Compactly supported (bi)orthogonal wavelets generated by interpolatory refinable functions

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Abstract: This paper provides several constructions of compactly supported wavelets generated by interpolatory refinable functions. It was shown in [D1] that there is no real compactly supported orthonormal symmetric dyadic refinable function, except the trivial case; and also shown in [L] and [GM] that there is no compactly supported interpolatory orthonormal dyadic refinable function. Hence, for the dyadic dilation case, compactly supported wavelets generated by interpolatory refinable functions have to be biorthogonal wavelets. The key step to construct the biorthogonal wavelets is to construct a compactly supported dual function for a given interpolatory refinable function. We provide two explicit iterative constructions of such dual functions with desired regularity. When the dilation factors are larger than 3, we provide several examples of compactly supported interpolatory orthonormal symmetric refinable functions from a general method. This leads to several examples of orthogonal symmetric (anti-symmetric) wavelets generated by interpolatory refinable functions.

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1. Introduction

The multiresolution analysis starts with a compactly supported refinable function $\phi \in L_2(\mathbb{R})$, whose shifts form a Riesz basis or an orthonormal basis of the closed shift invariant subspace $S(\phi)$ of $L_2(\mathbb{R})$ generated by ϕ. Recall that a compactly supported function $\phi \in L_2(\mathbb{R})$ is m-**refinable**, if the function ϕ satisfies the following refinement equation with **dilation factor** m:

$$\phi = \sum_{\alpha \in \mathbb{Z}} m a(\alpha) \phi(m \cdot - \alpha) \tag{1.1}$$

for a finitely supported sequence a. The sequence a is called the **mask** of ϕ. When $m = 2$, we simply call ϕ refinable. Taking the Fourier transform of (1.1), the refinement equation (1.1) can be written as

$$\widehat{\phi}(\omega) = \hat{a}(\omega/m)\widehat{\phi}(\omega/m), \tag{1.2}$$

where $\hat{a}(\omega) = \sum_{\alpha \in \mathbb{Z}} a(\alpha) \exp(-i\alpha\omega)$. We also call $\hat{a}$ the mask of the refinable function ϕ.

It was shown in [BDR] (also see [JS]) that the sequence of subspaces of $L_2(\mathbb{R})$ defined by

$$S^k(\phi) := \{f(m^k \cdot) : f \in S(\phi)\}; \quad k \in \mathbb{Z},$$

satisfies

$$\cup_{k \in \mathbb{Z}} S^k(\phi) = L_2(\mathbb{R}) \quad \text{and} \quad \cap_{k \in \mathbb{Z}} S^k(\phi) = \{0\}.$$

Hence, if ϕ and its shifts form an orthonormal or a Riesz basis of $S(\phi)$, the sequence of the subspaces $S^k(\phi)$, $k \in \mathbb{Z}$ forms a multiresolution of $L_2(\mathbb{R})$. Here we recall that a sequence $S^k(\phi)$ forms a multiresolution, when the following conditions are satisfied: (i) $S^k(\phi) \subset S^{k+1}(\phi)$; (ii) $\cup_{k \in \mathbb{Z}} S^k(\phi) = L_2(\mathbb{R})$ and $\cap_{k \in \mathbb{Z}} S^k(\phi) = \{0\}$; (iii) ϕ and its shifts form an orthonormal or a Riesz basis of $S(\phi)$.

If $\phi \in L_2(\mathbb{R})$ and its shifts form a Riesz basis of $S(\phi)$, we call ϕ is **stable**, and if $\phi \in L_2(\mathbb{R})$ and its shifts form an orthonormal basis of $S(\phi)$, then ϕ is called **orthonormal**. Finally, we say a continuous function ϕ is **interpolatory**, when ϕ satisfies $\phi(\alpha) = \delta_\alpha$, $\alpha \in \mathbb{Z}$.

In practice, a signal is sampled by an element in S^k for some k. Hence we call S^k, $k \in \mathbb{Z}$ sampling spaces. This sampling process is done by using another set of sampling basis to form a projection. This sampling basis is normally generated by kth dilation of another refinable function ϕ^d and its proper shifts. The function ϕ^d is also required to satisfy the following dual conditions:

$$\langle \phi, \phi^d(\cdot - \alpha) \rangle = \delta_\alpha; \quad \alpha \in \mathbb{Z}. \tag{1.3}$$

A stable function $\phi^d \in L_2(\mathbb{R})$ is called a **dual function** of ϕ, when (1.3) holds.

It is clear that when ϕ is orthonormal, $\phi = \phi^d$. When ϕ is only stable, then ϕ^d is not equal to ϕ and ϕ^d may or may not be in the space $S(\phi)$.

Images are often first represented by samples in the sampling space $S^k(\phi)$. When the pixel values of an image f are given, an image is normally (or easily) represented by

$$f_k = \sum_{\alpha \in \mathbb{Z}^s} f(\alpha/m^k) m^{k/2} \phi(m^k \cdot - \alpha),$$

for a certain dilation level k. However, to apply the decomposition and reconstruction algorithm, one should use the function

$$\sum_{\alpha \in \mathbb{Z}^s} \langle f_k, m^{k/2} \phi^d(m^k \cdot - \alpha) \rangle m^{k/2} \phi(m^k \cdot - \alpha).$$

This function is not the function f_k, unless the refinable function ϕ satisfies the condition $\phi(\alpha) = \delta_\alpha$. Hence, by using the sampling space generated by interpolatory refinable functions, one simplifies (or reduces the errors of) the first step of the decomposition and reconstruction algorithm.

In §3, we will construct examples of smooth compactly supported interpolatory orthonormal symmetric refinable functions ϕ with the dilation factor ≥ 3 from a general method. However, as it was shown in [D1], [GM] and [L], it is impossible to construct any compactly supported real orthonormal symmetric or interpolatory orthonormal refinable functions with the dyadic dilation other than the characteristic function of $[0, 1]$. Hence, to construct wavelets with dyadic dilation from compactly supported interpolatory refinable functions, one has to construct biorthogonal wavelets. The key step to this is to construct a compactly supported dual function for a given interpolatory refinable function. In §2, we will give two general constructions of compactly supported refinable functions with a required regularity which is dual to a given interpolatory refinable function.

Examples of compactly supported interpolatory refinable functions were first given in [Du] in the context of interpolatory subdivision schemes, and a general construction of interpolatory refinable functions was given by [D1] in the context of wavelets. Examples of compactly supported orthonormal symmetric refinable functions with the dilation factor 3 were given in [CL]. Examples of compactly supported interpolatory orthonormal refinable function with dilation factor ≥ 3 were given in [BDS]. Both examples in [CL] and [BDS] are continuous. Examples in [CL] are not interpolatory, while examples in [BDS] are not symmetric. Biorthogonal wavelet theory was established in [CDF] and [CD]. Methods of constructions of biorthogonal wavelets were also given in [CDF]. Here we focus on constructions of dual refinable functions from given interpolatory refinable functions. In [Sw] a lifting scheme was used to construct a "dual" mask $\hat{a}^d$ to the mask $\hat{a}$ of an given interpolatory refinable function. The dual mask constructed satisfies

$$\hat{a}(\omega)\overline{\hat{a}^d(\omega)} + \hat{a}(\omega+\pi)\overline{\hat{a}^d(\omega+\pi)} = 1. \tag{1.4}$$

A similar construction of [Sw] was also given in [R]. We remark that (1.4) is only a necessary condition that the mask $\hat{a}^d$ should satisfy, to make the corresponding refinable function ϕ^d a dual function of ϕ.

The constructions of refinable functions with orthonormal shifts corresponding to an arbitrary dilation m were also discussed in [Ha]. However, here we are interested in those constructions of refinable functions which not only have orthonormal shifts, but also are interpolatory.

In the rest of this section, we collect some basic facts that will be used in this paper.

First, a function $\phi \in L_2(\mathbb{R})$ is stable, if and only if the inequality

$$c \le \sum_{\alpha\in\mathbb{Z}} |\widehat{\phi}(\omega + 2\pi\alpha)|^2 \le C; \quad \text{a.e.} \quad \omega \in \mathbb{R} \tag{1.5}$$

holds for some constants $0 < c \le C < \infty$. We say a compactly supported distribution ϕ is **pre-stable**, if the inequality

$$c \le \sum_{\alpha\in\mathbb{Z}} |\widehat{\phi}(\omega + 2\pi\alpha)|^2$$

holds for some constant $0 < c$. It is clear that if a compactly supported pre-stable function is in $L_2(\mathbb{R})$, then it is stable.

A function $\phi \in L_2(\mathbb{R})$ is orthonormal, if and only if the equality

$$\sum_{\alpha\in\mathbb{Z}} |\widehat{\phi}(\omega + 2\pi\alpha)|^2 = 1; \quad \text{a.e.} \quad \omega \in \mathbb{R}$$

holds.

A compactly supported continuous function is interpolatory, if and only if the equality

$$\sum_{\alpha\in\mathbb{Z}} \widehat{\phi}(\omega + 2\pi\alpha) = 1$$

holds.

From the above observations, one can obtain easily that a compactly supported interpolatory function is stable; and the autocorrelation of an orthonormal function is interpolatory.

Let $\phi \in L_2(\mathbb{R})$ be an m-refinable function with the mask $\hat{a}$. If ϕ is orthonormal, then its mask $\hat{a}$ satisfies

$$\sum_{\nu\in\mathbb{Z}_m} |\hat{a}(\omega + \pi\nu)|^2 = 1, \quad \omega \in \mathbb{R}, \tag{1.6}$$

where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ is a quotient group ($\mathbb{Z}$ over the subgroup $m\mathbb{Z}$). The integers $\mathbb{Z}$ can be decomposed into thee disjoint sets (cosets) $\{\nu + m\mathbb{Z}\}$, where $\nu \in \mathbb{Z}/m\mathbb{Z}$.

If the m-refinable compactly supported continuous function ϕ is interpolatory, then its mask $\hat{a}$ satisfies

$$\sum_{\nu \in \mathbb{Z}_m} \hat{a}(\omega + \pi\nu) = 1, \quad \omega \in \mathbb{R}. \tag{1.7}$$

Condition (1.7) is called **the interpolatory condition** for the mask on $\mathbb{Z}_m$. Here we remark that Condition (1.7) is only a necessary condition for the corresponding refinable function is interpolatory. A necessary and sufficient condition in terms of the refinement mask is given in [LLS2] (cf §3).

2. Dual functions of interpolatory refinable functions

We start with the refinable functions provided in [D1] and [D2] whose mask is given by

$$H_N(\omega) := (\cos\omega/2)^N \left(\sum_{k=0}^{N/2-1} \binom{N/2-1+k}{k} \sin^{2k}(\omega/2) \right), \tag{2.1}$$

where N is an even number. The number N is the **order** of H_N. It was shown in [D1] and [D2] that the corresponding refinable function ϕ for the nonnegative mask H_N is interpolatory and symmetric. It was further shown in [D1] [D2] that the regularity of ϕ increases linearly with N. The Fourier transform of the corresponding refinable function ϕ is given by

$$\widehat{\phi} = \Pi_{k=1}^{\infty} H_N(\omega/2^k). \tag{2.2}$$

The main purpose of this section is to find compactly supported symmetric dual functions of a given interpolatory refinable function ϕ, such as the one defined in (2.2).

2.1. Construction

The constructions of compactly supported refinable functions that are dual to a given compactly supported interpolatory refinable function ϕ are based on the following proposition:

Proposition 2.3. *Let P and Q be two 2π periodic functions that satisfy the interpolatory condition (1.7) on $\mathbb{Z}_2$. Define H by either*

$$H := P + 2Q(1-P), \tag{2.4}$$

or

$$H := P^2 + 3Q(1-P). \tag{2.5}$$

Then the function PH satisfies the interpolatory condition (1.7) on $\mathbb{Z}_2$.

Proof. Let H be the function defined as (2.4). Then

$$\begin{aligned} P(\omega)H(\omega) &= P(\omega)(-2P(\omega)Q(\omega) + 2Q(\omega) + P(\omega)) \\ &= 2P(\omega)Q(\omega)(1-P(\omega)) + P^2(\omega) \\ &= 2P(\omega)P(\omega+\pi)Q(\omega) + P^2(\omega), \end{aligned}$$

and

$$P(\omega+\pi)H(\omega+\pi) = 2P(\omega+\pi)P(\omega)Q(\omega+\pi) + P^2(\omega+\pi).$$

Therefore

$$\begin{aligned} &P(\omega)H(\omega) + P(\omega+\pi)H(\omega+\pi) \\ &= 2P(\omega)P(\omega+\pi)Q(\omega) + P^2(\omega) + 2P(\omega+\pi)P(\omega)Q(\omega+\pi) + P^2(\omega+\pi) \\ &= 1. \end{aligned}$$

The proof for the H defined by (2.5) is similar. □

Remark. If $P = Q$, then the definitions of H in (2.4) and (2.5) coincide and

$$H = P(3-2P).$$

Remark. In general, for each given integer K define

$$H := \sum_{j=0}^{K-1} \binom{2K}{j} P^{2K-1-j}(1-P)^j + \binom{2K}{K} P^K(1-P)^K. \tag{2.6}$$

Then, the mask PH satisies (1.7), whenever P satisfies (1.7). Furthermore, assume that $P \geq 0$ and the corresponding refinable function ϕ is interpolatory, then the corresponding refinable function ϕ^d with mask H is a dual function of ϕ whenever it is in $L_2(\mathbb{R})$. The regularity of ϕ^d increase as K does. This is the starting point of [JRS], where more general multivariate cases with an arbitrary dilation matrix are studied. In fact, [JRS] was motivated by the observation here. The interested reader should counsult [JRS] for the details. Since it is complicated to extract mask coefficients from H, when K is large, we suggest here some iterative constructions from (2.6) for lower K below. □

Let ϕ be the symmetric interpolatory refinable function whose mask $\hat{a} := H_N$. We use Proposition 2.3 to construct the dual function of ϕ. First, pick a mask $H_{N'}$ defined by (2.1) with order N'. Then define the dual mask either by

$$\hat{a}^{d1} := -2\hat{a}H_{N'} + 2H_{N'} + \hat{a}$$

or by

$$\hat{a}^{d2} := -3\hat{a}H_{N'} + 3H_{N'} + \hat{a}^2.$$

Since $\hat{a}$ is nonnegative and satisfies the interpolatory condition (1.7) on $\mathbb{Z}_2$, we have that $0 \leq \hat{a} \leq 1$. Therefore $\hat{a}^{d1}$ and $\hat{a}^{d2}$ are nonnegative by the fact that both $\hat{a}$ and $H_{N'}$ are nonnegative.

Proposition 2.7. *Suppose that the functions ϕ^{d1} and ϕ^{d2} defined by*

$$\widehat{\phi}^{d1} = \Pi_{k=1}^{\infty}\hat{a}^{d1}(\cdot/2^k), \qquad \widehat{\phi}^{d2} = \Pi_{k=1}^{\infty}\hat{a}^{d2}(\cdot/2^k)$$

are in $L_2(\mathbb{R})$. Then ϕ^{d1} and ϕ^{d2} are dual functions of ϕ.

Proof. It follows from Proposition 2.3 that $\hat{a}\hat{a}^{d1}$ satisfies the interpolatory condition (1.7) for $\mathbb{Z}_2$. A result of [CD] (see also Theorem 3.14 of [S]) states that if $\hat{a}\hat{a}^{d1}$ satisfies the interpolatory condition (1.7) for $\mathbb{Z}_2$, then the corresponding refinable function ϕ^{d1} is a dual function of ϕ if and only if ϕ^{d1} is stable. Since $\phi^{d1} \in L_2(\mathbb{R})$ is compactly supported function, the right hand side inequality of (1.5) holds. Therefore, to show that ϕ^{d1} is a dual function of ϕ, one only needs to show that ϕ^{d1} is pre-stable.

The proof for ϕ^{d_2} is the same, replacing d_1 by d_2 everywhere. Since $0 \leq H_{N'} \leq 1$, the set of zeros of $\hat{a}^{d1}$ ($\hat{a}^{d2}$) is a subset of zeros of $\hat{a}$. Since the refinable function ϕ corresponding to the mask $\hat{a}$ is stable, the refinable function ϕ^{d1} (ϕ^{d2}) corresponding to the mask $\hat{a}^{d1}$ ($\hat{a}^{d2}$) is stable. $\square$

Next, we provide two iterative methods. Each of them gives a construction of compactly supported dual refinable functions of a given interpolatory refinable function. We will further show in the next subsection that the dual functions can be constructed to have a required regularity.

Iterative Construction 1. *Let ϕ be a given interpolatory refinable function whose mask $\hat{a}$ of order N is given by (2.1). Let $P_0 = \hat{a}$. For $k = 1, 2, \ldots$ do*

(i) define

$$P_k := P_{k-1}^2(3 - 2P_{k-1}),$$

(ii) define $\hat{a}_k^{d1} := \frac{P_k}{\hat{a}} = \frac{P_k}{H_N}$,

(iii) define

$$\phi_k^{d1}(\omega) := \Pi_{j=1}^{\infty}\hat{a}_k^{d1}(\omega/2^j).$$

Iterative Construction 2. *Let ϕ be a given interpolatory refinable function whose mask $\hat{a}$ of order N is given by (2.1). Let $P_0 = \hat{a}$ and $H_{3^{k-1}2N}$ be the mask of the order $(3^{k-1})2N$ defined by (2.1). For $k = 1, 2, \ldots$ do*

(i) define

$$P_k := P_{k-1}(-3P_{k-1}H_{3^{k-1}2N} + 3H_{3^{k-1}2N} + P_{k-1}^2), \tag{2.8}$$

(ii) define $\hat{a}_k^{d2} := \frac{P_k}{\hat{a}} = \frac{P_k}{H_N}$,

(iii) define

$$\phi_k^{d2}(\omega) := \Pi_{j=1}^{\infty}\hat{a}_k^{d2}(\omega/2^j).$$

Proof and explanation of the both constructions. It follows from Proposition 2.3 that each P_k, $k = 1, 2, \ldots$ satisfies the interpolatory condition (1.7) for $\mathbb{Z}_2$. Since each P_k, $k = 1, 2, \ldots$ has $\hat{a}$ as a factor, $\hat{a}_k^{d1}$ ($\hat{a}_k^{d2}$) is a real valued trigonometric polynomial for all k. By Proposition 2.7, the corresponding refinable function ϕ_k^{d1} (ϕ_k^{d2}) is a dual function of ϕ whenever ϕ_k^{d1} (ϕ_k^{d2}) is in $L_2(\mathbb{R})$. $\square$

Remark. In Construction 2, P_k is constructed by P_{k-1} and $H_{3^{k-1}2N}$. With this choice of the order of the mask $H_{3^{k-1}2N}$ defined by (2.1), we are able to show in the next subsection that the regularity of the the dual function ϕ_k^{d2} increases linearly with its support.

The following lemma will be used in the next section.

Lemma 2.9. *Let P_k, $k = 0, 1, \ldots$ be the masks defined in above two constructions. Then*

$$0 \le P_k \le 1.$$

Proof. We first note that P_k is nonnegative, whenever P_{k-1} is nonnegative. Since P_0 is nonnegative, the mask P_k is nonnegative for all k. The inequality $P_k \le 1$ follows directly from the facts that P_k is nonnegative and $P_k(\omega) + P_k(\omega + \pi) = 1$. $\square$

An immediate consequence of this lemma is that the mask $\hat{a}^{d1}$ ($\hat{a}^{d2}$) is real. Hence the Fourier transform of ϕ^{d1} (ϕ^{d2}) is real. Therefore the dual function ϕ^{d1} (ϕ^{d2}) is symmetric.

Next, we calculate the supports of the dual functions obtained by Construction 1 and 2. For a given trigonometric polynomial

$$\hat{p} = \sum_{\alpha\in\mathbb{Z}} p(\alpha)\exp(-i\alpha\omega),$$

the **length** of $\hat{p}$, denoted by len($\hat{\text{p}}$), is the difference between the highest degree and the lowest degree of $\hat{p}$. If a refinable function φ is symmetric to the origin, the supports of the refinable function φ and its mask $\hat{a}$ are $[-\text{len}(\hat{\text{a}})/2, \text{len}(\hat{\text{a}})/2]$ and $[-\text{len}(\hat{\text{a}})/2, \text{len}(\hat{\text{a}})/2] \cap \mathbb{Z}$. For example, the length of H_N is $2(N-1)$ (see e.g. [D2]).

To obtain the support of the symmetric refinable functions in Construction 1 and 2, one only needs to calculate the length of the corresponding masks.

Proposition 2.10. *Let $\hat{a}_k^{d1}$ and $\hat{a}_k^{d2}$ be the mask constructed at the kth iterate in Construction 1 and 2. Then for $k \geq 1$,*

$$\mathrm{len}(\hat{\mathrm{a}}_{\mathrm{k}}^{\mathrm{d1}}) = (3^{\mathrm{k}} - 1)(2\mathrm{N} - 2)$$

and

$$\mathrm{len}(\hat{\mathrm{a}}_{\mathrm{k}}^{\mathrm{d2}}) = 4\mathrm{N}(3^{\mathrm{k}} - 2^{\mathrm{k}-1} - 1/2) - (2^{\mathrm{k}+2} - 4).$$

Proof. We only give the proof of the second identity here, the proof of the first being even easier.

Let P_k be the mask defined by (2.8) in Construction 2. Then $\hat{a}_k^{d2} = P_k/\hat{a} = P_k/H_N$. The length of P_k satisfies

$$\mathrm{len}(\mathrm{P}_{\mathrm{k}}) \leq \mathrm{len}(\mathrm{H}_{3^{\mathrm{k}}2\mathrm{N}}). \tag{2.11}$$

Indeed, (2.11) is true, when $k = 1$. Suppose (2.11) holds for the case $k-1$. Then the longest length term of P_k defined by (2.8) is $P_{k-1}^2 H_{3^{k-1}2N}$. Hence

$$\mathrm{len}(\mathrm{P}_{\mathrm{k}}) = 2\mathrm{len}(\mathrm{P}_{\mathrm{k}-1}) + \mathrm{len}(\mathrm{H}_{3^{\mathrm{k}-1}2\mathrm{N}}) \leq 3\mathrm{len}(\mathrm{H}_{3^{\mathrm{k}-1}2\mathrm{N}}) \leq \mathrm{len}(\mathrm{H}_{3^{\mathrm{k}}2\mathrm{N}}).$$

Therefore the longest length term of P_k defined by (2.8) is always $P_{k-1}^2 H_{3^{k-1}2N}$ for all k. Using this fact, we can prove inductively that

$$\mathrm{len}(\mathrm{P}_{\mathrm{k}}) = 4\mathrm{N}(3^{\mathrm{k}} - 2^{\mathrm{k}-1}) - (2^{\mathrm{k}+2} - 2), \quad \mathrm{k} = 1, 2, \ldots. \tag{2.12}$$

The identity of the length of $\hat{a}_k^{d2}$ follows directly from the identity

$$\mathrm{len}(\hat{\mathrm{a}}_{\mathrm{k}}^{\mathrm{d2}}) = \mathrm{len}(\mathrm{P}_{\mathrm{k}}) - (2\mathrm{N} - 2)$$

and (2.12). $\square$

Finally, we remark that for a given pair of the dual refinable functions ϕ and ϕ^d with masks $\hat{a}$ and $\hat{a}^d$, the construction of the corresponding biorthogonal wavelets is straightforward. Let

$$\widehat{\psi}(\omega) := \exp(-i\omega)\overline{\hat{a}^d(\omega + \pi)\widehat{\phi}(\omega)}; \quad \widehat{\psi}^d(\omega) := \exp(-i\omega)\overline{\hat{a}(\omega + \pi)\widehat{\phi}^d(\omega)}.$$

It is easy to show that the functions ϕ and ψ and their shifts form a dual Riesz basis of the functions ϕ^d and ψ^d and their shifts (see e.g. [CDF], [RiS2]). To show that ψ and ψ^d are the biorthogonal wavelets, it remains to check whether the functions

$$2^{k/2}\psi(2^k \cdot - \alpha); \quad k \in \mathbb{Z}, \alpha \in \mathbb{Z}$$

and

$$2^{k/2}\psi^d(2^k\cdot-\alpha);\quad k\in\mathbb{Z},\alpha\in\mathbb{Z}$$

form a biorthogonal Riesz basis of $L_2(\mathbb{R})$.

It was shown in [RiS2] that as long as the functions

$$R_E(\omega):=\sum_{\alpha\in 2\pi\mathbb{Z}}|\widehat{\psi}(\omega+\alpha)|,\quad R_E^d(\omega):=\sum_{\alpha\in 2\pi\mathbb{Z}}|\widehat{\psi}^d(\omega+\alpha)|,$$

and

$$R_D(\omega):=\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^k\omega)|;\quad R_D^d(\omega):=\sum_{k\in\mathbb{Z}}|\widehat{\psi}^d(2^k\omega)|$$

are in L_∞, the functions

$$2^{k/2}\psi(2^k\cdot-\alpha);\quad k\in\mathbb{Z},\alpha\in\mathbb{Z}$$

and

$$2^{k/2}\psi^d(2^k\cdot-\alpha);\quad k\in\mathbb{Z},\alpha\in\mathbb{Z}$$

form a biorthogonal basis of $L_2(\mathbb{R})$

The conditions R_E and R_E^d in $L_\infty(\mathbb{R})$ are clearly satisfied, since ψ and ψ^d are stable. The conditions R_D and R_D^d in $L_\infty(\mathbb{R})$ will be satisfied, provided the functions ψ and ψ^d have certain decay at the infinity and have a zero of certain order at origin. Both conditions are satisfied in our constructions.

2.2. Regularity

In this subsection, we will give an asymptotical analysis of the regularity of the two constructions.

A function $\varphi\in C^\alpha$ for $n<\alpha<n+1$, provided that $\varphi\in C^n$ and

$$|D^\gamma\varphi(x+t)-D^\gamma(x)|\le C|t|^{\alpha-n},\quad\text{for all }\gamma=n\text{ and }|t|\le 1 \tag{2.13}$$

for some constant C independent of x. It is well known that

$$|\widehat{\varphi}(\omega)|\le C(1+|\omega|)^{-\gamma-1-\varepsilon}\Longrightarrow\varphi\in C^\gamma. \tag{2.14}$$

The analysis of the first construction depends on the following proposition. Since all the choice of the constants in the following proposition do not depend on ω, for simplicity, we denote all the constants by C even though the value C may change with each occurance.

Proposition 2.15. *Let φ_k^1 be the refinable function corresponding to the mask P_k in Construction 1. Suppose that*

$$\widehat{\varphi}_k^1(\omega) \le C(1+|\omega|)^{-\gamma_k}.$$

Then

$$\widehat{\varphi}_{k+1}^1(\omega) \le C(1+|\omega|)^{-2\gamma_k+1.6}.$$

Proof. Define

$$L_k = 3 - 2P_k.$$

Observe that $0 \le P_k \le 1$, $1 \le L_k(\omega) \le 3 < 2^{1.6}$, and $P_k(0) = 1$ implies $L_k(0) = 1$. Hence, $L_k(\omega) \le 1 + C|\omega|$, consequently,

$$\sup_{|\omega|\le 1} \Pi_{j=1}^{\infty} L_k(\omega/2^j) \le \sup_{|\omega|\le 1} \Pi_{j=1}^{\infty} \exp(|C2^{-j}\omega|) \le C.$$

For any $2^{K-1} \le |\omega| \le 2^K$, we have that

$$\Pi_{j=1}^{\infty} L_k(\omega/2^j) \le \Pi_{j=1}^{K} L_k(\omega/2^j) \Pi_{j=K}^{\infty} \exp(|C2^{-j}\omega|) \le C2^{1.6K} \le C(1+|\omega|)^{1.6}.$$

Since

$$\widehat{\varphi}_{k+1}^1(\omega) = (\widehat{\varphi}_k^1(\omega))^2 \Pi_{j=1}^{\infty} L_k(\omega/2^j),$$

we have that

$$\widehat{\varphi}_{k+1}^1(\omega) \le C(1+|\omega|)^{-2\gamma_k+1.6}.$$

□

Proposition 2.15 shows that as long as the Fourier transform of refinable function corresponding to the mask P_0 has the decay component larger than 1.6, then φ_k^1 will have an arbitrary large decay component for the sufficiently large k. By (2.14), we obtain that function φ_k^1 can have an arbitrary high regularity as long as k is sufficiently large. Note that

$$\varphi_k^1 = \varphi_0^1 * \phi_k^{d1},$$

we conclude that ϕ_k^{d1} has an arbitrary high regularity as long as k is sufficiently large. Altogether, we have the following result:

Theorem 2.16. *Let ϕ be a given interpolatory refinable function with mask $\hat{a} = H_N$ satisfying*

$$|\widehat{\phi}(\omega)| \le C(1+|\omega|)^{-(1.6+\epsilon)}.$$

For an arbitrary $\gamma \ge 0$, there exists a constant K such that for all $k \ge K$, the refinable function φ_k^1 corresponding to the mask P_k and the dual function ϕ_k^{d1} corresponding to the mask $\hat{a}_k^{d1}$ constructed in Construction 1 are in C^{γ}.

We use the decay estimates from the invariant cycles to give an asymptotical analysis of the regularity of Construction 2. Our analysis is based on the following result which was stated in Lemma 7.16 of [D2].

Result 2.17. *Let $\hat{a}$ be the mask of the refinable function φ. The mask $\hat{a}$ is factorized to the form*

$$\hat{a}(\omega) := (\cos(\omega/2))^J L(\omega).$$

Suppose that $[-\pi,\pi] = D_1 \cup D_2$ and that there exists $q > 0$ such that

$$|L(\omega)| \leq q, \quad \omega \in D_1;$$

$$|L(\omega)L(2\omega)| \leq q^2, \quad \omega \in D_2.$$

Then $|\widehat{\phi}(\omega)| \leq C(1+|\omega|)^{-J+\kappa}$, with $\kappa = \log q/\log 2$. In particular, when $q = 2^{J(1-\mu)-1}$ with $\mu > 0$, $\phi \in C^{\mu J-\varepsilon}$.

Example 2.18. *Let the mask $\hat{a} = H_N$ be the one given by (2.1), i.e.,*

$$\hat{a}(\omega) := (\cos\omega/2)^N \left(\sum_{k=0}^{N/2-1} \binom{N/2-1+k}{k} \sin^{2k}(\omega/2) \right) =: (\cos\omega/2)^N Q(\omega),$$

where N is an even number. It was shown in [D2] that

$$Q(\omega) \leq 2^{N-2}, \quad \text{for } \omega \in [-\pi,\pi], \tag{2.19}$$

and that always

$$Q(\omega) \leq \frac{1}{3} 2^{\frac{\log 3}{2\log 2}N}, \ |\omega| \leq \frac{2}{3}\pi \quad \text{and} \quad Q(\omega)Q(2\omega) \leq \frac{1}{9} 2^{\frac{\log 3}{\log 2}N}, \ |\omega| > \frac{2}{3}\pi.$$

Therefore the corresponding refinable function $\phi \in C^{\mu N}$, with $\mu = 1 - \frac{\log 3}{2\log 2} \approx .2075$.

The mask P_k obtained in Construction 2 can be factorized to the form

$$P_k(\omega) = (cos(\omega/2))^{3^k N} L_k(\omega).$$

It can be shown easily by part (i) of Construction 2 that

$$L_k = L_{k-1}(-3P_{k-1}Q_{k-1} + 3Q_{k-1} + L_{k-1}^2), \quad k = 1, 2, \cdots, \tag{2.20}$$

where $Q_{k-1} = H_{3^{k-1}2N}/(cos(\omega/2))^{3^{k-1}2N}$.

Proposition 2.21. *Let*

$$P_k(\omega) = (cos(\omega/2))^{3^k N} L_k(\omega)$$

be the mask obtained by the kth iterates in Construction 2.

Then

$$(2.22)\qquad \begin{aligned} L_k(\omega) &\le 2^{(1-\tau)3^k N-1}, \quad |\omega| \le \frac{2}{3}\pi; \\ \mathbf{q}_k(\omega) :=& |L_k(\omega)L_k(2\omega)| \le 2^{(1-\tau)3^k 2N-2}, \quad |\omega| > \frac{2}{3}\pi \end{aligned}$$

with $\tau \le 0.05$.

Proof. First, we prove the first inequality in (2.22) with the larger constant $\tau_1 = 0.1$. It is easy to show that the inequality holds for $k = 1$ and $\tau = \tau_1$. Suppose that $L_{k-1}(\omega) \le 2^{(1-\tau_1)3^{k-1}N-1}$, $|\omega| \le \frac{2}{3}\pi$. Applying (2.20) and Lemma 2.9 and Example 2.18, we have

$$\begin{aligned} L_k &= L_{k-1}(3Q_{k-1}(1-P_{k-1}) + L_{k-1}^2) \\ &\le L_{k-1}(3Q_{k-1} + L_{k-1}^2) \\ &\le 2^{(1-\tau_1)3^{k-1}N-1}(2^{(1-\mu)3^{k-1}2N} + \frac{1}{4}2^{(1-\tau_1)3^{k-1}2N}) \\ &\le 2^{(1-\tau_1)3^k N-1} \end{aligned}$$

for all $|\omega| \le \frac{2}{3}\pi$.

Next, we show that

$$L_k(\omega) \le 2^{3^k N-1}, k = 1, 2 \cdots.$$

It is clear that the inequality holds for $k = 1$. Suppose that $L_{k-1}(\omega) \le 2^{3^{k-1}N-1}$. Applying (2.20), (2.19) and Lemma 2.9, we have that for all $\omega \in [-\pi, \pi]$,

$$\begin{aligned} L_k &\le L_{k-1}(3Q_{k-1} + L_{k-1}^2) \\ &\le 2^{3^{k-1}N-1}(3 \cdot 2^{3^{k-1}2N-2} + 2^{3^{k-1}2N-2}) \\ &\le 2^{3^k N-1}. \end{aligned}$$

Note that when $|\omega| > \frac{2}{3}\pi$, we have $|2\omega| \subset [-\frac{2}{3}\pi, \frac{2}{3}\pi] \pmod{2\pi}$. Hence

$$\max_{|\omega|>\frac{2}{3}\pi} Q(2\omega) \le \max_{|\omega|\le\frac{2}{3}\pi} Q(\omega), \quad \text{and} \quad \max_{|\omega|>\frac{2}{3}\pi} L(2\omega) \le \max_{|\omega|\le\frac{2}{3}\pi} L(\omega).$$

Finally, we prove the second inequality in (2.22). When $k = 1$, the second inequality in (2.22) holds. Assume that the second inequality in (2.22) holds for $k - 1$. For any

$|\omega| > \frac{2}{3}\pi$,

$$\begin{aligned}
\mathbf{q}_k(\omega) &= |L_k(\omega)L_k(2\omega)| \\
&\leq \mathbf{q}_{k-1}(\omega)(9Q_{k-1}(\omega)Q_{k-1}(2\omega) + 3Q_{k-1}(\omega)L^2_{k-1}(2\omega) \\
&+ 3Q_{k-1}(2\omega)L^2_{k-1}(\omega) + \mathbf{q}^2_{k-1}(\omega)) \\
&\leq \mathbf{q}_{k-1}(\omega)(9Q_{k-1}(\omega)Q_{k-1}(2\omega) + 3Q_{k-1}(\omega)\max_{|\omega|\leq\frac{2}{3}\pi} L^2_{k-1}(\omega) \\
&+ 3(\max_{|\omega|\leq\frac{2}{3}\pi} Q_{k-1}(\omega))L^2_{k-1}(\omega) + \mathbf{q}^2_{k-1}(\omega)) \\
&\leq \mathbf{q}_{k-1}(\omega)(2^{(1-\mu)3^{k-1}4N} + \frac{3}{4}2^{3^{k-1}2N}2^{(1-\tau_1)3^{k-1}2N-2} \\
&+ 2^{(1-\mu)3^{k-1}2N}2^{3^{k-1}2N-2} + \mathbf{q}^2_{k-1}(\omega)) \\
&< \mathbf{q}_{k-1}(\omega)(2^{(1-\mu)3^{k-1}4N} + \frac{3}{16}2^{(1-\frac{\tau_1}{2})3^{k-1}4N} + \frac{1}{4}2^{(1-\frac{\mu}{2})3^{k-1}4N} + \mathbf{q}^2_{k-1}(\omega)) \\
&\leq \mathbf{q}_{k-1}(\omega)(\frac{3}{4}2^{(1-\frac{\tau_1}{2})3^{k-1}4N} + \mathbf{q}^2_{k-1}(\omega)) \\
&\leq 2^{(1-\tau)3^{k-1}2N-2}(\frac{3}{4}2^{(1-\frac{\tau_1}{2})3^{k-1}4N} + \frac{1}{4}2^{(1-\tau)3^{k-1}4N}) \\
&= 2^{(1-\tau)3^{k-1}2N-2}(\frac{3}{4}2^{(1-\tau)3^{k-1}4N} + \frac{1}{4}2^{(1-\tau)3^{k-1}4N}) \\
&= 2^{(1-\tau)3^k 2N-2}.
\end{aligned}$$

□

Proposition 2.21 together with Result 2.17 gives the following corollary.

Corollary 2.23. *The refinable function φ corresponding to the mask P_k obtained at the kth iterate of Construction 2 is in $C^{\tau 3^k N}$ with $\tau = 0.05$.*

Proposition 2.10 and Corollary 2.23 indicate that the support of the refinable function φ^2_k corresponding to the mask P_k and its regularity have the same growth order. Hence the regularity of φ^2_k increases linearly with its support.

Corollary 2.24. *Let ϕ be the interpolatory refinable function with the mask $\hat{a}$ of order N defined by (2.1). Let ϕ^{d2}_k, $k = 1, 2, \ldots$ be the dual functions of ϕ constructed at the kth step of Construction 2. The regularity of the function ϕ^{d2}_k increases linearly with its support.*

Theorem 2.16 and Corollary 2.24 show that for a given interpolatory refinable function with mask $\hat{a} = H_N$ defined by (2.1), dual refinable functions with the required regularity can be obtained by using either Construction 1 or Construction 2.

Remark. Construction 1 and 2 also say that interpolatory refinable functions with desired regularity can be constructed iteratively from a simple interpolatory refinable function.

Theorem 2.16 and 2.23 provide an asymptotical regularity analysis for both constructions. However, we use the following sharper estimates to calculate the regularities of the examples given in this paper.

The quantities to be used to measure smoothness are the ones used in [D2], [RiS1], [J], [CGV] and [RS]. Define

$$\kappa_p := \sup\{\kappa : \int_{\mathbb{R}} (1+|\omega|^p)^{\kappa} |\widehat{\varphi}(\omega)|^p \, d\omega < \infty\}. \tag{2.25}$$

We are only concerned with κ_1 and κ_2. When $p=2$, finiteness of the integral in (2.25) defines the function φ to be in the Sobolev space $W_2^{\kappa}(\mathbb{R}^s)$, and the critical exponent is taken as a measure of the L_2 smoothness of φ. Further, φ is at least in $\in C^{\kappa_1-\varepsilon}$. Since $\kappa_1 \geq \kappa_2 - 1/2$, φ is always in $C^{\kappa_2-1/2}$.

The criterion to be used to find the critical exponents is contained in the following statement(see [J], [RiS3]): For an integer k, let

$$V_k := \big\{ v \in \ell_0(\mathbb{Z}) : \sum_{\alpha\in\mathbb{Z}^s} p(\alpha)v(\alpha) = 0, \quad \forall\, p \in \Pi_k \big\},$$

where Π_k denotes the polynomials of degree k. If for the m-refinable function ϕ the mask $\hat{a}$ satisfies

$$\hat{a}(0) = 1; \qquad D^{\beta}\hat{a}(\nu\pi) = 0 \quad \text{for } |\beta| \leq \rho \text{ and } \nu \in \mathbb{Z}_m,$$

then (i) $V_{\rho-1}$ is invariant under the matrix

$$\mathbb{H} := \Big[a(m\alpha - \beta) \Big]_{\alpha\in[-N,N]}$$

where the mask a of ϕ is supported in $[-N,N]$, and (ii) $V_{2\rho-1}$ is invariant under the matrix

$$\mathbb{H}_{\text{au}} := \Big[h(m\alpha - \beta) \Big]_{\alpha\in[-2N,2N]}$$

for the mask of $\phi * \phi(-\cdot)$

$$|\hat{a}|^2/m = \sum_{\alpha\in\mathbb{Z}} h(\alpha)\exp(-i\alpha\omega).$$

Let λ and λ_{au} be the spectral radius of $\mathbb{H}|_{V_{\rho-1}}$ and $\mathbb{H}_{\text{au}}|_{V_{2\rho-1}}$, respectively. The critical indices satisfy

$$\kappa_2(\phi) \geq -\frac{\log(\lambda_{\text{au}})}{2\log(m)},$$

and the equality holds, when ϕ is stable. If the mask $\hat{a}$ is nonnegative,

$$\kappa_1(\phi) \geq -\frac{\log(\lambda)}{\log(m)}.$$

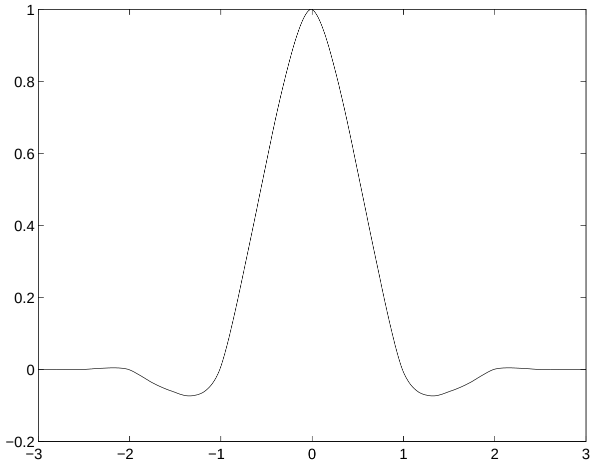

Figure 2.1 The plot of ϕ

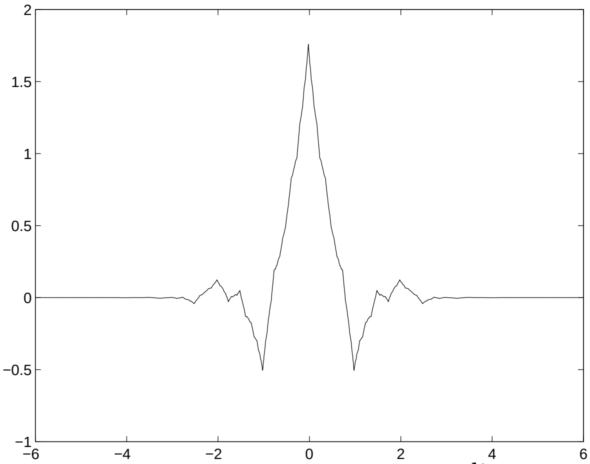

Figure 2.2. The plot of ϕ^{d1}

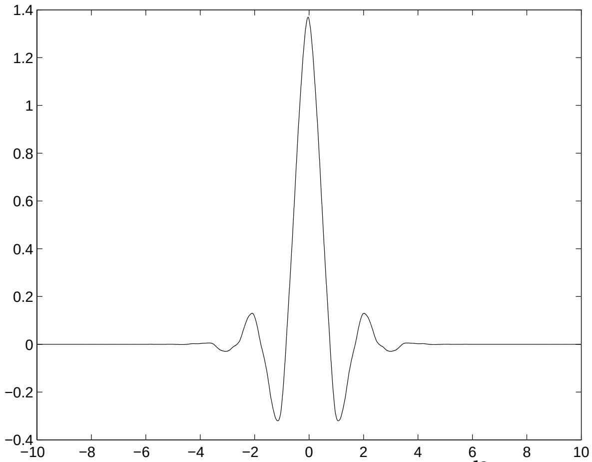

Figure 2.3. The plot of ϕ^{d2}

2.3. Examples

Let $\hat{a}$ be the mask of order 4 defined by (2.1), and let ϕ be the corresponding interpolatory refinable function. Figures 2.1, 2.2, and 2.3 are the function ϕ and its dual functions ϕ_1^{d1} and ϕ_1^{d2} from Construction 1 and Construction 2 respectively. Finally, $\kappa_1(\phi_1^{d1}) = 0.8582$ and $\kappa_1(\phi_1^{d2}) = 2.1704$.

3. Interpolatory refinable function with orthonormal shifts

Let ϕ be an m-refinable function satisfying the equation

$$\phi = \sum_{\alpha \in \mathbb{Z}} b(\alpha)\phi(m \cdot - \alpha), \tag{3.1}$$

where b is a finitely supported sequence. The sequence b is also called the mask of ϕ.

Define

$$B(z) := \sum_{\alpha\in\mathbb{Z}} b(\alpha)z^\alpha, \quad z \in \mathbb{C},$$

and for each ν, $\nu = 1, 2, \ldots m-1$, define Laurent polynomials

$$B_\nu(z) = \sum_{\alpha\in\mathbb{Z}} b(\nu + m\alpha)z^\alpha, \quad z \in \mathbb{C}. \tag{3.2}$$

Denote $B(\exp(-i\omega))$ by $\hat{b}(\omega)$. Each of the functions $B(z)$ and $\hat{b}(\omega)$ will also be called the mask of ϕ. We note that the righthand side of refinement equation (3.1) differs from that of (1.1) by a constant m. In particular, we have the identity $m\hat{a}(\omega) = \hat{b}(\omega)$.

It follows from (1.6) that the orthonormality of the refinable function ϕ implies

$$\sum_{\nu\in\mathbb{Z}_m} |B_\nu(z)|^2 = m, \quad |z| = 1. \tag{3.3}$$

If the refinable function ϕ is interpolatory, then (1.7) gives

$$B_0(z) = 1. \tag{3.4}$$

3.1. Construction

Our idea here is to use masks of some orthonormal $(m-1)$-refinable functions to construct masks satisfying (3.3) and (3.4). The details of the construction are as follows:

Construction 3. *Let*

$$Q(z) := \sum_{\alpha\in\mathbb{Z}} q(\alpha)z^\alpha$$

be the mask of an orthonormal $(m-1)$-refinable function φ, i.e., the function φ satisfies

$$\varphi = \sum_{\alpha\in\mathbb{Z}} q(\alpha)\varphi((m-1)\cdot - \alpha).$$

Define

$$Q_\nu(z) = \sum_{\alpha\in\mathbb{Z}} q(\nu + (m-1)\alpha)z^\alpha, \quad \nu = 0, 1, \ldots, m-2.$$

Do:

(i) define $B_0(z) = 1$ and $B_\nu = Q_{\nu-1}$, $\nu = 1, 2, \ldots, m-1$;

(ii) define $B(z) = \sum_{\nu=0}^{m-1} z^\nu B_\nu(z^2)$;

(iii) define $\hat{b}(\omega) = B(\exp(-i\omega))$ and

$$\widehat{\phi}(\omega) = \Pi_{k=1}^{\infty}(\hat{b}(\omega/m^k)/m).$$

The mask B and the corresponding B_ν satisfy both (3.3) and (3.4). Since B is a Laurent polynomial, the mask b is finitely supported. We assume that the support of b is $\mathbb{Z}\cap[-N, N]$. Offhand, ϕ defined as above may only be a distribution supported on $[-N, N]$ and its mask satisfies (3.3) and (3.4). Hence ϕ is a possible candidate of an interpolatory orthonormal m- refinable function.

To show that the function ϕ is interpolatory, one first needs to show that ϕ is continuous. We use the method briefly described at the end of §2.2 to check the regularities of our examples in the next subsection. All examples in the next subsection are continuous.

After knowing that ϕ is continuous, we use Theorem 2.3 of [LLS2] to check whether the function ϕ is interpolatory. It was proven in [LLS2] that a continuous m-refinable function, whose mask satisfies (3.4), is interpolatory if and only if 1 is the simple eigenvalue of the matrix

$$((1/m)b(mp - q))_{-N\le p,q\le N}. \tag{3.5}$$

Once we know ϕ is interpolatory, we further check whether function ϕ is orthonormal. This can be done immediately by a result of [LLS2] again. Indeed, Proposition 2.1 of [LLS2] states that a refinable function ϕ is orthonormal if and only if (i) ϕ is stable, (ii) the corresponding mask satisfies (3.3). Since the interpolatory function is stable and since the mask of ϕ satisfies (3.3), ϕ is orthonormal. Altogether, we have the following proposition:

Proposition 3.6. *Let ϕ be a continous m-refinable function with the mask $B(z) = \sum_{-N}^{N} b(\alpha)z^\alpha$. Suppose $B(z)$ satisfies the conditions (3.3) and (3.4). Then ϕ is interpolatory and orthonormal if and only if 1 is the simple eigenvalue of the matrix (3.5).*

Finally, the algorithm in [LLS1] can be used to construct the corresponding orthonormal wavelets from a given orthonormal refinable function. However, for our examples in the next subsection, we use a direct method to construct the corresponding wavelets. This method constructs symmetric (or anti-symmetric) wavelets from the symmetric refinable functions.

3.2. Examples

In this section, we give examples of interpolatory orthonormal 3-refinable functions and interpolatory orthonormal symmetric 4-refinable functions with a proper regularity. The corresponding wavelets are also constructed.

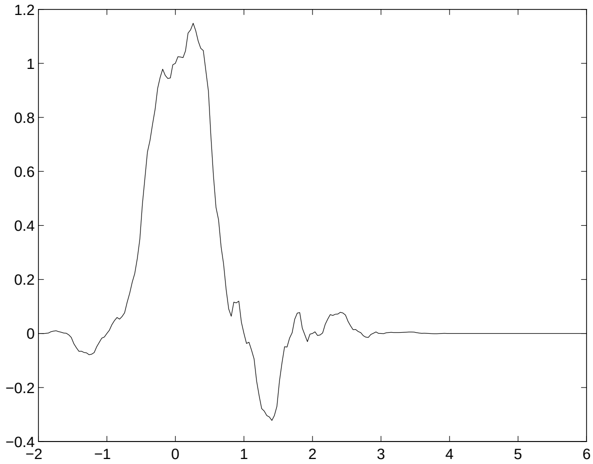

Figure 3.1 The plot of ϕ

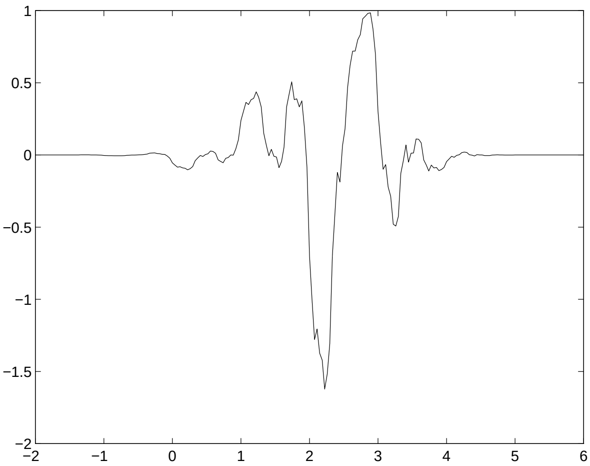

Figure 3.2. The plot of ψ_1

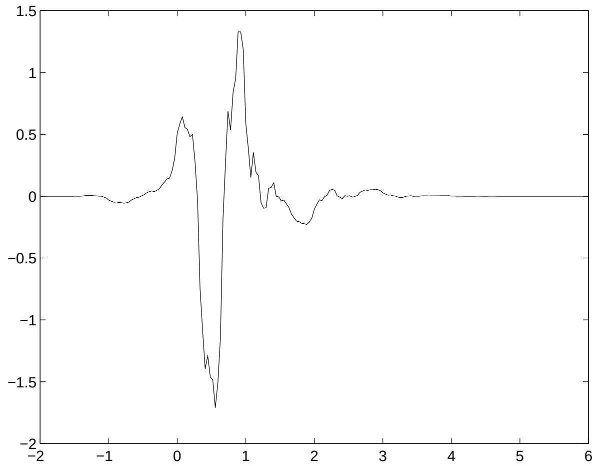

Figure 3.3. The plot of ψ_2

As suggested by Construction 3, we use orthonormal dyadic refineble functions to construct interpolatory orthonormal 3-refinable functions. By using Daubechies 6 points orthonormal refinable functions, we construct an example of interpolatory orthonormal 3-refinable function. Figures 3.1, 3.2 and 3.3 are the interpolatory refinable function ϕ and its wavelets ψ_1 and ψ_2. The corresponding Sobolev exponents is 1.0981.

Next, we give three examples of interpolatory orthonormal symmetric 4-refinable functions. The corresponding masks have the factors $(1-z^4)^N/(1-z)^N$ for $N = 2, 3, 4$.

In each example, we first construct an orthonormal symmetric 3-refinable function, then we use Construction 3 to obtain an interpolatory orthonormal symmetric 4-refinable function. The corresponding Sobolev exponent κ_2 of each example is 0.8904, 1.0057 and 1.3034 respectively.

Table 3.1 lists half of the masks of the interpolatory orthonormal symmetric 4-refinable functions and the corresponding wavelets for $N = 2, 3, 4$. The whole mask of each example can be obtained according to the symmetry or antisymmetry of the mask. We note here that ϕ, ψ_2 and ψ_3 are symmetric. ψ_1 is antisymmetric.

Figures 3.4, 3.5, 3.6, 3.7 give interpolatory orthonormal symmetric 4-refinable function ϕ with $N = 4$ and the corresponding wavelets ψ_1, ψ_2, ψ_3.

Finally, we describe the method used to construct the symmetric (anti-symmetric) wavelets from the symmetric refinable functions constructed. Suppose that the m-refinable function ϕ and its mask B are constructed from an $(m-1)$-refinable function and the corresponding mask Q by Construction 3. The Laurent polynomials B_ν, $\nu = 0, 2, \ldots, m-1$ are defined as (3.2) and the Laurent polynomials Q_ν, $\nu = 0, 1, \ldots, (m-1)$ are defined as in Construction 3. It is known that the construction of wavelets from an m-refinable function is equivalent to extending the polyphase $1 \times m$ row $1/\sqrt{m}(B_\nu(z))$ to a paraunitary matrix (see e.g. [LLS1]). Such extensions become simplier for our examples, because of the fact $B_0(z) = 1$.

The idea here is similar to that of Construction 3. In fact, we will use the paraunitary matrix E^{m-1} of order $m-1$ with the first row $1/\sqrt{(m-1)}Q_\nu$ to construct the paraunitary matrix E^m of order m with the first row $1/\sqrt{m}B_\nu$. Suppose that we have the paraunitary matrix E^{m-1} with first row $1/\sqrt{(m-1)}Q_\nu$ in hands. The matrix E^m is constructed as following:

$$
\begin{aligned}
&E^m(1) = 1/\sqrt{m}(B_\nu);\\
&E^m(2) = m^{-1/2}(-\sqrt{m-1}, E^{m-1}(1));\\
&E^m(j) = (0, E^{m-1}(j-1)); \quad j = 3, \cdots, m,
\end{aligned}
\tag{3.7}
$$

where $E^m(j)$ and $E^{m-1}(j)$ are the j-th rows of E^m and E respectively. One can verify that E^m is the paraunitary matrix. Therefore we can obtain the wavelets easily. Further, if $E^{(m-1)}$ leads symmetric or antisymmetric wavelets, so does E^m in all our examples.

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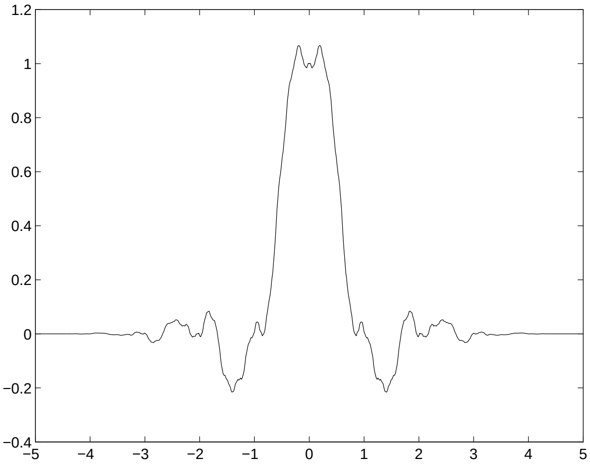

Figure 3.4. The plot of ϕ

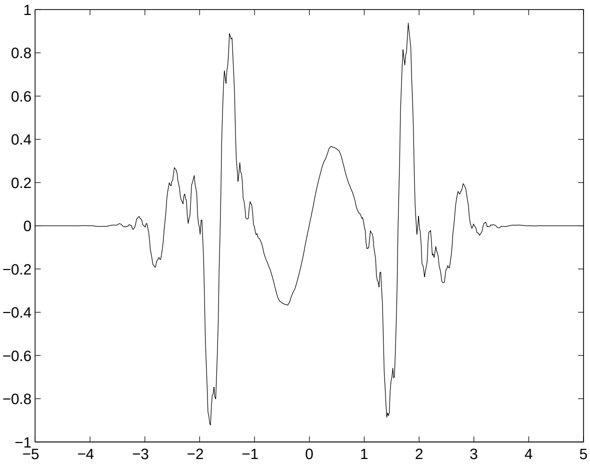

Figure 3.5. The plot of ψ_1

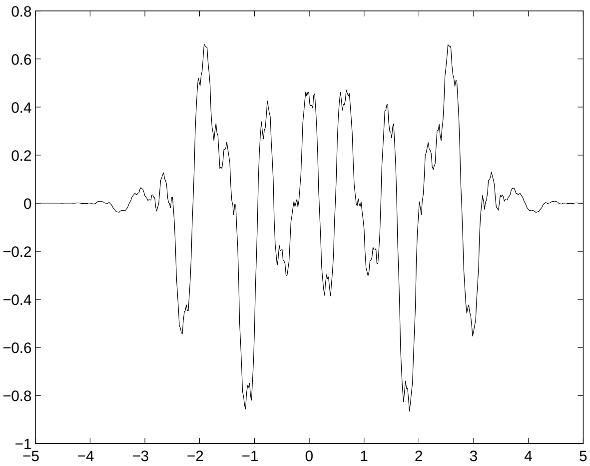

Figure 3.6. The plot of ψ_2

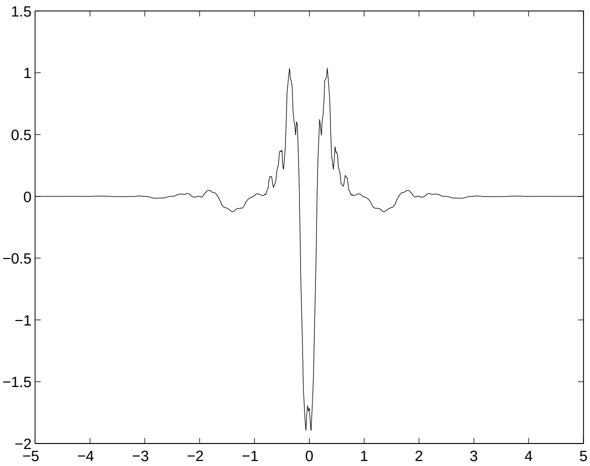

Figure 3.7. The plot of ψ_3

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Table 3.1. The centered masks of 4-refinable functions and wavelets

N	j	ϕ	ψ_1	ψ_2	ψ_3
2	-7	.00136994542	-.001369945366		.0007909383568
	-6	.02219632712	-.02219632706		.01281505543
	-5	-.1798162725	.1798162720	.006575504759	-.1038169733
	-4	.0000000000	.0000000000	.1065385948	.0000000000
	-3	.06744383661	-.08322448894	-.08630875230	.03893871721
	-2	.4778036728	-.7334872395	.0000000000	.2758600790
	-1	1.111002491	.9603341642	.1494052376	.6414375870
	0	1.000000000	.0000000000	1.201136375	-1.732050808
3	-11	-.02503989158	-.02503602134		-.01445678814
	-10	.03407840260	.03407309542		.01967517491
	-9	.02318969100	.02318610889	.1706725185	.01338857434
	-8	.0000000000	.0000000000	-.2322789604	.0000000000
	-7	.07060913900	-.5111290409	-.1580615204	.04076620540
	-6	-.1647352120	.6269956681	.0000000000	-.09510991896
	-5	-.1588085600	.3799561257	.8656580561	-.09168816484
	-4	.0000000000	.0000000000	-.7102589035	.0000000000
	-3	.07147210000	-.7118398990	-.1649414440	.04126443616
	-2	.6306568090	.1019633067	.0000000000	.3641098783
	-1	1.018577500	-.4264023249	-.006215638203	.5880759936
	0	1.000000000	.0000000000	.4708673599	-1.732050808
4	-15	.002583256408	-.002583571458		.001491443782
	-14	-.003409971160	.003410386733		-.001968747767
	-13	-.002250628523	.002250903031	-.03068840418	-.001299400983
	-12	.0000000000	.0000000000	.04050955362	.0000000000
	-11	-.02576255772	-.15650224014	.02673687302	-.01487401963
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	-8	.0000000000	.0000000000	.5238605493	.0000000000
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	-6	-.1615255652	.7072100429	.0000000000	-.09325682852
	-5	-.1681283527	.2572091065	-.7666165500	-.09706894966
	-4	.0000000000	.0000000000	.2991928958	.0000000000
	-3	.08376910294	-.1768222273	-.1950449473	.04836411412
	-2	.6251070818	-.3557274930	.0000000000	.3609057418
	-1	1.020377721	-.2831027482	.4076364047	.5891153517
	0	1.000000000	.0000000000	-.3129135448	-1.732050808