin^{2m}(\xi/2) = e^{-i\xi}\overline{\hat a(\xi+\pi)}.$$

Hence, $X(\psi)$ forms a Riesz basis of $L_2(\mathbb{R})$ by Theorem 2.2. Since all the other masks $\widehat{b^\ell}$, $1 \leqslant \ell < 2m$ vanish at both 0 and π, a similar discussion as above shows that $X(\psi^\ell)$, $1 \leqslant \ell < 2m$ cannot be a Riesz basis of $L_2(\mathbb{R})$.

Here come some remarks on the two extension principles mentioned above. The interested reader can find more details in [20, 9, 2]. Both the unitary extension principle of [20] and the oblique extension principle of [9] are derived from the characterization of MRA-based frames given in [20] (also see [11]). The unitary extension principle leads to the first set of the examples of compactly supported spline tight frames defined in (4.5). As pointed out in [9], the approximation order (see [9] for the definitions) of the truncated tight frame expansion of $X(\Psi)$, where Ψ is defined in (4.5), cannot be over 2. The attempt to derive spline tight frame systems whose truncated expansions have a better approximation order leads to oblique extension principle of [9] which is a generalization of the unitary extension principle. This leads to a systematic construction of spline tight wavelet frame systems whose truncated expansion has an arbitrary pre-assigned approximation order. Functions defined in (4.2) and (4.3) are examples of constructions given in [9]. In a similar fashion, [2] obtained the oblique extension principle independently by an attempt to improve the vanishing moments of the functions obtained from the unitary extension principle of [20].

In the rest of this section, we discuss some relations between Riesz wavelet bases constructed in this paper and bi-frames constructed in [8, 9]. In order to do so, let us recall a result from [8, Corollary 3.4] (also see [9]). Let ϕ be a compactly supported refinable function in $L_2(\mathbb{R})$ with a finitely supported mask a and $\hat{\phi}(0) \neq 0$. For any finitely supported sequence b on $\mathbb{Z}$ such that

$$\hat{b}(0) = 0 \quad \text{and} \qquad \lim_{\xi \to \pi} \frac{\hat{a}(\xi)}{\hat{b}(\xi)} = 0, \tag{4.6}$$

define $\hat{\psi}(2\xi) = \hat{b}(\xi)\hat{\phi}(\xi)$, then $X(\{\psi, \psi(\cdot - 1/2)\})$ forms a wavelet frame in $L_2(\mathbb{R})$. Moreover, there exists $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ of compactly supported functions, which are derived via the mixed oblique extension principle (see [9]) from an MRA generated by an arbitrarily chosen compactly supported refinable function $\tilde{\phi} \in L_2(\mathbb{R})$ whose mask contains the factor $(1 + e^{-i\xi})^\ell$, where ℓ is the smallest integer that is greater than the multiplicity of zeros of $\hat{b}(\xi)$ at $\xi = 0$, such that $(X(\{\psi, \psi(\cdot - 1/2)\}), X(\{\tilde{\psi}^1, \tilde{\psi}^2\}))$ is a pair of bi-frames in $L_2(\mathbb{R})$. Since such a pair of bi-frames is MRA-based and all the wavelet masks are finitely supported, a fast frame transform (see [9]) associated with the bi-frames for both decomposition and reconstruction of functions is available.

Now let us discuss some relations between Riesz wavelet bases and bi-frames. Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with $\hat{\phi}(0) \neq 0$ and a finitely supported mask a such that $\hat{a}(0) = 1$ and $\hat{a}(\pi) = 0$. Let b be a finitely supported sequence on $\mathbb{Z}$ such that $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$, where ψ is defined to be $\hat{\psi}(2\xi) = \hat{b}(\xi)\hat{\phi}(\xi)$. For example, such a wavelet mask b and a function ψ may be chosen as in Theorem 3.2. In particular, when ϕ is a B-spline of order m, the wavelet mask b can be chosen to be

$$e^{-i\xi}\Big(\frac{1 - e^{i\xi}}{2}\Big)^m.$$

Since $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$, the compactly supported function ψ must satisfy $\hat{\psi}(0) = 0$ and $[\hat{\psi}, \hat{\psi}](0) \neq 0$. Since $\hat{\phi}(0) \neq 0$ and $\hat{\psi}(0) = 0$, it follows from the definition $\hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2)$ that $\hat{b}(0) = 0$. On the other hand, we must have $\hat{b}(\pi) \neq 0$ since otherwise $\hat{b}(0) = \hat{b}(\pi) = 0$ implies $[\hat{\psi}, \hat{\psi}](0) = 0$, which is a contradiction. Now by $\hat{a}(\pi) = 0$, it is evident that the wavelet mask b must satisfy the conditions in (4.6). It follows from our discussion above (see [8, Corollary 3.4]) that $X(\{\psi, \psi(\cdot - 1/2)\})$ is a wavelet frame. Note that $X(\{\psi, \psi(\cdot - 1/2)\}) = X(\psi) \cup X(\psi(\cdot - 1/2))$ with $X(\psi)$ being a Riesz basis. Moreover, there exists a set $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ of compactly supported smooth L_2 functions, which are derived via the mixed oblique extension principle (see [9]) from an MRA generated by an arbitrarily chosen smooth refinable function, such that $(X(\{\psi, \psi(\cdot - 1/2)\}), X(\{\tilde{\psi}^1, \tilde{\psi}^2\}))$ forms a pair of bi-frames. For this MRA based bi-frame pair, one can derive fast decomposition and reconstruction algorithms as given in [9].

On the other hand, since $X(\psi)$ is a Riesz basis, it has a unique Riesz dual basis, say $X(\tilde{\psi})$. Since the unique Riesz dual wavelet $\tilde{\psi}$, as indicated by Theorem 3.2, is not compactly supported and has low order of smoothness. This indicates that $X(\psi)$ being a Riesz basis does not imply that the decomposition (analysis) operator must have an inverse in a function space other than the space $L_2(\mathbb{R})$ that is of interest to us in this paper, since its dual wavelet may have lower smoothness. Furthermore, even the decomposition operator may have an inverse in the function space, it may not have a fast reconstruction algorithm due to the slow decay of the dual masks with infinite support. However, the above discussions show that by introducing redundancy into a Riesz wavelet basis $X(\psi)$, one can obtain a compactly supported dual frame system which has the same smoothness (in some case, it can be extended to a tight frame wavelet system). Hence, the frame decomposition operator can be invertible in various function spaces and a fast decomposition and reconstruction algorithm can be obtained based on the compactly supported dual frame.

The following example illustrates some of the above discussions. The reader is referred to [8, 9] for more details on bi-frames.

Example 4.1. Let B_3 be the B-spline function of order 3. The refinement mask for the refinable function B_3 is $\hat{a}(\xi) = 2^{-3}(1+e^{-i\xi})^3$. Define $\psi = \frac{1}{4}B_3(2\cdot-1) - \frac{3}{4}B_3(2\cdot) + \frac{3}{4}B_3(2\cdot+1) - \frac{1}{4}B_3(2\cdot+2)$, that is, $\hat{\psi}(2\xi) = 2^{-3}e^{-i\xi}(1-e^{i\xi})^3\widehat{B_3}(\xi)$. By Theorem 2.2, $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$. The function ψ has 3 vanishing moments and the regularity 5/2. By Theorem 2.2, its unique Riesz dual basis $X(\tilde{\psi})$ must have a non-compactly supported generator $\tilde{\psi}$. However, the frame system $X(\psi) \cup X(\psi(\cdot - 1/2))$ can have a compactly supported smooth dual frame as defined below. We choose $\tilde{\phi} = B_3$. Let $\tilde{b}^1$ and $\tilde{b}^2$ be two finitely supported sequences on $\mathbb{Z}$ given by

$$\widehat{\tilde{b}^1}(\xi) := \frac{(z-1)^3}{1920}\Big[13\,(z^{-6}+z^2) + 78\,(z^{-5}+z) + 356\,(z^{-4}+1) + 1226\,(z^{-3}+z^{-1}) + 2334\,z^{-2}\Big],$$

$$\widehat{\tilde{b}^2}(\xi) := \frac{(z-1)^3}{960}\Big[39\,(z^{-4}+z^2) + 234\,(z^{-3}+z) + 613\,(z^{-2}+1) + 948z^{-1}\Big],$$

where $z := e^{-i\xi}$. Define the dual wavelet functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$ by

$$\widehat{\tilde{\psi}^1}(\xi) = \widehat{\tilde{b}^1}(\xi/2)\widehat{B_3}(\xi/2), \qquad \widehat{\tilde{\psi}^2}(\xi) = \widehat{\tilde{b}^2}(\xi/2)\widehat{B_3}(\xi/2).$$

Then it has been proved in [8, 9] that $(X(\{\psi, \psi(\cdot - 1/2)\}), X(\{\tilde{\psi}^1, \tilde{\psi}^2\}))$ is a pair of bi-frames in $L_2(\mathbb{R})$. Note that both $\tilde{\psi}^1$ and $\tilde{\psi}^2$ are compactly supported and antisymmetric. Moreover, they have the regularity 5/2 and 3 vanishing moments. See Figure 3 for the graphs of all the functions ψ, $\tilde{\psi}^1$ and $\tilde{\psi}^2$.

5. Other Examples of Riesz Wavelets with Short Support

In this section, we first show that one can further greatly shorten the support of the Riesz spline wavelets with the same order of smoothness given in Section 2 by reducing the order of vanishing moments of the wavelets. The second part gives examples which are derived from interpolatory refinable functions.

Theorem 5.1. *Let B_m be the B-spline function of order m. For any integer $\tilde{m}$ such that $\tilde{m} \geqslant m\log_6(8/3)$ (note that $\log_6(8/3) \approx 0.5474$) and $\tilde{m}+m$ is an even integer, define $\hat{\psi}(2\xi) := e^{i\xi(m-\tilde{m}-2)/2}(\frac{1-e^{i\xi}}{2})^{\tilde{m}}\widehat{B_m}(\xi)$. Then $X(\psi)$ is a Riesz wavelet basis in $L_2(\mathbb{R})$.*

Proof. We use Corollary 3.3. For this, we first note that the refinement mask of the spline B_m is $\hat{a}(\xi) = (\frac{1+e^{-i\xi}}{2})^m\hat{A}(\xi)$ with $\hat{A}(\xi) = 1$. Let $\hat{b}(\xi) := e^{i\xi(m-\tilde{m}-2)/2}(\frac{1-e^{i\xi}}{2})^{\tilde{m}}$ and define $\hat{\psi}(2\xi) = \hat{b}(\xi)\widehat{B_m}(\xi)$. Then

$$\hat{d}(\xi) := \hat{a}(\xi)\hat{b}(\xi+\pi) - \hat{a}(\xi+\pi)\hat{b}(\xi) = e^{-i\xi}(-1)^{(m-\tilde{m}-2)/2}[\cos^{m+\tilde{m}}(\xi/2) + \sin^{m+\tilde{m}}(\xi/2)] \neq 0. \quad (5.1)$$

The mask $\hat{\mathring{a}}$ in Theorem 3.2 is

$$\hat{\mathring{a}}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{m}}\hat{\mathring{A}}(\xi) \quad \text{with} \quad \hat{\mathring{A}}(\xi) := \frac{e^{-i\xi(m-\tilde{m})/2}}{\cos^{m+\tilde{m}}(\xi/2) + \sin^{m+\tilde{m}}(\xi/2)}.$$

Consider $f_n(x) := x^n + (1-x)^n, 0 \leqslant x \leqslant 1$ and $n \in \mathbb{N}$. Since $f_n'(x) = n[x^{n-1} - (1-x)^{n-1}]$, f_n decreases on $[0, 1/2]$ and increases on $[1/2, 1]$. Therefore, $f_n(x) \geqslant f_n(1/2)$ for all $x \in [0, 1]$.

Since f_n decreases on $[0, 1/4]$, we have

$$f_n((2x-1)^2)f_n(x) \geqslant f_n(1/2)f_n(x) \geqslant f_n(1/2)f_n(1/4) \qquad \forall\, x \in [0, 1/4].$$

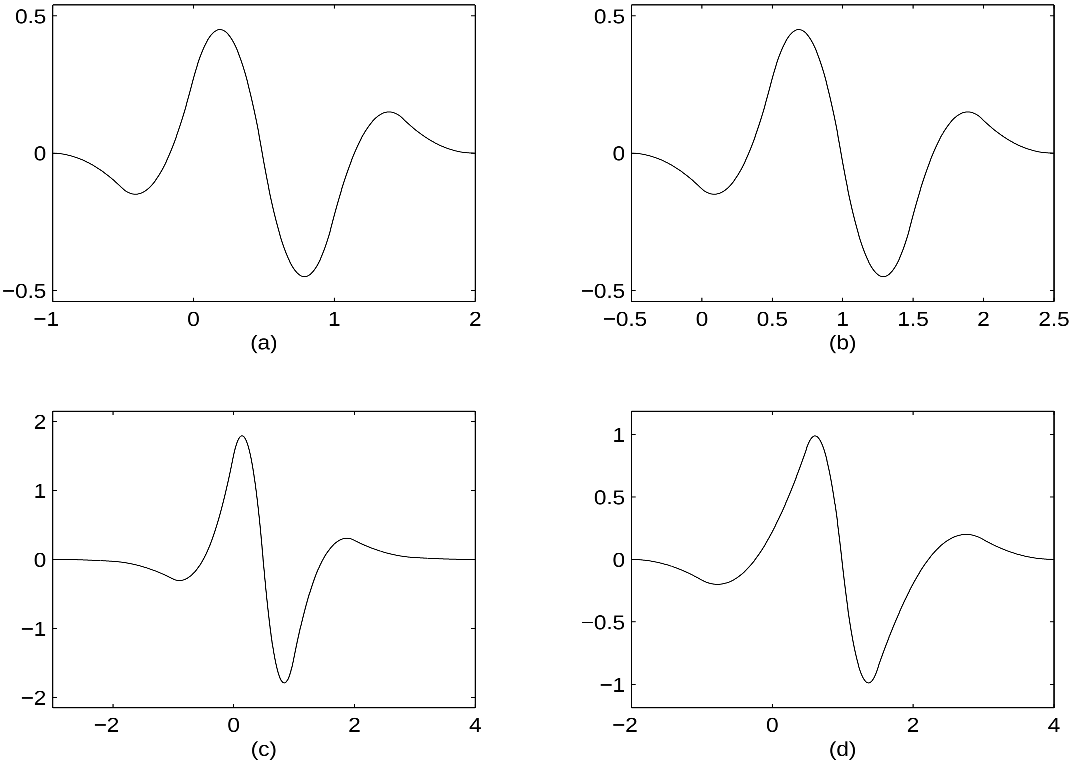

FIGURE 3. The graphs of the functions ψ and $\psi(\cdot - 1/2)$ (the top row) and the graphs of the dual functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$ (the bottom row) in Example 4.1. All the functions ψ, $\psi(\cdot - 1/2)$, $\tilde{\psi}^1$ and $\tilde{\psi}^2$ have 3 vanishing moments and the regularity 5/2. The system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$ and $(X(\{\psi, \psi(\cdot - 1/2)\}), X(\{\tilde{\psi}^1, \tilde{\psi}^2\}))$ is a pair of bi-frames derived from the B-spline function B_3 of order 3.

On the other hand, since the map $x \mapsto (2x-1)^2$ maps the interval $[1/4, 1/2]$ onto the interval $[0, 1/4]$, we have

$$f_n((2x-1)^2)f_n(x) \geqslant f_n(1/4)f_n(x) \geqslant f_n(1/2)f_n(1/4) \qquad \forall\, x \in [1/4, 1/2].$$

By the symmetry of f_n on $[0,1]$, we conclude that

$$f_n((2x-1)^2)f_n(x) \geqslant f_n(1/2)f_n(1/4) = 2^{1-n}(4^{-n} + 3^n 4^{-n}) > 2^{1-3n}3^n \qquad \forall\, x \in [0,1], n \in \mathbb{N}.$$

Now, since $m + \tilde{m}$ is an even integer, we observe that for all $\xi \in \mathbb{R}$,

$$|\hat{\tilde{A}}(2\xi)\hat{\tilde{A}}(\xi)| = \Big[f_{(m+\tilde{m})/2}((2\cos^2(\xi) - 1)^2)f_{(m+\tilde{m})/2}(\cos^2(\xi/2))\Big]^{-1} < 2^{3(m+\tilde{m})/2-1}3^{-(m+\tilde{m})/2}.$$

Since $2^{3(m+\tilde{m})/2-1}3^{-(m+\tilde{m})/2} \leqslant 2^{2\tilde{m}-1}$ for all $\tilde{m} \geqslant m \log_6(8/3)$, we conclude that $\rho_{\tilde{A}} < 2^{\tilde{m}-1/2}$ for all $\tilde{m} \geqslant m \log_6(8/3)$. It is clear that $\rho_A = 1 < 2^{m-1/2}$. Hence, $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$ by Corollary 3.3. ■

The assumption that $m + \tilde{m}$ is an even integer in Theorem 5.1 is used only to guarantee that (5.1) holds. A more refined analysis can be employed in Theorem 5.1 to show that $\rho_{\tilde{A}} \leqslant (4/3)^{(m+\tilde{m})/2}$ by estimating $\hat{\tilde{A}}_3$. Therefore, $\tilde{m} > m \log_3(4/3) + \log_3 2$ is enough and $\log_3(4/3) \approx 0.26186$.

Example 5.2. Let $m = 4$ and $\tilde{m} = 2$. Define $\hat{b}(\xi) = (1 - e^{i\xi})^2/4$ and $\hat{\psi}(2\xi) = \hat{b}(\xi)\widehat{B_4}(\xi)$. Since $m + \tilde{m} = 6$ is an even number and $\tilde{m} > m \log_3(4/3) + \log_3 2 \approx 1.6784$, $X(\psi)$ is a Riesz basis

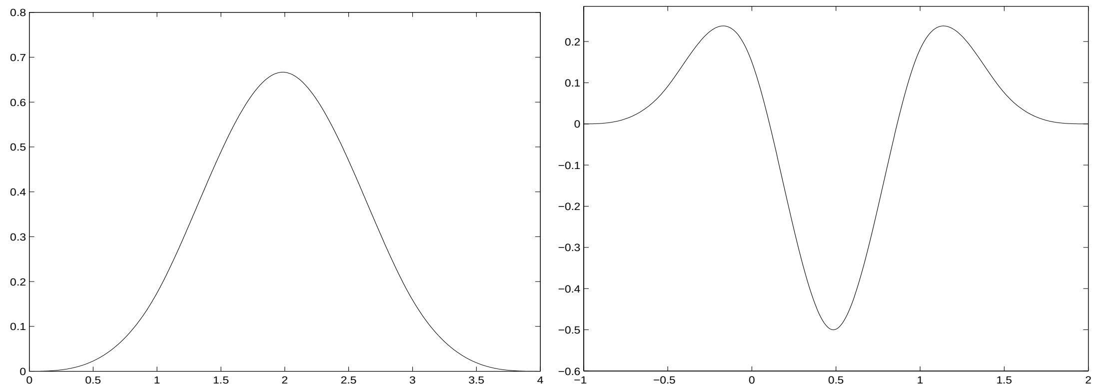

FIGURE 4. The graph of the spline B_4 (left) and the graph of the function ψ (right) in Example 5.2. The function ψ has 2 vanishing moments and the regularity 7/2. The system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

for $L_2(\mathbb{R})$. The function ψ has 2 vanishing moments and the regularity 7/2. See Figure 4 for the graphs of the functions B_4 and ψ.

In the following, we consider a continuous refinable function ϕ that satisfies the condition $\phi(k) = \delta(k)$, $k \in \mathbb{Z}$. Such a refinable function is called an *interpolatory refinable function*. Clearly, the centered piecewise linear B-spline B_2 is interpolatory. However, the higher order B-splines are not interpolatory. Smooth interpolatory refinable functions can be obtained by a convolution of a B-spline with a distribution. More precisely, for any positive integer m, the mask a given by

$$\hat{a}(\xi) = \cos^{2m}(\xi/2) P_m(\sin^2(\xi/2)), \tag{5.2}$$

where

$$P_m(x) := \sum_{j=0}^{m-1} \frac{(m-1+j)!}{j!(m-1)!} x^j, \qquad x \in \mathbb{R}, \tag{5.3}$$

defines an interpolatory refinable function ϕ with the refinement mask a in (5.2).

This set of masks for interpolatory refinable functions were provided in [6]. Each of them is the autocorrelation of a refinement mask of some refinable function whose shifts form an orthonormal system. Interpolatory masks were first constructed, then the masks of the compactly supported orthonormal refinable functions were obtained as a square root of $\hat{a}$ in [6]. More details about this construction can be found in [6] and [7].

Theorem 5.3. *Let m be a positive integer. Let ϕ be the interpolatory refinable function with the mask a defined in* (5.2). *Define $\hat{\psi}(2\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)}\hat{\phi}(\xi)$. Then $X(\psi)$ is a Riesz basis in $L_2(\mathbb{R})$.*

Proof. To apply Lemma 2.1, we first note that

$$\hat{a}(\xi) = 2^{-2m}(1+e^{-i\xi})^{2m}\hat{A}(\xi) \quad \text{with} \quad \hat{A}(\xi) := e^{im\xi} P_m(\sin^2(\xi/2)),$$

and

$$\hat{\tilde{A}}(\xi) = \frac{\hat{A}(\xi)}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2}.$$

Then, [7, Lemmas 7.1.7 and 7.1.8] says that $\rho_A \leqslant P_m(3/4) \leqslant 3^{m-1}$ for all $m \in \mathbb{N}$ (see [7, page 226]. Therefore, $\rho_A \leqslant 3^{m-1} < 2^{2m-1/2}$ for all $m \in \mathbb{N}$.

Since the mask of any interpolatory refinable function must satisfy

$$\hat{a}(\xi) + \hat{a}(\xi+\pi) = 1, \qquad \xi \in \mathbb{R},$$

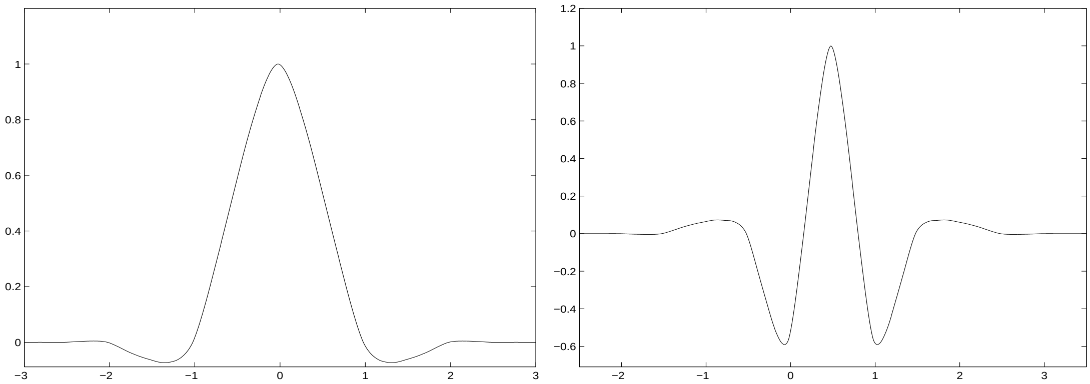

FIGURE 5. The graph of the interpolatory refinable function ϕ (left) and the graph of the function ψ (right) in Example 5.4. The function ψ has 4 vanishing moments and the regularity 2.44077. The system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

we have $|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 \geqslant 1/2$ for all $\xi \in \mathbb{R}$. Thus, $|\hat{\tilde{A}}| \leqslant 2|\hat{A}|$. This leads to that

$$\rho_{\tilde{A}} \leqslant 2\rho_A \leqslant 2 \times 3^{m-1} < 2^{2m-1/2} \qquad \forall\, m \in \mathbb{N}.$$

Hence, $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$ by Lemma 2.1. ■

Example 5.4. Let ϕ be the interpolatory refinable function with the mask $\hat{a}(\xi) = \frac{1}{2} + \frac{9}{16}\cos(\xi) - \frac{1}{16}\cos(3\xi)$, that is, the mask a is given in (5.2) with $m = 2$. Define $\hat{b}(\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)}$ and $\hat{\psi}(2\xi) = \hat{b}(\xi)\hat{\phi}(\xi)$. By Theorem 5.3, $X(\psi)$ is a Riesz wavelet basis for $L_2(\mathbb{R})$. The function ψ has 4 vanishing moments and the regularity 2.44077. See Figure 5 for the graphs of the interpolatory refinable function ϕ and the function ψ.

The support of the function in Theorem 5.3 can be further shortened by reducing the order of vanishing moments as shown in the next result.

Theorem 5.5. *Let m be a positive integer. Let ϕ be the interpolatory refinable function with the mask a given in* (5.2). *Define $\hat{\psi}(2\xi) = (\frac{1-e^{i\xi}}{2})^2\hat{\phi}(\xi)$. Then $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.*

Proof. We apply Corollary 3.3. For this, we note that

$$\hat{a}(\xi) = 2^{-2m}(1+e^{-i\xi})^{2m}\hat{A}(\xi) \quad \text{with} \quad \hat{A}(\xi) := e^{im\xi}P_m(\sin^2(\xi/2)), \qquad \hat{b}(\xi) := (1-e^{i\xi})^2/4$$

and

$$\hat{d}(\xi) = e^{i\xi}[\cos^{2m+2}(\xi/2)P_m(\sin^2(\xi/2)) + \sin^{2m+2}(\xi/2)P_m(\cos^2(\xi/2))] \neq 0 \qquad \forall\, \xi \in \mathbb{R}.$$

Then $\hat{\tilde{a}}(\xi) = 2^{-2}(1+e^{-i\xi})^2\hat{\tilde{A}}(\xi)$ with

$$\hat{\tilde{A}}(\xi) := \frac{e^{i\xi}}{\cos^{2m+2}(\xi/2)P_m(\sin^2(\xi/2)) + \sin^{2m+2}(\xi/2)P_m(\cos^2(\xi/2))}.$$

To apply Corollary 3.3, it remains to estimate $\hat{\tilde{A}}$. We note that for positive numbers a_1, a_2, a_3, a_4, it is easy to verify that $\frac{a_1}{a_2} \leqslant \frac{a_3}{a_4}$ implies $\frac{a_1}{a_2} \leqslant \frac{a_1+a_3}{a_2+a_4} \leqslant \frac{a_3}{a_4}$. Since $P_m(x) = \sum_{j=0}^{m-1} c_j x^j$, where $c_j := \frac{(m-1+j)!}{j!(m-1)!} > 0$, it follows from

$$\frac{c_0}{c_0} \leqslant \frac{c_1(1-x)}{c_1 x} \leqslant \cdots \leqslant \frac{c_{m-1}(1-x)^{m-1}}{c_{m-1}x^{m-1}} = \frac{(1-x)^{m-1}}{x^{m-1}}, \qquad x \in (0, 1/2].$$

that

$$\frac{P_m(1-x)}{P_m(x)} \leqslant \frac{(1-x)^{m-1}}{x^{m-1}} \leqslant \frac{(1-x)^m}{x^m} \qquad \forall\, x \in (0, 1/2].$$

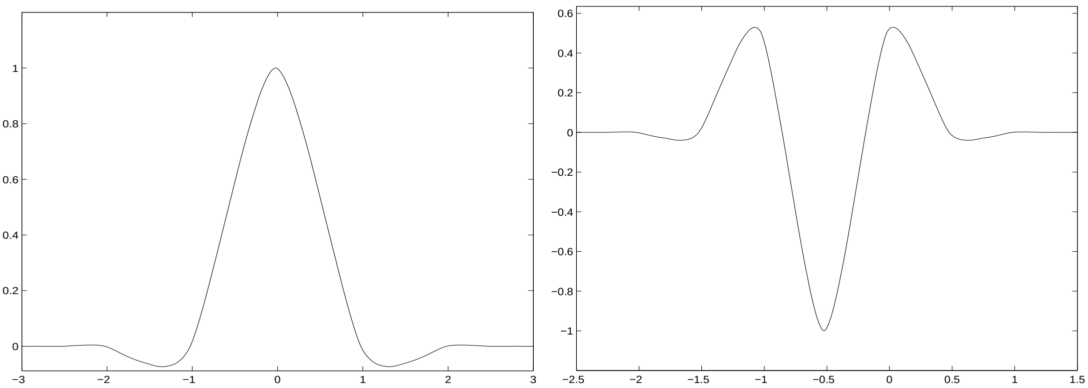

FIGURE 6. The graph of the interpolatory refinable function ϕ (left) and the graph of the function ψ (right) in Example 5.6. The function ψ has 2 vanishing moments and the regularity 2.44077. The system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

In other words, we have $x^m P_m(1-x) \leqslant (1-x)^m P_m(x)$ for all $x \in [0, 1/2]$. So, we deduce that

$$(1-x-x)x^m P_m(1-x) \leqslant (1-x-x)(1-x)^m P_m(x) \qquad \forall\, x \in [0,1/2],$$

which is equivalent to

$$(1-x)x^m P_m(1-x) - x^{m+1}P_m(1-x) \leqslant (1-x)^{m+1}P_m(x) - x(1-x)^m P_m(x) \qquad \forall\, x \in [0,1/2].$$

Hence,

$$(1-x)x^m P_m(1-x) + x(1-x)^m P_m(x) \leqslant x^{m+1}P_m(1-x) + (1-x)^{m+1}P_m(x) \qquad \forall\, x \in [0,1/2].$$

Note that $x^m P_m(1-x) + (1-x)^m P_m(x) = 1$ for all $x \in [0, 1/2]$. For any $x \in [0, 1/2]$, we have

$$\begin{aligned} 1 &= (1-x+x)[x^m P_m(1-x) + (1-x)^m P_m(x)] \\ &= [(1-x)x^m P_m(1-x) + x(1-x)^m P_m(x)] + [x^{m+1}P_m(1-x) + (1-x)^{m+1}P_m(x)] \\ &\leqslant 2[x^{m+1}P_m(1-x) + (1-x)^{m+1}P_m(x)]. \end{aligned}$$

Consequently, by symmetry, we deduce that

$$x^{m+1}P_m(1-x) + (1-x)^{m+1}P_m(x) \geqslant 1/2 \qquad \forall\, x \in [0,1].$$

Using the above inequality and taking $x = \cos^2(\xi/2)$, we have

$$|\hat{\tilde{A}}(\xi)| = [x^{m+1}P_m(1-x) + (1-x)^{m+1}P_m(x)]^{-1} \leqslant 2 \qquad \forall\, \xi \in \mathbb{R}.$$

It follows from the definition of $\rho_{\tilde{A}}$ that $\rho_{\tilde{A}} \leqslant 2 < 2^{2-1/2}$. Hence, $X(\psi)$ must be a Riesz basis for $L_2(\mathbb{R})$ by Corollary 3.3. ■

Example 5.6. Let ϕ be the interpolatory refinable function with the mask $\hat{a}(\xi) = \frac{1}{2} + \frac{9}{16}\cos(\xi) - \frac{1}{16}\cos(3\xi)$, that is, the mask a is given in (5.2) with $m = 2$. Define $\hat{b}(\xi) = (1-e^{i\xi})^2/4$ and $\hat{\psi}(2\xi) = \hat{b}(\xi)\hat{\phi}(\xi)$. By Theorem 5.5, $X(\psi)$ is a Riesz wavelet basis for $L_2(\mathbb{R})$. The function ψ has 2 vanishing moments and the regularity 2.44077. See Figure 6 for the graphs of the interpolatory refinable function ϕ and the function ψ.

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