Extension of matrices with Laurent polynomial entries

Zuowei Shen
Department of Mathematics
National University of Singapore
Singapore 119260
matzuows@leonis.nus.sg

September 1996

Abstract: Let P and Q be $n\times r$, $r \le n$ matrices with Laurent polynomial entries. Suppose that $Q^T(z)P(z) = I_r$, $z \in \mathbb{C}\backslash\{0\}$. This note provides an algorithmatic construction of two $n\times(n-r)$ matrices G and H with Laurent polynomial entries such that the $n\times n$ matrices $X := (P \quad G)$ and $Y := (Q \quad H)$ satisfy

$$Y^T(z)X(z) = I_n, \quad z \in \mathbb{C}\backslash\{0\}.$$

AMS Subject Classification: Primary 41A15, 15A54 65D07 65F30

Keywords: Matrix extension, Laurent polynomial Biorthogonal (multi)wavelets.

Extension of matrices with Laurent polynomial entries

Zuowei Shen
Department of Mathematics
National University of Singapore
Singapore 119260

1. Introduction

This note provides an algorithmatic solution to the following problem:

Matrix Extension Problem 1.1. *Let P and Q be $n \times r$, $r \le n$ matrices in $\mathcal{P}$ where $\mathcal{P}$ is the set of all finite order matrices with Laurent polynomial entries. Suppose that $Q^T(z)P(z) = I_r$, $z \in \mathbb{C}\backslash\{0\}$, where Q^T is the transpose of the matrix Q and I_r is the identity matrix of order r. How to obtain $n \times (n-r)$ matrices G and H in $\mathcal{P}$ such that the $n \times n$ matrices $X := (P \quad G)$ and $Y = (Q \quad H)$ satisfy*

$$Y^T(z)X(z) = I_n, \quad z \in \mathbb{C}\backslash\{0\}?$$

For the special case that $P = Q$, this problem has been solved in [LLS] in the context of the construction of compactly supported orthonormal (multi)wavelets from a multiresolution. In fact, [LLS] provides an algorithmatic method to find G such that the matrix $X = (P \quad G)$ satisfy $X^*X = I_n$ on the torus.

Another special case is that when the $n \times r$ matrix $P \in \mathcal{P}$ is nonsingular on $\mathbb{C}\backslash\{0\}$, how to extend this matrix to an $n \times n$ nonsigular matrix on $\mathbb{C}\backslash\{0\}$ in $\mathcal{P}$ with P as its first r columns. By the Quillen-Suslin Theorem, such an extension exists. An algorithmatic method is provided in [LLS] in the context of the construction of compactly supported (multi)prewavelet from a multiresolution.

However, Problem 1.1 is not that well studied in the literature. The solution to Problem 1.1 can be applied to construct compactly supported biorthonormal (multi)wavelets from a pair of dual multiresolutions. It also can be used to design a pair of high pass filter banks from a given pair of low pass filter banks in the area of the signal processing. The interested reader should consult [SN] for the details.

We remark that with this note together with [LLS] the problems of extending matrices related to the construction of the compactly supported orthonormal (multi)wavelets, (multi)prewavelets ([LLS]) and biorthonormal (multi)wavelets have been completely solved for the univarate case, although there may still have some possibilities to improve the algorithm. It should be mentioned that the corresponding matrix extension problems for the multivariate case are much more challenging. However, some results have been obtained in this direction (see [JS], [RS1], and [RS2]).

2. Extension of matrices

Denote by $\mathbb{D}$ the set of all finite order square diagonal matrices whose diagonal elements are of the form cz^α with $c \in \mathbb{C}\backslash\{0\}$ for some $\alpha \in \mathbb{Z}$. We also denote by $\mathbf{i}_j$ the column vector whose jth entry is 1 and all other entries are 0.

We start with the following Proposition:

Proposition 2.1. *Let p be an $n \times 1$ matrix in $\mathcal{P}$ with the first entry being 1. Then, there exists an $n \times n$ matrix A in $\mathcal{P}$ which is nonsingular on $\mathbb{C}\backslash\{0\}$, such that*

$$A(z)p(z) = \mathbf{i}_1.$$

Proof. By multiplying a proper diagonal matrix D_0 in $\mathbb{D}$, we may assume that

$$p = \sum_{i=0}^{N} \mathbf{a}_i z^i,$$

where $\mathbf{a}_i$ is an $n \times 1$ constant column vector. The assumption that the first entry of p is 1 implies that the first entry of $\mathbf{a}_0$ is 1 and the first entry of $\mathbf{a}_i$ is 0, for $i = 1, \ldots, N$. Hence, there is a nonsigular constant matrix C derived from row operations, so that $C_1\mathbf{a}_0 = \mathbf{i}_1$. The column vector $C_1p(z)$ has the form that the first entry is 1 and the other entries are either 0 or a polynomial with its least degree at least 1. Then, there is a matrix $D_1 \in \mathbb{D}$, such that the column vector D_1C_1p has degree at most $N-1$, $N > 0$. This implies that D_1C_1p has lower degree polynomials as its entries in comparison with p and it has 1 as its first entry as well. Hence, after a finite number of steps,

we have

$$D_kC_k \cdots D_1C_1D_0p = \mathbf{i}_1.$$

Let $A = D_kC_k \cdots D_1C_1D_0$. Then $A \in \mathcal{P}$ is nonsingular on $\mathbb{C}\backslash\{0\}$ and $Ap = \mathbf{i}_1$. □

A similar argument, with some modifications, leads to the following two results of [LLS].

Result 2.2. *Let $p(z)$ be an $n \times 1$ matrix in $\mathcal{P}$. Then there is an invertable matrix $A(z)$ in $\mathcal{P}$ such that*

$$A(z)p(z) = (f(z), 0, \ldots, 0)^T. \tag{2.3}$$

where $f(z) \in \mathcal{P}$ (i.e. f is a Laurent polynomial). Further, A is a product of (constant) matrices corresponding to row operations and diagonal matrices in $\mathbb{D}$.

Proof. The proof provided in [LLS] is elementary and the matrix A can be obtained constructively. The interested reader should consult [LLS] for the details. □

Directly applying this result, in [LLS] the following result was obtained (see Theorem 3.2 [LLS]) :

Result 2.4. *Let $P = (p_1 \dots p_r)$ be a $n \times r$ matrix in $\mathcal{P}$. Then, there exists an $n \times n$ matrix $A(z)$ in $\mathcal{P}$ which is nonsingular on $\mathbb{C}\backslash\{0\}$, such that*

$$A(z)P(z) = \begin{pmatrix} U(z) \\ \mathbf{0} \end{pmatrix},$$

where $U(z)$ is an upper triangular $r \times r$ matrix in $\mathcal{P}$ and $\mathbf{0}$ is the $(n-r) \times r$ zero matrix.

We need the following Proposition to solve Problem 1.1.

Proposition 2.5. *Let $P = (p_1 \dots p_r)$ be a $n \times r$ matrix in $\mathcal{P}$ which has rank r on $\mathbb{C}\backslash\{0\}$. Then, one can (constructively) find an $n \times n$ matrix $A(z)$ in $\mathcal{P}$ which is nonsingular on $\mathbb{C}\backslash\{0\}$, such that*

$$A(z)P(z) = \begin{pmatrix} I_r \\ \mathbf{0} \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and $\mathbf{0}$ is the $(n-r) \times r$ zero matrix.

Proof. Applying Result 2.4, one can construct a matrix $A_0(z) \in \mathcal{P}$ which is nonsingular on $\mathbb{C}\backslash\{0\}$ such that

$$A_0(z)P(z) = \begin{pmatrix} U(z) \\ \mathbf{0} \end{pmatrix},$$

where $U(z)$ is an upper triangular $r \times r$ matrix in $\mathcal{P}$ and $\mathbf{0}$ is the $(n-r) \times r$ zero matrix. Since P has rank r on $\mathbb{C}\backslash\{0\}$, when multiplying by a proper $D \in \mathbb{D}$, we may assume that the diagonal entries of U are 1. Hence, the first column of A_0P is $\mathbf{i}_1$. Suppose that we have constructed a matrix A_{j-1}, such that the matrix

$$A_{j-1}(z)A_0(z)P(z) = \begin{pmatrix} U_{j-1}(z) \\ \mathbf{0} \end{pmatrix},$$

where $U_{j-1}(z)$ is an upper triangular $r \times r$ matrix in $\mathcal{P}$ and the matrix $A_{j-1}(z)A_0(z)p(z)$ has the form

$$A_{j-1}(z)A_0(z)P(z) = (\mathbf{i}_1, \dots, \mathbf{i}_{j-1}, s_j(z) \dots, s_r(z)),$$

where each column s_k has 1 as its kth entry and the lth entry, $l > k$, is 0. Applying Proposition 2.1, one obtains a matrix $S_j \in \mathcal{P}$ which is nonsingular on $\mathbb{C}\backslash\{0\}$, such that $S_j s_j = \mathbf{i}_j$. Since S_j is a product of constant matrices corresponding to row operations (const jth row $\pm$ ith row, $i < j$) and diagonal matrices from $\mathbb{D}$, we have that columns $S_j\mathbf{i}_i = \mathbf{i}_i$ for $i < j$ (by multiplying an (extra) proper diagonal matrix from $\mathbb{D}$, if it is necessary) and the lth entry, $l > j$ of the column $S_j s_k$, $k > j$, does not change. Let $A_j = S_j A_{j-1}$, then the matrix $A_{j-1}(z)A_0(z)p(z)$ has the form that

$$A_j(z)A_0(z)P(z) = (\mathbf{i}_1, \dots \mathbf{i}_j, S_j(z), s_{j+1}(z), \dots, S_j(z)s_r(z)),$$

where the kth entry of the column $S_j s_k$ is 1 and the lth entry, $l > k$ of $S_j s_k$ is 0. Therefore, by a finite number of inductive steps, we find a matrix $A \in \mathcal{P}$, such that

$$A(z)P(z) = \begin{pmatrix} I_r \\ \mathbf{0} \end{pmatrix}$$

□

Finally, we reach to the solution to Problem 1.1.

Theorem 2.6. *Let P and Q be $n \times r$, $r \leq n$ matrices in $\mathcal{P}$ with $Q^T(z)P(z) = I_r$, $z \in \mathbb{C}\backslash\{0\}$. Then there exist two $n \times (n-r)$ matries G and H in $\mathcal{P}$ such that the $n \times n$ matrices $X := (P \quad G)$ and $Y = (Q \quad H)$ satisfy*

$$Y^T(z)X(z) = I_n, \quad z \in \mathbb{C}\backslash\{0\}.$$

Further, the matrices G and H can be obtained by an algorithmatic construction.

Proof. Applying Result 2.4, one can find a matrix $A \in \mathcal{P}$ which is nonsingular on $\mathbb{C}\backslash\{0\}$, such that

$$A(z)P(z) = \begin{pmatrix} U(z) \\ \mathbf{0} \end{pmatrix},$$

where $U(z)$ is an upper triangular $r \times r$ matrix in $\mathcal{P}$ and $\mathbf{0}$ is the $(n-r) \times r$ zero matrix. Denote by the last $n-r$ rows of $A(z)$ $a_1^T(z), \dots, a_{n-r}^T$ and define an $n \times (n-r)$ matrix $H = (a_1 \cdots a_{n-r})$. Then $H^T(z)P(z) = \mathbf{0}$, $z \in \mathbb{C}\backslash\{0\}$ and the matrix $H(z)$ has rank $n-r$ for all $z \in \mathbb{C}\backslash\{0\}$. Let $Y = (Q \quad H)$. Then, the matrix $Y(z)$ is nonsingular for all $z \in \mathbb{C}\backslash\{0\}$, because the matrix

$$\begin{aligned} Y^T(z)(P(z) \quad H(z)) &= \begin{pmatrix} Q^T(z) \\ H^T(z) \end{pmatrix} (\, P(z) \quad H(z)\,) \\ &= \begin{pmatrix} I_r & Q^T(z)H(z) \\ \mathbf{0} & H^T H \end{pmatrix} \end{aligned}$$

is nonsingular on $\mathbb{C}\backslash\{0\}$.

Applying Proposition 2.5, one obtains a matrix $B \in \mathcal{P}$, such that

$$B(z)Y(z) = B(z)(Q(z) \quad H(z)) = I_n, \quad z \in \mathbb{C}\backslash\{0\}. \tag{2.7}$$

Denote the last $n-r$ rows of $B(z)$ $b_1^T(z), \dots, b_{n-r}^T(z)$ and define the $n \times (n-r)$ by matrix $G(z) = (b_1(z) \cdots b_{n-r}(z))$. Then, $Q^T(z)G(z) = \mathbf{0}$ and $H^T(z)G(z) = I_{n-r}$, $z \in \mathbb{C}\backslash\{0\}$.

Let $X(z) = (P(z) \quad G(z))$. Recall the fact that $H^T(z)P(z) = \mathbf{0}$ and $Q^T(z)P(z) = I_r$, we have

$$\begin{aligned} Y^T(z)X(z) &= \begin{pmatrix} Q^T(z) \\ H^T(z) \end{pmatrix} (P(z) \quad G(z)) \\ &= \begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & I_{n-r} \end{pmatrix} \\ &= I_n. \end{aligned}$$

The last statement of this theorem follows from the fact that the Results and Propositions used here are all obtained by algorithmatic constructions. □

Remark 2.8. *The matrix $B(z)$ in (2.7) is the inverse matrix of $Y(z)$. Since $Y(z)$ is nonsingular on $\mathbb{C}\backslash\{0\}$, $\det Y(z) = z^\alpha$, for some $\alpha \in \mathbb{Z}$. Therefore, Y^{-1} has Laurent polynomial entries and one may obtain $B(z)$ without using Proposition 2.5. However, Proposition 2.5 provides a constructive method to obtain Y^{-1} which is implementable on computers. This is in particular useful when n is large.* □

Reference

[JS] Jia, Rong-Qing and Zuowei Shen, Multiresolution and wavelets, Proc. Edinburgh Math. Soc. **37** (1994), 271-300.

[LLS] Lawton, W., S.L. Lee and Zuowei Shen, An algorithm for matrix extension and wavelet construction, Math. Comp. **65** (1996) 723–737.

[RS1] Riemenschneider, S.D. and Zuowei Shen, Box splines, cardinal series and wavelets, in "Approximation Theory and Functional Analysis", C. K. Chui, ed., Academic Press, 1991, pp. 133–149.

[RS2] Riemenschneider, S. D. and Zuowei Shen, Wavelets and pre-wavelets in low dimensions, J. Approx. Theory **71** (1992), 18–38.

[SN] Strang G.and Truong Nguyen, Wavelets and Filter Banks, Wellesley-Cambridge Press (1996).