HYPERBOLIC CONE-SURFACES, GENERALIZED MARKOFF MAPS, SCHOTTKY GROUPS AND McSHANE'S IDENTITY

YING ZHANG

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To the memory of my father

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Summary

In this thesis we study hyperbolic cone-surfaces, generalized Markoff maps and classical Schottky groups to obtain generalizations and variations of McShane's identity and hence generalize the work of McShane and Bowditch.

We study hyperbolic cone-surfaces with cusps and/or geodesic boundary and obtain a generalized McShane's identity for such hyperbolic cone-surfaces with all cone angles less than or equal to π. As applications we derive some related identities. We reformulate the generalized identity as a unified identity in terms of complex lengths of the geodesic boundary components and cone points.

We also study generalized Markoff maps and extend the generalized identity for one-hole hyperbolic tori to an identity for general representations of the once-punctured torus group in $\mathrm{PSL}(2, \mathbf{C})$ satisfying certain conditions. Applying the techniques to representations stabilized by a hyperbolic element in the mapping class group of the punctured torus we derive a formula for the complex length of a longitude in the torus boundary of a once-punctured torus bundle M over the circle with an incomplete hyperbolic structure. This applies to the case of a closed hyperbolic 3-manifold which is obtained by performing hyperbolic Dehn surgery on such a bundle M.

Finally, we extend the generalized McShane's identity obtained for compact hyperbolic surfaces with geodesic boundary to an identity for marked classical Schottky groups by analytic continuation along paths in the marked classical Schottky space. This gives some new identities for fuchsian Schottky groups.

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Chapter 1

Introduction

The following conventions are assumed throughout this thesis.

- All surfaces considered are connected and orientable.
- When used, $|\gamma|$ always denotes the hyperbolic length of γ if γ is a simple closed geodesic or a simple geodesic arc on a hyperbolic (cone-)surface.
- For $u \in \mathbf{C}\backslash\{0\}$, we always assume
  - (i) $\sqrt{u}$ is the square root of u which has positive real part if $u \notin \mathbf{R}_{<0}$ and positive imaginary part if $u \in \mathbf{R}_{<0}$, whereas
  - (ii) $u^{1/2}$ is any once for all choice of one of the two square roots of u.
- For $z \in \mathbf{C}$, we always choose values for $\log z \,(z \neq 0)$, $\cosh^{-1}(z)$, $\sinh^{-1}(z)$ and $\tanh^{-1}(z)$ so that:
  - (i) $\Im \log z \in (-\pi, \pi]$;
  - (ii) $\Re \cosh^{-1}(z) \geq 0$ and $\Im \cosh^{-1}(z) \in (-\pi, \pi]$;
  - (iii) $\Im \sinh^{-1}(z)$, $\Im \tanh^{-1}(z) \in (-\pi/2, \pi/2]$.

1.1 The original McShane's identities

Greg McShane discovered in his Ph.D. thesis [29] the following striking identity concerning the lengths of all simple closed geodesics on a hyperbolic torus—in this thesis by a *hyperbolic torus* we mean a once-punctured torus equipped with a complete hyperbolic structure of finite area.

Theorem 1.1 (McShane [29]) *In a hyperbolic torus T,*

$$\sum_{\gamma} \frac{1}{1+e^{|\gamma|}} = \frac{1}{2}, \tag{1.1}$$

where the sum extends over all simple closed geodesics on T and where $|\gamma|$ denotes the length of γ in the given hyperbolic structure.

Later in [30] McShane extended his identity to more general hyperbolic surfaces with cusps as follows.

Theorem 1.2 (McShane [30]) *In a finite area complete hyperbolic surface M with cusps and with no boundary,*

$$\sum \frac{1}{1+e^{(|\alpha|+|\beta|)/2}} = \frac{1}{2}, \tag{1.2}$$

where the sum is taken over all unordered pairs of simple closed geodesics α, β (where α or β might be a cusp treated as a simple closed geodesic of length 0) on M such that α, β bound with a distinguished cusp point an embedded pair of pants on M.

Remark 1.3 Theorem 1.1 can be regarded as a special case of Theorem 1.2 where α, β are the same for each pair α, β. Note that the identities (1.1) and (1.2) hold for arbitrary finite area complete hyperbolic structures on the respective surfaces.

In [31] McShane demonstrated three other closely related identities for the lengths of simple closed geodesics in each of the three Weierstrass classes on a hyperbolic torus. Recall that a hyperbolic torus T has a unique elliptic involution η which maps each simple closed geodesic on T onto itself with orientation reversed. The three *Weierstrass points* are the fixed points of η. Each simple closed geodesic on T passes through exactly two of the three Weierstrass points. The simple closed geodesics on T which lie in the *Weierstrass class* dual to a Weierstrass point x are precisely all the simple closed geodesics on T which miss the Weierstrass point x .

Theorem 1.4 (McShane [31]) *In a hyperbolic torus T,*

$$\sum_{\gamma \in \mathcal{A}} \sin^{-1}\left(\frac{1}{\cosh \frac{1}{2}|\gamma|}\right) = \frac{\pi}{2}, \tag{1.3}$$

where the sum is taken over all simple closed geodesics γ in a given Weierstrass class $\mathcal{A}$ on T .

1.2 Other extensions and generalizations

Bowditch's work. On the other hand, Brian Bowditch gave an alternative proof of Theorem 1.1 using Markoff maps [6] and extended the identity (1.1) in Theorem 1.1 to an identity for certain type-preserving representations, including quasifuchsian representations, of the once-punctured torus group into $\mathrm{PSL}(2, \mathbf{C})$ [8]. He also derived variations of the identity for once-punctured torus bundles M with complete hyperbolic structures of finite volume [7]. More precisely, he obtained a formula which expresses the modulus of the cusp of M as a sum similar to that in (1.1) except that his sum is taken over part of the closed geodesics in M which are freely homotopic to simple closed curves on a once-punctured torus fiber and the lengths of these closed geodesics are their complex lengths in M.

Akiyoshi-Miyachi-Sakuma's work. There are also some other generalizations along these directions by Hirotaka Akiyoshi, Hideki Miyachi and Makoto

Sakuma, see [2], [1] and [39]. In [1] they gave a formula which expresses the "width" of the limit set of a geometrically finite once-punctured torus group in terms of the complex lengths of the closed geodesics which correspond to essential simple closed curves on the punctured torus. In [2] they generalized Bowditch's "modulus" identity for once-punctured torus bundles to a "modulus" identity, with summands as in (1.2), for punctured-surface bundles with complete hyperbolic structures of finite volume . They also generalized their "width" formula for quasifuchsian once-punctured torus groups to a "width" formula for quasifuchsian punctured surface groups.

Mirzakhani's work. Maryam Mirzakhani in [34] has generalized McShane's identity to an identity for compact hyperbolic surfaces with geodesic boundary and/or cusps. She then found beautiful applications for the generalized identity by obtaining a recursive formula for the Weil-Petersson volumes of moduli spaces of such Riemann surfaces.

1.3 Outline of main results

In this thesis we give further generalizations of McShane's identity and Bowditch's variations by studying hyperbolic cone-surfaces, generalized Markoff maps and classical Schottky groups.

In Chapter 3 we further generalize McShane's identity (1.2) to an identity for compact hyperbolic cone-surfaces with possibly cusps and geodesic boundary. The main result is Theorem 3.5 or reformulated as Theorem 3.17. As corollaries we derive McShane's three Weierstrass identities for hyperbolic tori and the generalizations of them to identities for hyperbolic one cone/one-hole tori. Applying the main theorem to some quotient orbifolds, we derive an identity for genus two closed hyperbolic surfaces (see Theorem 3.12) which is also obtained by McShane

[33] using a somewhat different method.

In Chapter 4 we extend the generalized McShane's identity obtained in Chapter 3 for hyperbolic one-hole/one-cone tori to an identity for general representations in $\mathrm{PSL}(2, \mathbf{C})$ of the once-punctured torus group which satisfy certain conditions set by Bowditch (we call them BQ-conditions). This is done via studying generalized Markoff maps and generalizes Bowditch's work in [8]. See Theorem 4.3 for the statement of the result in terms of representations and Theorem 4.10 in terms of generalized Markoff maps.

In Chapter 5 we generalize Bowditch's McShane-type formula in [7] for the modulus of the cusp of a complete hyperbolic once-punctured torus bundle M to a formula for the complex length of a longitude of ∂M when M is given an incomplete hyperbolic structure. The main results are Theorem 5.3, 5.4 and Corollary 5.5.

In Chapter 6 we extend the complexified generalized McShane's identity (3.19) obtained in Chapter 3 to an identity for the marked classical Schottky groups. This is achieved by analytic continuation along paths in the marked classical Schottky space.

As preparations, a brief review of Fenchel's theory on oriented lines in the hyperbolic 3-space and some calculations that are needed in Chapters 3, 4 and 5 are presented in Chapter 2.

Chapter 2

Calculations in Hyperbolic Geometry

2.1 Fenchel's theory of oriented lines

In [18] Werner Fenchel used the notion of the complex length, or width, from one oriented line to another along an oriented common normal of them to derive the general sine and cosine rules for oriented right-angled hexagons in $\mathbb{H}^3$. The aim of this section is to give a brief review of part of Fenchel's theory that will be needed in this thesis. We exclude the degenerate cases for clarity. Proofs of the results (except Lemmas 2.15 and 2.17) in this section can be found in [18].

We work in the upper-half-space model of the hyperbolic 3-space $\mathbb{H}^3$. Hence the extended complex plane $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ is regarded as the ideal boundary of $\mathbb{H}^3$. The points in $\mathbf{C}_\infty$ are the **ideal points** of $\mathbb{H}^3$. By a **line** we mean a complete geodesic in $\mathbb{H}^3$. A line has two ideal points as its endpoints. By specifying for a line its starting and ending ideal endpoints one gets an **oriented line** in $\mathbb{H}^3$. An oriented line with starting ideal point u and ending ideal point u' is denoted $[u, u']$.

The fractional linear transformations on $\mathbf{C}_\infty$ are exactly the **motions**, i.e. the orientation-preserving isometries of $\mathbb{H}^3$ (as shown in [18] by identifying $\mathbb{H}^3$ with the set of quaternions with positive $\mathbf{k}$ components). Thus the group of motions is identified with $\mathrm{PSL}(2,\mathbf{C})$, and each motion, f, is determined by exactly two matrices, $\pm\mathbf{f}$, in $\mathrm{SL}(2,\mathbf{C})$.

Definition 2.1 For an ordered quadruple (z_1, z_2, z_3, z_4) of points of $\mathbf{C}_\infty$, no three of which coincide, we define its **cross-ratio** $R\!\!\!/(z_1, z_2; z_3, z_4)$ by

$$R\!\!\!/(z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} \Big/ \frac{z_4 - z_1}{z_4 - z_2} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \in \mathbf{C}_\infty. \tag{2.1}$$

This is the choice used by Fenchel [18]. As is well-known, the cross-ratio is invariant under fractional linear transformations, that is, for every fractional linear transformation f one has

$$R\!\!\!/\big(f(z_1), f(z_2); f(z_3), f(z_4)\big) = R\!\!\!/(z_1, z_2; z_3, z_4).$$

Thus if $w, w' \in \mathbf{C}_\infty$ are respectively the repulsive and attractive fixed ideal points of a fractional linear transformation f, then for every $x, y \in \mathbf{C}_\infty$, one has

$$R\!\!\!/(w', w; x, y) = R\!\!\!/\big(w', w; f(x), f(y)\big).$$

Multiplying by $R\!\!\!/(w', w; y, f(x))$ on both sides, one gets

$$R\!\!\!/(w', w; x, f(x)) = R\!\!\!/\big(w', w; y, f(y)\big);$$

that is, $R\!\!\!/\big(w', w; x, f(x)\big)$ is independent of x.

Definition 2.2 The number

$$m(f) = R\!\!\!/\big(w', w; x, f(x)\big) \in \mathbf{C}\backslash\{0\}$$

is called the **multiplier** of f.

Note that the multiplier $m(f^{-1})$ of f^{-1} is equal to $m(f)^{-1}$.

Definition 2.3 With a motion f we associate the **displacement** or **complex translation length** of f,

$$l(f) = \log m(f) \in \mathbf{C}/2\pi i\mathbf{Z}.$$

The real part of $l(f)$ is the distance through which the axis is translated and its imaginary part the angle, measured in radian, through which the half-planes bounded by the axis of f are rotated. If $\mathbf{f} \in \mathrm{SL}(2, \mathbf{C})$ determines f, one has

$$2\cosh l(f) = m(f) + m(f)^{-1} = \mathrm{tr}(\mathbf{f}^2). \tag{2.2}$$

Definition 2.4 The rotation through the angle π about a line is called the **half-turn** about the line. Thus a motion f is a half-turn if and only if $l(f) = \pi i$. Equivalently, $\mathbf{f} \in \mathrm{SL}(2, \mathbf{C})$ determines a half-turn if and only if $\mathrm{tr}(\mathbf{f}) = 0$.

Definition 2.5 A **line matrix** is a 2×2 non-singular complex matrix $\mathbf{l}$ such that $\mathrm{tr}(\mathbf{l}) = 0$. It is called **normalized** if $\det(\mathbf{l}) = 1$.

Hence a normalized line matrix $\mathbf{l} \in \mathrm{SL}(2, \mathbf{C})$ determines a line in the upper half-space model of the hyperbolic 3-space, the fixed axis of the half turn it represents.

Note that there are exactly two normalized line matrices, $\pm\mathbf{l}$, which determine a given unoriented line L. Thus Fenchel [18] used normalized line matrices to consistently represent oriented lines in the upper half-space model of the hyperbolic 3-space.

Definition 2.6 The normalized line matrix $\mathbf{l}$ which represents the oriented line $[u, u']$, where $u, u' \in \mathbf{C}_\infty$, is chosen as

$$\mathbf{l} = \frac{i}{u' - u}\begin{pmatrix} u' + u & -2u'u \\ 2 & -u' - u \end{pmatrix}.$$

This convention is consistent in the sense that it is preserved under motions: if $\mathbf{l}$ represents $[u, u']$ and f is a motion determined by $\mathbf{f} \in \mathrm{SL}(2, \mathbf{C})$, then $\mathbf{f}\,\mathbf{l}\,\mathbf{f}^{-1}$ represents $[f(u), f(u')]$.

We say two lines are normal to each other, or, one is a normal of the other, if they intersect orthogonally. Note that two oriented lines $[u, u']$ and $[v, v']$ are normal to each other if and only if $\not R(u, u'; v, v') = -1$.

Definition 2.7 An ordered triple $(L, M; N)$ is called a **double cross** if L, M, N are oriented lines in $\mathbb{H}^3$ such that N is a common normal of L and M.

Definition 2.8 The **width**

$$\sigma = \sigma(L, M; N) \in \mathbf{C}/2\pi i\mathbf{Z}$$

of a double cross $(L, M; N)$, where $L = [u, u'], M = [v, v']$ and $N = [w, w']$, is defined by

$$\exp(\sigma) = \not R(w', w; u, v) = \not R(w', w; u', v'), \tag{2.3}$$

We also call $\sigma(L, M; N)$ the **complex length** from L to M along N.

Let $\mathbf{l}, \mathbf{m}, \mathbf{n}$ be the normalized line matrices representing L, M, N respectively. Then it can be easily checked that

$$\cosh \sigma = \frac{1}{2}\mathrm{tr}(-\mathbf{ml}), \quad \sinh \sigma = \frac{1}{2i}\mathrm{tr}(\mathbf{mnl}). \tag{2.4}$$

Remark 2.9 The following useful facts about line matrices can be easily proved.

(i) A line L with line matrix $\mathbf{l}$ is a normal of the axis of a motion f with matrix $\mathbf{f}$ if and only if

$$\mathrm{tr}(\mathbf{f}\,\mathbf{l}) = 0.$$

In particular, two lines with matrices $\mathbf{l}$ and $\mathbf{m}$ are normal to each other if and only if $\mathrm{tr}(\mathbf{m}\,\mathbf{l}) = 0$. Actually, in this case one has $\mathbf{m}\,\mathbf{l} = -\mathbf{l}\,\mathbf{m}$.

(ii) Let f and g be motions with matrices $\mathbf{f}, \mathbf{g} \in \mathrm{SL}(2, \mathbf{C})$ and with disjoint axes. Then $\mathbf{fg} - \mathbf{gf}$ is a line matrix determining the common normal of the axes of f and g.

Now we turn to Fenchel's cosine and sine rules of right-angled hexagons in hyperbolic 3-space.

Definition 2.10 An **oriented right-angled hexagon** $\mathcal{H} = (S_n, n \bmod 6)$ is a cyclically ordered 6-tuplet of oriented lines S_n, $n \bmod 6$, such that for every $n \bmod 6$ the lines S_n and S_{n+1} are normal to each other. For non-degeneracy we require that for every $n \bmod 6$ the lines S_{n-1} and S_{n+1} do not coincide. The oriented lines S_n, $n \bmod 6$, are called the **side-lines** of $\mathcal{H}$.

Suppose $\mathbf{s}_n$, $n \bmod 6$ are the normalized line matrices corresponding to the side-lines S_n of $\mathcal{H}$. Then they satisfy the following conditions for $n \bmod 6$:

$$\mathrm{tr}(\mathbf{s}_{n+1}\mathbf{s}_n) = 0, \quad \mathbf{s}_{n-1} \text{ and } \mathbf{s}_{n+1} \text{ are linearly independent.} \tag{2.5}$$

Conversely, six cyclically ordered normalized line matrices $(\mathbf{s}_n, n \bmod 6)$, satisfying conditions (2.5) determine the side lines of an oriented right-angled hexagon.

Definition 2.11 For every $n \bmod 6$ the three successive side-lines S_{n-1}, S_n, S_{n+1} of an oriented right-angled hexagon $\mathcal{H}$ form a double cross $(S_{n-1}, S_{n+1}; S_n)$. Its width $\sigma_n \in \mathbf{C}/2\pi i\mathbf{Z}$ is called the n-th **side length** of $\mathcal{H}$.

With the above notation, one has (observing that $\mathbf{s}_{n+1}\mathbf{s}_n = -\mathbf{s}_n\mathbf{s}_{n+1}$)

$$\cosh\sigma_n = \frac{1}{2}\mathrm{tr}(-\mathbf{s}_{n+1}\mathbf{s}_{n-1}), \quad \sinh\sigma_n = \frac{1}{2i}\mathrm{tr}(\mathbf{s}_{n+1}\mathbf{s}_n\mathbf{s}_{n-1}). \tag{2.6}$$

In Chapter VI, [18] Fenchel derived the following useful sine and cosine rules for an oriented right-angled hexagon in hyperbolic 3-space.

Proposition 2.12 (Fenchel [18]) *The side-lengths σ_n of an oriented right-angled hexagon $(S_n, n \bmod 6)$ satisfy*

$$\frac{\sinh\sigma_1}{\sinh\sigma_4} = \frac{\sinh\sigma_3}{\sinh\sigma_6} = \frac{\sinh\sigma_5}{\sinh\sigma_2} \tag{2.7}$$

and

$$\cosh\sigma_{n+3} = \cosh\sigma_{n-1}\cosh\sigma_{n+1} + \sinh\sigma_{n-1}\sinh\sigma_{n+1}\cosh\sigma_n. \tag{2.8}$$

for all $n \bmod 6$. □

Notation: $\Delta_{\mathbf{n}}(\mathbf{l},\mathbf{m})$. For oriented lines $\mathbf{l},\mathbf{m},\mathbf{n}$ in $\mathbb{H}^3$ such that $\mathbf{n}$ is an oriented common normal to $\mathbf{l}$ and $\mathbf{m}$, we shall use $\Delta_{\mathbf{n}}(\mathbf{l},\mathbf{m})$ to denote the complex distance from $\mathbf{l}$ to $\mathbf{m}$ along $\mathbf{n}$, that is, the width $\sigma(\mathbf{l},\mathbf{m};\mathbf{n})$ of the double cross $(\mathbf{l},\mathbf{m};\mathbf{n})$, as defined in Definition 2.8.

Definition 2.13 For each non-parabolic element $A \in \mathrm{SL}(2,\mathbf{C})\backslash\{\pm I\}$, its **naturally oriented axis** $\mathbf{a}(A)$ is defined as follows:

(i) when A represents a loxodromic (including hyperbolic) isometry of $\mathbb{H}^3$, the orientation of $\mathbf{a}(A)$ is directed from its repulsive fixed ideal point to its attractive fixed ideal point;

(ii) when A represents an elliptic isometry of $\mathbb{H}^3$ but not an involution, the orientation of $\mathbf{a}(A)$ is defined so that A has rotation angle in $(0,\pi)$ with respect to $\mathbf{a}(A)$; and

(iii) when A is an involution, i.e. $A^2=-I$, then the orientation of $\mathbf{a}(A)$ is the same as the oriented line that A represents.

Remark 2.14 Note that $\mathbf{a}(A^{-1})$ always has the opposite orientation as $\mathbf{a}(A)$. If A is not an involution then $\mathbf{a}(-A)=\mathbf{a}(A)$, whereas if A is an involution then $\mathbf{a}(-A)=\mathbf{a}(A^{-1})$ since $A^{-1}=-A$.

We have observed that two oriented lines in $\mathbb{H}^3$ with line matrices $\mathbf{l}$ and $\mathbf{m}$ are normal to each other if and only if $\mathrm{tr}\,(\mathbf{lm})=0$. This can be extended as follows.

Lemma 2.15 *Given a non-parabolic $K \in \mathrm{SL}(2,\mathbf{C})$ and an oriented line in $\mathbb{H}^3$ with line matrix L, we have that $\mathbf{a}(K)\perp\mathbf{a}(L)$ if and only if $\mathrm{tr}(KL)=0$.*

Proof. It is easy to see that $K-K^{-1}$ is a line matrix for $\mathbf{a}(K)$. Noticing that $L^2=-I$, we have $\mathbf{a}(K)\perp\mathbf{a}(L)$ if and only if $\mathrm{tr}[(K-K^{-1})L]=0$, if and only if $\mathrm{tr}(KL)=\mathrm{tr}(K^{-1}L)=\mathrm{tr}[(K^{-1}L)^{-1}]=\mathrm{tr}(-LK)=-\mathrm{tr}(KL)$, that is, $\mathrm{tr}(KL)=0$. □

Remark 2.16 The conclusion of Lemma 2.15 is not true for general $K, L \in \mathrm{SL}(2,\mathbf{C})$. Actually, given non-parabolic elements $K, L \in \mathrm{SL}(2,\mathbf{C})$ such that $\mathbf{a}(K) \perp \mathbf{a}(L)$, we have $\mathrm{tr}(KL) = 0$ if and only if at least one of K and L is a line matrix. This can be proved easily by direct calculations after a suitable normalization.

Finally, we prove a useful property which relates the complex translation length $l(K)$ of an element K of $\mathrm{SL}(2,\mathbf{C})$ to the action on $\mathrm{SL}(2,\mathbf{C})$ by conjugation by K. This will be used in §2.7, in the proof of Lemma 2.40.

Lemma 2.17 *Given a non-parabolic element $K \in \mathrm{SL}(2,\mathbf{C})$ and a line matrix $L \in \mathrm{SL}(2,\mathbf{C})$ such that $\mathbf{a}(K) \perp \mathbf{a}(L)$, we have KLK^{-1} is a line matrix such that $\mathbf{a}(K) \perp \mathbf{a}(KLK^{-1})$, and the complex translation length of K is given by*

$$l(K) = \Delta_{\mathbf{a}(K)}\big(\mathbf{a}(L), \mathbf{a}(KLK^{-1})\big).$$

Proof. First, KLK^{-1} is a line matrix since $\mathrm{tr}(KLK^{-1}) = \mathrm{tr}(L) = 0$. By Lemma 2.15, we have $\mathbf{a}(K) \perp \mathbf{a}(KLK^{-1})$ since $\mathrm{tr}[(KLK^{-1})K] = \mathrm{tr}(KL) = 0$.

Now suppose the motion of $\mathbb{H}^3$ represented by K maps the oriented line $\mathbf{a}(L)$ onto the oriented line $\mathbf{a}(M)$ with line matrix $M \in \mathrm{SL}(2,\mathbf{C})$. Then by definition, the complex translation length of K is given by

$$l(K) = \Delta_{\mathbf{a}(K)}\big(\mathbf{a}(L), \mathbf{a}(M)\big).$$

Hence it is suffices to show that $M = KLK^{-1}$. By Fenchel [18], V.3, page 68, we have $-K^2 = ML$. Hence we only need to show that $-K^2 = KLK^{-1}L$, or equivalently, $KLKL = -I$. This is equivalent to that KL is a line matrix, or $\mathrm{tr}(KL) = 0$, which is true by Lemma 2.15. □

2.2 The functions G and S

The two functions G and S defined below will be used in the next chapter.

Definition 2.18 We define functions $G, S : \mathbf{C}^3 \to \mathbf{C}$ as follows:

$$G(x,y,z) = 2\tanh^{-1}\left(\frac{\sinh(x)}{\cosh(x)+\exp(y+z)}\right), \tag{2.9}$$

$$S(x,y,z) = \tanh^{-1}\left(\frac{\sinh(x)\sinh(y)}{\cosh(z)+\cosh(x)\cosh(y)}\right). \tag{2.10}$$

Here for a complex number x, $\tanh^{-1}(x)$ is defined to have imaginary part in $(-\pi/2, \pi/2]$.

Remark 2.19 Although the two functions are not defined on a subset of $\mathbf{C}^3$, this will not cause any problems as in this thesis we will only consider values of x, y and z for which they are defined. Using the identity

$$x = \frac{1}{2}\log\frac{1+\tanh(x)}{1-\tanh(x)} \mod \pi i,$$

it is easy to check that the two functions also have the following expressions:

$$G(x,y,z) = \log\left(\frac{\exp(x)+\exp(y+z)}{\exp(-x)+\exp(y+z)}\right), \tag{2.11}$$

$$S(x,y,z) = \frac{1}{2}\log\left(\frac{\cosh(z)+\cosh(x+y)}{\cosh(z)+\cosh(x-y)}\right), \tag{2.12}$$

as used by Mirzakhani in [34]. (She uses different notations $\mathcal{D}, \mathcal{R}$ as explained below.) Here for a non-zero complex number x, $\log(x)$ is assumed to have imaginary part in $(-\pi, \pi]$. We shall see that both expressions of the functions are useful.

Geometric meanings of G and S. For $x, y, z > 0$, the geometrical meanings of $G(x,y,z)$ and $S(x,y,z)$ are as follows. Let $\mathcal{P}(2x, 2y, 2z)$ be the unique hyperbolic pair of pants whose boundary components X, Y, Z are simple closed geodesics of lengths $2x, 2y, 2z$ respectively. Then $S(x,y,z)$ is half the length of the orthogonal projection of the boundary geodesic Y onto X in $\mathcal{P}(2x, 2y, 2z)$ and $S(x,z,y)$ is half the length of the orthogonal projection of the boundary geodesic Z onto X

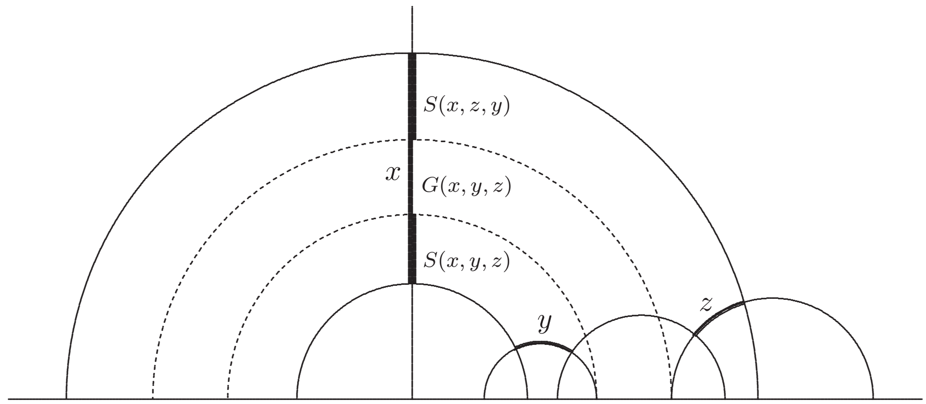

Figure 2.1: The functions G and S

in $\mathcal{P}(2x, 2y, 2z)$, and $G(x, y, z)$ is the length of each of the two gaps between these two projections on X. See Figure 2.1. We have therefore the identity

$$G(x, y, z) + S(x, y, z) + S(x, z, y) = x \tag{2.13}$$

for all $x, y, z \geq 0$. Note that the same identity holds modulo πi for all $x, y, z \in \mathbf{C}$.

Remark 2.20 The relations between our functions G, S and Mirzakhani's functions $\mathcal{D}, \mathcal{R}$ are

$$G(x, y, z) = \mathcal{D}(2x, 2y, 2z)/2, \tag{2.14}$$

$$S(x, y, z) = x - \mathcal{R}(2x, 2z, 2y)/2. \tag{2.15}$$

Evaluating $G + S$. The identities in the following lemma will be used in §3.8 for proving the complexified reformulation of the generalized McShane's identity.

Lemma 2.21 (i) *For $x, z \geq 0$ and $y \in [0, \pi/2]$,*

$$G(x, yi, z) + S(x, yi, z) = x - \tanh^{-1}\left(\frac{\sinh(x)\sinh(z)}{\cos(y) + \cosh(x)\cosh(z)}\right). \tag{2.16}$$

(ii) *For $x, y \in [0, \pi/2]$ and $z \geq 0$,*

$$G(xi, yi, z) + S(xi, yi, z) = \left[x - \tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y) + \cos(x)\cosh(z)}\right)\right] i. \tag{2.17}$$

Remark 2.22 The identities (2.16) and (2.17) are extensions of (2.13). The proof given below is just a justification of this via careful calculations.

Proof. (i) The identity (2.16) follows from the following two identities since $\Re S(x, yi, z) = 0$:

$$\Re G(x, yi, z) = x - \tanh^{-1}\left(\frac{\sinh(x)\sinh(z)}{\cos(y) + \cosh(x)\cosh(z)}\right), \tag{2.18}$$

$$\Im G(x, yi, z) + \Im S(x, yi, z) = 0. \tag{2.19}$$

Proof of (2.18) *and* (2.19): By definition,

$$\begin{aligned} G(x, yi, z) &= \log \frac{\exp(x) + \exp(yi + z)}{\exp(-x) + \exp(yi + z)} \\ &= \log \frac{[\exp(x) + \cos(y)\exp(z)] + i[\sin(y)\exp(z)]}{[\exp(-x) + \cos(y)\exp(z)] + i[\sin(y)\exp(z)]}. \end{aligned}$$

Hence

$$\begin{aligned} \Re G(x, yi, z) &= \frac{1}{2}\log \frac{[\exp(x) + \cos(y)\exp(z)]^2 + [\sin(y)\exp(z)]^2}{[\exp(-x) + \cos(y)\exp(z)]^2 + [\sin(y)\exp(z)]^2} \\ &= \frac{1}{2}\log \frac{\exp(2x) + \exp(2z) + 2\exp(x)\cos(y)\exp(z)}{\exp(-2x) + \exp(2z) + 2\exp(-x)\cos(y)\exp(z)} \\ &= \frac{1}{2}\log\left(\frac{\cosh(x-z) + \cos(y)}{\cosh(x+z) + \cos(y)}\,\frac{\exp(x+z)}{\exp(-x+z)}\right) \\ &= x - \frac{1}{2}\log \frac{\cosh(x+z) + \cos(y)}{\cosh(x-z) + \cos(y)} \\ &= x - \tanh^{-1}\left(\frac{\sinh(x)\sinh(z)}{\cos(y) + \cosh(x)\cosh(z)}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\Im G(x, yi, z) \\ = &\tan^{-1}\left(\frac{\sin(y)\exp(z)}{\exp(x) + \cos(y)\exp(z)}\right) - \tan^{-1}\left(\frac{\sin(y)\exp(z)}{\exp(-x) + \cos(y)\exp(z)}\right) \\ = &\tan^{-1}\left(\frac{[\exp(-x) - \exp(x)]\sin(y)\exp(z)}{[\exp(x) + \cos(y)\exp(z)][\exp(-x) + \cos(y)\exp(z)] + [\sin(y)\exp(z)]^2}\right) \\ = &\tan^{-1}\left(\frac{[\exp(-x) - \exp(x)]\sin(y)\exp(z)}{1 + \exp(2z) + [\exp(x) + \exp(-x)]\cos(y)\exp(z)}\right) \\ = &-\tan^{-1}\left(\frac{\sinh(x)\sin(y)}{\cosh(z) + \cosh(x)\cos(y)}\right) \\ = &-\Im S(x, yi, z), \end{aligned}$$

since

$$\begin{aligned} S(x,yi,z) &= \tanh^{-1}\left(\frac{\sinh(x)\sinh(yi)}{\cosh(z)+\cosh(x)\cosh(yi)}\right) \\ &= i\,\tan^{-1}\left(\frac{\sinh(x)\sin(y)}{\cosh(z)+\cosh(x)\cos(y)}\right). \end{aligned}$$

(ii) The identity (2.17) will follow from the following two identities:

$$\Im\, G(xi,yi,z) = x - \tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y)+\cos(x)\cosh(z)}\right), \tag{2.20}$$

$$\Re\, G(xi,yi,z) + S(xi,yi,z) = 0. \tag{2.21}$$

Proof of (2.20) *and* (2.21): By definition,

$$\begin{aligned} G(xi,yi,z) &= \log\frac{\exp(xi)+\exp(yi+z)}{\exp(-xi)+\exp(yi+z)} \\ &= \log\frac{[\cos(x)+\cos(y)\exp(z)]+i[\sin(x)+\sin(y)\exp(z)]}{[\cos(x)+\cos(y)\exp(z)]+i[-\sin(x)+\sin(y)\exp(z)]}. \end{aligned}$$

Hence

$$\begin{aligned} \Re\, G(xi,yi,z) &= \frac{1}{2}\log\frac{[\cos(x)+\cos(y)\exp(z)]^2+[\sin(x)+\sin(y)\exp(z)]^2}{[\cos(x)+\cos(y)\exp(z)]^2+[-\sin(x)+\sin(y)\exp(z)]^2} \\ &= \frac{1}{2}\log\frac{1+\exp(2z)+\cos(x-y)\exp(z)}{1+\exp(2z)+\cos(x+y)\exp(z)} \\ &= \frac{1}{2}\log\frac{\cosh(z)+\cos(x-y)}{\cosh(z)+\cos(x+y)} \\ &= -\frac{1}{2}\log\frac{\cosh(z)+\cosh(xi+yi)}{\cosh(z)+\cosh(xi-yi)} \\ &= -S(xi,yi,z). \end{aligned}$$

On the other hand,

$$\begin{aligned} I &:= \Im\, G(xi,yi,z) \\ &= \tan^{-1}\left(\frac{\sin(x)+\sin(y)\exp(z)}{\cos(x)+\cos(y)\exp(z)}\right) - \tan^{-1}\left(\frac{-\sin(x)+\sin(y)\exp(z)}{\cos(x)+\cos(y)\exp(z)}\right) \\ &= \tan^{-1}\left(\frac{2\sin(x)[\cos(x)+\cos(y)\exp(z)]}{[\cos(x)+\cos(y)\exp(z)]^2-[\sin(x)]^2+[\sin(y)\exp(z)]^2}\right) \\ &= \tan^{-1}\left(\frac{\sin(2x)+2\sin(x)\cos(y)\exp(z)}{\cos(2x)+\exp(2z)+2\cos(x)\cos(y)\exp(z)}\right). \end{aligned}$$

Hence

$$iI = \tanh^{-1}\left(\frac{i\sin(2x) + 2i\sin(x)\cos(y)\exp(z)}{\cos(2x) + \exp(2z) + 2\cos(x)\cos(y)\exp(z)}\right),$$

or

$$\frac{\exp(2iI) - 1}{\exp(2iI) + 1} = \frac{i\sin(2x) + 2i\sin(x)\cos(y)\exp(z)}{\cos(2x) + \exp(2z) + 2\cos(x)\cos(y)\exp(z)}.$$

Hence

$$\begin{aligned}
\exp(2iI) &= \frac{\exp(2xi) + \exp(2z) + 2\exp(xi)\cos(y)\exp(z)}{\exp(-2xi) + \exp(2z) + 2\exp(-xi)\cos(y)\exp(z)} \\
&= \frac{\cosh(xi - z) + \cos(y)}{\cosh(xi + z) + \cos(y)}\,\frac{\exp(xi + z)}{\exp(-xi + z)} \\
&= \exp(2xi)\,\frac{\cosh(yi) + \cosh(xi - z)}{\cosh(yi) + \cosh(xi + z)}.
\end{aligned}$$

Thus

$$\begin{aligned}
iI &= xi - \frac{1}{2}\log\frac{\cosh(yi) + \cosh(xi + z)}{\cosh(yi) + \cosh(xi - z)} \\
&= xi - \tanh^{-1}\left(\frac{\sinh(xi)\sinh(z)}{\cosh(yi) + \cosh(xi)\cosh(z)}\right) \\
&= xi - i\tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y) + \cos(x)\cosh(z)}\right).
\end{aligned}$$

Therefore

$$\Im G(xi, yi, z) = I = x - \tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y) + \cos(x)\cosh(z)}\right).$$

□

2.3 The functions $l/2$, h and $\mathfrak{h}$

In this section we give the definitions of the half-length function $l/2$, Bowditch's function h and our gap function $\mathfrak{h}$ and some simple properties of these functions.

The functions $l/2$ and l. For $x \in \mathbf{C}$, let $l(x)/2 \in \mathbf{C}/2\pi i\mathbf{Z}$ be defined by

$$l(x)/2 = \cosh^{-1}(x/2). \tag{2.22}$$

In particular, $\Re\big(l(x)/2\big) \geq 0$ and if $\Re\big(l(x)/2\big) = 0$ then $\Im\big(l(x)/2\big) \geq 0$. Note that $e^{l(x)/2} \in \mathbf{C}$ is well-defined. Its relation with the function h is shown in Lemma 2.24 below.

On the other hand, for $A \in \mathrm{SL}(2,\mathbf{C})$, we define

$$l(A)/2 = l(\mathrm{tr}A)/2 \in \mathbf{C}/2\pi i\mathbf{Z}. \tag{2.23}$$

It is clear that $l(-A)/2 = l(A)/2 + \pi i \mod 2\pi i$.

Hence one has

$$l(x) = 2\cosh^{-1}(x/2) = \cosh^{-1}[(x^2-2)/2\,] \in \mathbf{C}/2\pi i\mathbf{Z}. \tag{2.24}$$

Remark 2.23 Note that for $A \in \mathrm{PSL}(2,\mathbf{C})$, the translation length

$$l(A) = 2\cosh^{-1}(\mathrm{tr}A/2) = \cosh^{-1}[(\mathrm{tr}^2A-2)/2\,] \in \mathbf{C}/2\pi i\mathbf{Z}$$

is well-defined although $l(A)/2$ is not.

The function h. We define an even function $h : \mathbf{C}\backslash\{0\} \to \mathbf{C}$ by

$$h(x) = \frac{1}{2}\left(1 - \sqrt{1-\frac{4}{x^2}}\right). \tag{2.25}$$

Note that the square root here is assumed to have nonnegative real part. It is easy to check that

$$h(x)^2 - h(x) + x^{-2} = 0. \tag{2.26}$$

Actually, if $x \notin [-2,2]$ then $h(x)$ is the root of this quadratic equation which has smaller real part. By (2.29) below, we also have

$$h(x) = \frac{1}{1+e^{l(x)}}. \tag{2.27}$$

Lemma 2.24 *If $x \notin [-2,2]$ then*

$$e^{-l(x)/2} = xh(x). \tag{2.28}$$

Proof. Note that $\Re[l(x)/2] > 0$ since $x \notin [-2,2]$. Hence $\left|e^{l(x)/2}\right| > \left|e^{-l(x)/2}\right|$ and

$$\left|x^{-1}e^{l(x)/2}\right| > \left|x^{-1}e^{-l(x)/2}\right|.$$

Note that $\cosh[l(x)/2] = x/2$ implies that

$$e^{l(x)/2} + e^{-l(x)/2} = x.$$

Hence

$$\left(\frac{e^{l(x)/2}}{x}\right)^2 - \left(\frac{e^{l(x)/2}}{x}\right) + \frac{1}{x^2} = 0$$

and

$$\left(\frac{e^{-l(x)/2}}{x}\right)^2 - \left(\frac{e^{-l(x)/2}}{x}\right) + \frac{1}{x^2} = 0.$$

Then $\left|x^{-1}e^{-l(x)/2}\right| < \left|x^{-1}e^{l(x)/2}\right|$ implies that

$$x^{-1}e^{-l(x)/2} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{x^2}}\right) = h(x).$$

This proves Lemma 2.24. □

Corollary 2.25

$$e^{l(x)} = x^{-2}h(x)^{-2} = h(x)^{-1} - 1. \tag{2.29}$$

The function $\mathfrak{h}$. For $\tau \in \mathbf{C}$ fixed, we define a function

$$\mathfrak{h} = \mathfrak{h}_\tau : \mathbf{C}\backslash\{\pm\sqrt{\tau+2}\} \to \mathbf{C}$$

by the following equivalent expressions:

$$\mathfrak{h}(x) = 2\tanh^{-1}\left(\frac{\sinh\nu}{\cosh\nu + e^{l(x)}}\right) \tag{2.30}$$

$$= \log\left(\frac{e^{\nu} + e^{l(x)}}{e^{-\nu} + e^{l(x)}}\right) \tag{2.31}$$

$$= \log\left(\frac{1 + (e^{\nu} - 1)\,h(x)}{1 + (e^{-\nu} - 1)\,h(x)}\right) \quad (\text{when } \ x \neq 0), \tag{2.32}$$

where $\nu = \cosh^{-1}(-\tau/2)$.

Remark 2.26 (i) Note that $\mathfrak{h}(0)$ is well-defined if $\tau \neq -2$. Actually,

$$\mathfrak{h}(0) = \nu + \pi i. \tag{2.33}$$

(ii) Note also that $e^{\pm\nu} + e^{l(x)} = 0$ if and only if $\pm\nu + \pi i = l(x)$ which implies that $x^2 = \tau + 2$.

(iii) It can be shown by the proof of Lemma 2.24 that

$$e^{\nu} = -\frac{\tau}{2}\left(1 + \sqrt{1 - \frac{4}{\tau^2}}\right). \tag{2.34}$$

Hence

$$\sinh\nu = -\frac{\tau}{2}\sqrt{1 - \frac{4}{\tau^2}}. \tag{2.35}$$

2.4 The attractive fixed points

In this section we exhibit a definite formula, which will be used in §4.4, to find the attractive fixed ideal point in $\mathbf{C}_\infty$ for a given loxodromic/hyperbolic element of $\mathrm{PSL}(2,\mathbf{C})$ represented by $A \in \mathrm{SL}(2,\mathbf{C})$.

Suppose $A = \big(A_{ij}\big)_{2\times 2}$. The two fixed points of A in $\mathbf{C}_\infty$ are the roots of quadratic equation $A_{21}z^2 + (A_{22} - A_{11})z - A_{12} = 0$, hence given by

$$z = \left[(A_{11} - A_{22}) \pm \sqrt{(A_{11} + A_{22})^2 - 4}\,\right]/2A_{21}.$$

Recall that the square root here has been assumed to have positive real part.

In this form, however, it is *not* true that one sign always gives the attractive or the repulsive fixed point. We rewrite it as:

$$z = \big[(A_{11} - A_{22}) \pm (A_{11} + A_{22})\sqrt{1 - 4(A_{11} + A_{22})^{-2}}\,\big]/2A_{21}. \tag{2.36}$$

Then corresponding to the plus and minus signs in (2.36) are respectively the attractive and repulsive fixed points, $\mathrm{Fix}^+(A)$ and $\mathrm{Fix}^-(A)$.

Lemma 2.27 *Suppose $A \in \mathrm{SL}(2, \mathbf{C})$ is loxodromic (hyperbolic) as a motion in $\mathbb{H}^3$. Then*

$$\mathrm{Fix}^+(A) = \big[(A_{11} - A_{22}) + (A_{11} + A_{22})\sqrt{1 - 4(A_{11} + A_{22})^{-2}}\,\big]/2A_{21}, \tag{2.37}$$

$$\mathrm{Fix}^-(A) = \big[(A_{11} - A_{22}) - (A_{11} + A_{22})\sqrt{1 - 4(A_{11} + A_{22})^{-2}}\,\big]/2A_{21}. \tag{2.38}$$

Proof. This is true since we know that they correspond respectively the two eigenvalues of the matrix A:

$$\lambda^+ = (A_{11} + A_{22})\big[\,1 + \sqrt{1 - 4(A_{11} + A_{22})^{-2}}\,\big]/2, \tag{2.39}$$

$$\lambda^- = (A_{11} + A_{22})\big[\,1 - \sqrt{1 - 4(A_{11} + A_{22})^{-2}}\,\big]/2 \tag{2.40}$$

which have respectively norm greater and less than 1. □

Remark 2.28 It is easy to see that the above formulas for attractive and repulsive fixed ideal points of $A \in \mathrm{SL}(2, \mathbf{C})$ actually work for $A \in \mathrm{PSL}(2, \mathbf{C})$.

2.5 The gap from A to B along BA

In this section we define and determine the gap which will be used in the various generalized McShane's identities in this thesis.

Definition 2.29 Given two points $z_1, z_2 \in \mathbf{C}_\infty$ and an oriented line L in the upper-half space model of the hyperbolic 3-space, we define the **complex length** from z_1 to z_2 along L to be the complex length from $\lfloor L, z_1 \rfloor$ to $\lfloor L, z_2 \rfloor$ along L as defined in §2.1, where $\lfloor L, z_j \rfloor$, $j = 1, 2$, is the oriented line which is normal to L and has z_j as its ending ideal point.

Definition 2.30 For $A, B \in \mathrm{SL}(2, \mathbf{C})$ such that $\mathrm{tr}(BA) \neq \pm 2$, the **gap from** A **to** B **along** BA is defined as the complex length from $\mathrm{Fix}^+(A)$ to $\mathrm{Fix}^-(B)$ measured along the naturally oriented axis $\mathbf{a}(BA)$ of BA.

Lemma 2.31 *Suppose $A, B \in \mathrm{SL}(2, \mathbf{C})$ with $\mathrm{tr}(BA) \neq \pm 2$. Then the gap from A to B along BA is given by*

$$G\left(\frac{l(-BA)}{2}, \frac{l(A)}{2}, \frac{l(B)}{2}\right) = 2 \tanh^{-1}\left(\frac{\sinh \frac{l(-BA)}{2}}{\cosh \frac{l(-BA)}{2} + \exp \frac{l(A)+l(B)}{2}}\right). \tag{2.41}$$

Remark. The minus sign before BA in (2.41) is necessary if we do not wish to alter the function G. However, we can eliminate the difference in signs among $l(-BA)/2$, $l(A)/2$, $l(B)/2$ by adding minus signs before A and B.

Proof. Let us denote

$$\frac{\alpha}{2} := \frac{l(A)}{2}, \quad \frac{\beta}{2} := \frac{l(B)}{2}, \quad \frac{l}{2} := \frac{l(-BA)}{2}.$$

Note that there are three line matrices $Q, R, P \in \mathrm{SL}(2, \mathbf{C})$ such that $A = -RQ, B = -PR$; hence $BA = -PQ$. We may normalize A, B by simultaneous conjugation so that the axis of BA is the oriented line $[0, \infty]$. Furthermore, since $l(BA)/2 = l/2 + \pi i$, we may assume that the oriented lines corresponding to Q, R, P are respectively

$$[-1, 1], [e^b, e^a], [e^{l/2}, -e^{l/2}]$$

for some $a, b \in \mathbf{C}$. Then the axes of A, B are respectively the oriented lines

$$[e^{-x}, e^x], \ [e^{l/2-y}, e^{l/2+y}]$$

for some $x, y \in \mathbf{C}$. Note that $[e^b, e^a]$ is an oriented common normal of $[e^{-x}, e^x]$ and $[e^{l/2-y}, e^{l/2+y}]$. See Figure 2.2.

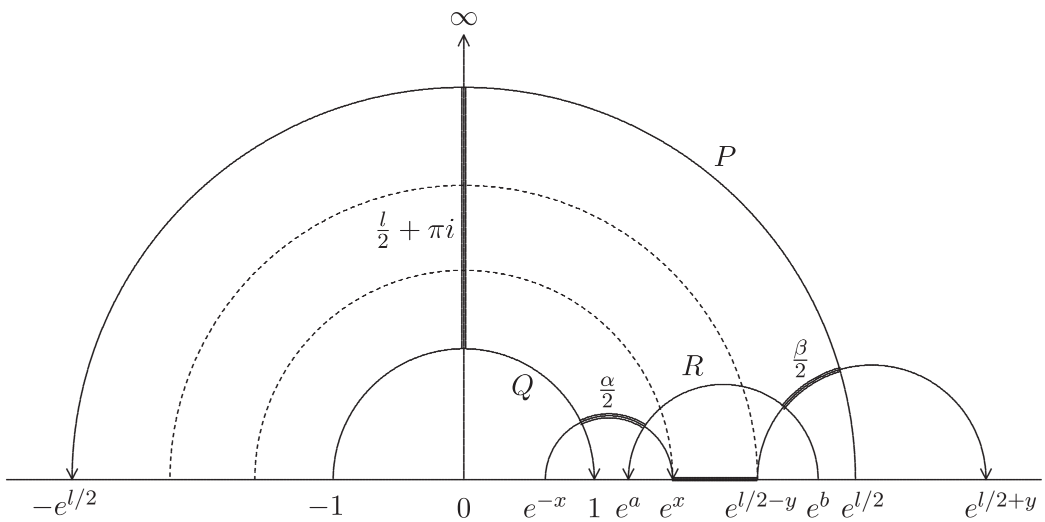

Figure 2.2: The gap from A to B along BA

By definition, the gap from A to B along BA is given by

$$\log R\!\!\!/(\infty, 0;\, e^x, e^{l/2-y}) = \log\left(\frac{e^{l/2-y}}{e^x}\right) = \frac{l}{2} - x - y. \tag{2.42}$$

Then (2.41) follows from the identity (2.43) addressed in Lemma 2.32 below since

$$\begin{aligned}
\tanh\left(\frac{l}{4} - \frac{x+y}{2}\right) &= \frac{\tanh\frac{l}{4} - \tanh\frac{x+y}{2}}{1 - \tanh\frac{l}{4}\tanh\frac{x+y}{2}} = \frac{\tanh\frac{l}{4}\left(1 - \tanh\frac{\alpha+\beta}{4}\right)}{1 - \tanh^2\frac{l}{4}\tanh\frac{\alpha+\beta}{4}} \\
&= \frac{2\tanh\frac{l}{4}\big/\left(\exp\frac{\alpha+\beta}{2} + 1\right)}{1 - \tanh^2\frac{l}{4}\left(\exp\frac{\alpha+\beta}{2} - 1\right)\big/\left(\exp\frac{\alpha+\beta}{2} + 1\right)} \\
&= \frac{2\tanh\frac{l}{4}}{\left(\exp\frac{\alpha+\beta}{2} + 1\right) - \tanh^2\frac{l}{4}\left(\exp\frac{\alpha+\beta}{2} - 1\right)} \\
&= \frac{2\tanh\frac{l}{4}}{\exp\frac{\alpha+\beta}{2}\left(1 - \tanh^2\frac{l}{4}\right) + \left(1 + \tanh^2\frac{l}{4}\right)} \\
&= \frac{\sinh\frac{l}{2}}{\exp\frac{\alpha+\beta}{2} + \cosh\frac{l}{2}}.
\end{aligned}$$

□

Lemma 2.32 *With the above notation,*

$$\tanh\frac{x+y}{2} = \tanh\frac{l}{4}\tanh\frac{\alpha+\beta}{4}. \tag{2.43}$$

Proof. Since $\alpha/2 = l(A)/2$ and $A = -RQ$, we have

$$e^{\frac{\alpha}{2}} \;=\; \not{R}(e^x, e^{-x};\, 1, e^a) \;=\; \not{R}(e^x, e^{-x};\, -1, e^b); \tag{2.44}$$

hence

$$\tanh\frac{a}{2} = \tanh\frac{x}{2}\tanh\frac{\alpha}{4}, \tag{2.45}$$

$$\tanh\frac{b}{2} = \tanh\frac{x}{2}\coth\frac{\alpha}{4}. \tag{2.46}$$

Similarly, since $\beta/2 = l(B)/2$ and $B = -PR$, we have

$$e^{\frac{\beta}{2}} \;=\; \not{R}(e^{l/2+y}, e^{l/2-y};\, e^a, -e^{l/2}) \;=\; \not{R}(e^{l/2+y}, e^{l/2-y};\, e^b, e^{l/2}); \tag{2.47}$$

hence

$$\tanh\left(\frac{l}{4} - \frac{a}{2}\right) = \tanh\frac{y}{2}\coth\frac{\beta}{4}, \tag{2.48}$$

$$\tanh\left(\frac{l}{4} - \frac{b}{2}\right) = \tanh\frac{y}{2}\tanh\frac{\beta}{4}. \tag{2.49}$$

It follows from (2.45) and (2.48) that

$$\begin{aligned} \tanh\frac{l}{4} &= \frac{\tanh(\frac{l}{4} - \frac{a}{2}) + \tanh\frac{a}{2}}{1 + \tanh(\frac{l}{4} - \frac{a}{2})\tanh\frac{a}{2}} \\ &= \frac{\tanh\frac{y}{2} + \tanh\frac{x}{2}\tanh\frac{\alpha}{4}\tanh\frac{\beta}{4}}{\tanh\frac{\beta}{4} + \tanh\frac{x}{2}\tanh\frac{y}{2}\tanh\frac{\alpha}{4}}. \end{aligned} \tag{2.50}$$

It follows from (2.46) and (2.49) that

$$\begin{aligned} \tanh\frac{l}{4} &= \frac{\tanh(\frac{l}{4} - \frac{b}{2}) + \tanh\frac{b}{2}}{1 + \tanh(\frac{l}{4} - \frac{b}{2})\tanh\frac{b}{2}} \\ &= \frac{\tanh\frac{x}{2} + \tanh\frac{y}{2}\tanh\frac{\alpha}{4}\tanh\frac{\beta}{4}}{\tanh\frac{\alpha}{4} + \tanh\frac{x}{2}\tanh\frac{y}{2}\tanh\frac{\beta}{4}}. \end{aligned} \tag{2.51}$$

Combining (2.50) and (2.51), we get

$$\begin{aligned} \tanh\frac{l}{4} &= \frac{(\tanh\frac{y}{2} + \tanh\frac{x}{2}\tanh\frac{\alpha}{4}\tanh\frac{\beta}{4}) + (\tanh\frac{x}{2} + \tanh\frac{y}{2}\tanh\frac{\alpha}{4}\tanh\frac{\beta}{4})}{(\tanh\frac{\beta}{4} + \tanh\frac{\alpha}{4}\tanh\frac{x}{2}\tanh\frac{y}{2}) + (\tanh\frac{\alpha}{4} + \tanh\frac{\beta}{4}\tanh\frac{x}{2}\tanh\frac{y}{2})} \\ &= \frac{(\tanh\frac{x}{2} + \tanh\frac{y}{2})\,(1 + \tanh\frac{\alpha}{4}\tanh\frac{\beta}{4})}{(1 + \tanh\frac{x}{2}\tanh\frac{y}{2})\,(\tanh\frac{\alpha}{4} + \tanh\frac{\beta}{4})} \\ &= \tanh\frac{x+y}{2}\coth\frac{\alpha+\beta}{4}, \end{aligned}$$

which is (2.43). □

2.6 The function Ψ and properties

In this section we define $\Psi(x, y, z) \in \mathbf{C}$ for $x, y, z \in \mathbf{C}$ which will be used in Chapters 4 and 5. It will replace the role of the complex weight z/xy which was intensively used in [8] in the study of Markoff maps. The geometric meaning of $\Psi(x, y, z)$ will be explored in §2.7.

The function Ψ. We define a function

$$\Psi : \mathbf{C}^3 \to \mathbf{C}$$

as follows. Given $x, y, z \in \mathbf{C}$, let

$$\mu = x^2 + y^2 + z^2 - xyz \quad \text{and} \quad \nu = \cosh^{-1}(1 - \mu/2). \tag{2.52}$$

Here we have in mind that $x = \mathrm{tr}A$, $y = \mathrm{tr}B$ and $z = \mathrm{tr}AB$ for some $A, B \in \mathrm{SL}(2, \mathbf{C})$; hence $\mu - 2 = \mathrm{tr}[A, B]$ by the trace identity (4.7) in $\mathrm{SL}(2, \mathbf{C})$ (where $[A, B] = ABA^{-1}B^{-1}$) and $\nu = l(-[A, B])/2 = l([A, B])/2 + \pi i$, the *other* half complex length of $[A, B]$.

Then $\Psi(x, y, z) \in \mathbf{C}$ is defined by

$$\Psi(x, y, z) = \log \frac{xy + (e^\nu - 1)z}{(x^2 - \mu)^{1/2}(y^2 - \mu)^{1/2}}, \tag{2.53}$$

or, equivalently, by

$$\cosh \Psi(x, y, z) = \frac{xy - (\mu/2)z}{(x^2 - \mu)^{1/2}(y^2 - \mu)^{1/2}} \tag{2.54}$$

and

$$\sinh \Psi(x, y, z) = \frac{(\sinh \nu)\, z}{(x^2 - \mu)^{1/2}(y^2 - \mu)^{1/2}}. \tag{2.55}$$

Note that we always use the principal branch of the log function, hence $\Psi(x, y, z)$ has imaginary part in $(-\pi, \pi]$.

Remark 2.33 (i) Note that $\Psi(x, y, z)$ is well-defined if $x^2 \neq \mu$ and $y^2 \neq \mu$. Actually, in this case we have $xy + (e^\nu - 1)z = [xy - (\mu/2)z] + [(\sinh \nu)z] \neq 0$ since $[xy - (\mu/2)z]^2 = (x^2 - \mu)(y^2 - \mu) + [(\sinh \nu)z]^2$.

(ii) It can be checked that

$$\frac{\partial}{\partial\nu}\Big|_{\nu=0} 2\,\Psi(x,y,z) = \frac{z}{xy}. \tag{2.56}$$

(iii) Recall our **convention** that for $u \in \mathbf{C}$, we use $u^{1/2}$ (in contrary to $\sqrt{u}$) to mean a certain (once for all) choice of one of the two square roots of a complex number $u \in \mathbf{C}$.

(iv) Note that the value $\Psi(x,y,z)$ depends on the choices of the square roots in its expression, thus it is only well-defined modulo πi without specifying the choices of the square roots. However, in Proposition 2.34 below, the appropriate sums there do not depend on the choices of square roots and hence are well-defined modulo $2\pi i$.

Properties of the function Ψ. The above defined function Ψ has the following useful properties.

Proposition 2.34 *Given $\mu \in \mathbf{C}$ with $\mu \neq 0, 4$, let $\nu = \cosh^{-1}(1-\mu/2)$. For $x, y, z \in \mathbf{C}$ such that $x^2+y^2+z^2-xyz=\mu$, we have*

(i) *if $x, y, z \neq \pm\sqrt{\mu}$, then*

$$\Psi(y,z,x) + \Psi(z,x,y) + \Psi(x,y,z) = \nu \mod 2\pi i; \tag{2.57}$$

(ii) *if $x, y \neq \pm\sqrt{\mu}$ and $w \in \mathbf{C}$ with $z + w = xy$, then*

$$\Psi(x,y,z) + \Psi(x,y,w) = \nu \mod 2\pi i; \tag{2.58}$$

(iii) *if $x, y \neq 0, \pm\sqrt{\mu}$ and z is the one of the two roots of the equation $x^2 + y^2 + z^2 - xyz = \mu$ which has smaller norm, that is,*

$$z = \frac{xy}{2}\left(1 - \sqrt{1 - 4\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{\mu}{x^2y^2}\right)}\,\right), \tag{2.59}$$

then

$$\lim_{y\to\infty} 2\Psi(x,y,z) = \mathfrak{h}(x) \mod 2\pi i, \tag{2.60}$$

where $\mathfrak{h} = \mathfrak{h}_\tau$ is the function defined in §2.3 with $\tau = \mu - 2$.

Proof. (i) Let

$$\alpha = \Psi(y,z,x), \;\; \beta = \Psi(z,x,y), \;\; \gamma = \Psi(x,y,z).$$

Then

$$\cosh\alpha = \frac{yz-(\mu/2)x}{(y^2-\mu)^{1/2}(z^2-\mu)^{1/2}}, \;\; \sinh\alpha = \frac{(\sinh\nu)x}{(y^2-\mu)^{1/2}(z^2-\mu)^{1/2}};$$

$$\cosh\beta = \frac{zx-(\mu/2)y}{(z^2-\mu)^{1/2}(x^2-\mu)^{1/2}}, \;\; \sinh\beta = \frac{(\sinh\nu)y}{(z^2-\mu)^{1/2}(x^2-\mu)^{1/2}};$$

$$\cosh\gamma = \frac{xy-(\mu/2)z}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}, \;\; \sinh\gamma = \frac{(\sinh\nu)z}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}.$$

Hence

$$\begin{aligned}\cosh(\alpha+\beta) &= \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \\ &= \frac{[yz-(\mu/2)x][zx-(\mu/2)y] + (\sinh\nu)^2 xy}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}\,(z^2-\mu)} \\ &= \frac{xy(z^2-\mu) + (\mu^2/2)xy - (\mu/2)(x^2+y^2)z}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}\,(z^2-\mu)}\end{aligned}$$

and

$$\begin{aligned}\cosh(\nu-\gamma) &= \cosh\nu\cosh\gamma - \sinh\nu\sinh\gamma \\ &= \frac{(1-\mu/2)[xy-(\mu/2)z] - (\mu^2/4-\mu)z}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}} \\ &= \frac{xy(1-\mu/2)+(\mu/2)z}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}.\end{aligned}$$

Thus

$$\cosh(\alpha+\beta) = \cosh(\nu-\gamma)$$

since

$$\begin{aligned}& [xy(1-\mu/2)+(\mu/2)z](z^2-\mu) \\ = \;& xy(z^2-\mu) + (\mu^2/2)xy - (\mu/2)(xyz+\mu-z^2)z \\ = \;& xy(z^2-\mu) + (\mu^2/2)xy - (\mu/2)(x^2+y^2)z.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sinh(\alpha+\beta) &= \sinh\alpha\cosh\beta+\cosh\alpha\sinh\beta\\ &= \frac{(\sinh\nu)x[zx-(\mu/2)y]+[yz-(\mu/2)x](\sinh\nu)y}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}\,(z^2-\mu)}\\ &= \frac{(\sinh\nu)[(x^2+y^2)z-\mu xy]}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}\,(z^2-\mu)}\end{aligned}$$

and

$$\begin{aligned}\sinh(\nu-\gamma) &= \sinh\nu\cosh\gamma-\cosh\nu\sinh\gamma\\ &= \frac{(\sinh\nu)[(xy-(\mu/2)z)-(1-\mu/2)z]}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}\\ &= \frac{(\sinh\nu)(xy-z)}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}.\end{aligned}$$

Thus

$$\sinh(\alpha+\beta)=\sinh(\nu-\gamma)$$

since

$$\begin{aligned}(xy-z)(z^2-\mu) &= (xyz+\mu-z^2)z-\mu xy\\ &= (x^2+y^2)z-\mu xy.\end{aligned}$$

Hence we have

$$\alpha+\beta=\nu-\gamma \mod 2\pi i.$$

(ii) Suppose $z+w=xy$. Let

$$\delta=\Psi(x,y,w).$$

Then

$$\cosh\delta=\frac{xy-(\mu/2)w}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}},\ \ \sinh\delta=\frac{(\sinh\nu)w}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}.$$

Hence

$$\begin{aligned}\cosh(\nu-\gamma) &= \frac{xy-(\mu/2)(xy-z)}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}\\ &= \frac{xy-(\mu/2)w}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}\\ &= \cosh\delta\end{aligned}$$

and

$$\begin{aligned}\sinh(\nu-\gamma) &= \frac{(\sinh\nu)(xy-z)}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}\\ &= \frac{(\sinh\nu)w}{(x^2-\mu)^{1/2}(y^2-\mu)^{1/2}}\\ &= \sinh\delta.\end{aligned}$$

Thus

$$\delta=\nu-\gamma \mod 2\pi i.$$

(iii) By definition

$$\begin{aligned}\mathfrak{h}(x) &= \log\,[e^{\nu}+e^{l(x)}]/[e^{-\nu}+e^{l(x)}]\\ &= \log\,[e^{\nu}+e^{l(x)}]^2/[(e^{-\nu}+e^{l(x)})(e^{\nu}+e^{l(x)})]\\ &= \log\,[e^{\nu}+e^{l(x)}]^2/[1+e^{l(x)}(2-\mu)+e^{2l(x)}]\\ &= \log\,[e^{\nu}+e^{l(x)}]^2/[e^{l(x)}(x^2-\mu)]\\ &= \log\;x^2[1+(e^{\nu}-1)h(x)]^2/(x^2-\mu),\end{aligned}$$

since it is easy to verify that

$$x^{-2}h(x)^{-1}=1-h(x).$$

On the other hand, since $\lim_{y\to\infty} z/xy = h(x)$ and

$$\begin{aligned}2\Psi(x,y,z) &= \log\,[xy+(e^{\nu}-1)z]^2/[(x^2-\mu)(y^2-\mu)]\\ &= \log\,[1+(e^{\nu}-1)(z/xy)]^2/[(1-\mu/x^2)(1-\mu/y^2)],\end{aligned}$$

we have

$$\begin{aligned}\lim_{y\to\infty} 2\Psi(x,y,z) &= \log\,[1+(e^{\nu}-1)h(x)]^2/(1-\mu/x^2)\\ &= \mathfrak{h}(x).\end{aligned}$$

This completes the proof of Proposition 2.34. □

2.7 Geometric meanings of $\mathfrak{h}$ and Ψ

In this section we explore the geometric meanings of the functions $\mathfrak{h}(x)$ and $\Psi(x, y, z)$ defined respectively in §2.3 and §2.6. The geometric interpretation of $\Psi(x, y, z)$ will be used in an essential way in the proof of Theorem 5.4.

Recall our notation $\Delta_{\mathbf{n}}(\mathbf{l}, \mathbf{m})$ for the complex distance from $\mathbf{l}$ to $\mathbf{m}$ along $\mathbf{n}$ where $\mathbf{l}, \mathbf{m}, \mathbf{n}$ are oriented lines in $\mathbb{H}^3$ such that $\mathbf{n}$ is an oriented common normal to $\mathbf{l}$ and $\mathbf{m}$.

Geometric meaning of the gap function $\mathfrak{h}$. The function $\mathfrak{h}$ has the following geometric interpretation as the gap function used in the generalized McShane's identity.

Let $A, B \in \mathrm{SL}(2, \mathbf{C})$. Consider the commutator $[B^{-1}, A^{-1}] = (B^{-1}A^{-1}B)A$ and let $\tau := \mathrm{tr}\,[B^{-1}, A^{-1}]$. Let $x = \mathrm{tr}\,A$, $y = \mathrm{tr}\,B$ and $z = \mathrm{tr}\,AB$ and set

$$\mu = x^2 + y^2 + z^2 - xyz. \tag{2.61}$$

Then the Fricke trace identity in $\mathrm{SL}(2, \mathbf{C})$ tells us that $\mu = \tau + 2$. Let $\nu = \cosh^{-1}(-\tau/2) = \cosh^{-1}(1 - \mu/2) \in \mathbf{C}/2\pi i\mathbf{Z}$.

If A is loxodromic (including hyperbolic), so is $B^{-1}A^{-1}B$. Let the attractive and repulsive fixed points of A be respectively denoted as $\mathrm{Fix}^+(A)$ and $\mathrm{Fix}^-(A)$. Similarly we have $\mathrm{Fix}^+(B^{-1}A^{-1}B)$ and $\mathrm{Fix}^-(B^{-1}A^{-1}B)$.

We denote the oriented line in $\mathbb{H}^3$ which is normal to the axis $\mathbf{a}(B^{-1}A^{-1}BA)$ and has $\mathrm{Fix}^+(A)$ as its ending ideal point by $\lfloor \mathbf{a}(B^{-1}A^{-1}BA), \mathrm{Fix}^+(A) \rfloor$. Similarly, we have another oriented line $\lfloor \mathbf{a}(B^{-1}A^{-1}BA), \mathrm{Fix}^-(B^{-1}A^{-1}B) \rfloor$.

Then the complex length from the oriented line $\lfloor \mathbf{a}(B^{-1}A^{-1}BA), \mathrm{Fix}^+(A) \rfloor$ to the oriented line $\lfloor \mathbf{a}(B^{-1}A^{-1}BA), \mathrm{Fix}^-(B^{-1}A^{-1}B) \rfloor$ along $\mathbf{a}(B^{-1}A^{-1}BA)$ is just given by $\mathfrak{h}(x) = \mathfrak{h}_\tau(x)$. See Figure 2.3 for an illustration in the cases where the isometries $A, B^{-1}A^{-1}B$ and $B^{-1}A^{-1}BA$ have coplanar non-intersecting axes.

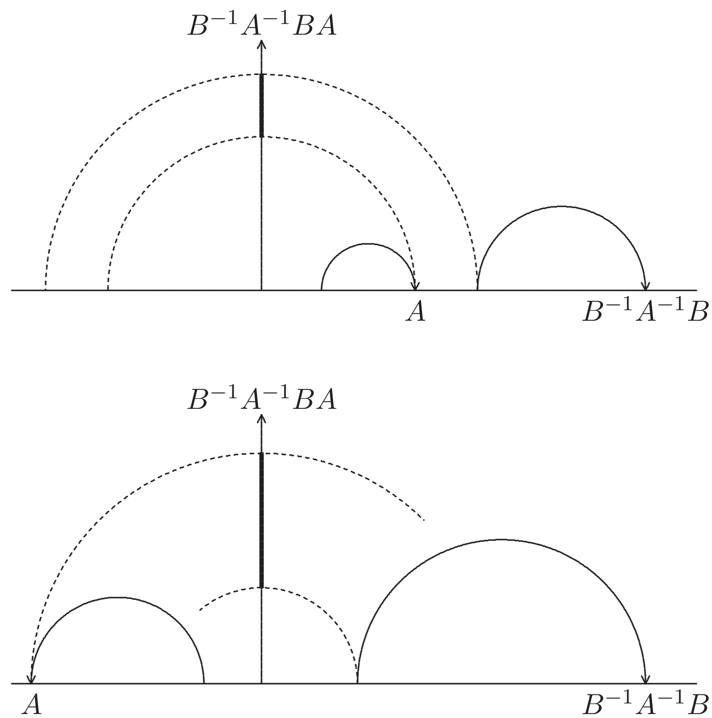

Figure 2.3: The complex gap

Lemma 2.35 *With the above notation and with $\mathfrak{h} = \mathfrak{h}_\tau$ as in (2.30), we have*

$$\mathfrak{h}(x) = \Delta_{\mathbf{a}(B^{-1}A^{-1}BA)}\big(\lfloor \mathbf{a}(B^{-1}A^{-1}BA), \mathrm{Fix}^+(A)\rfloor, \lfloor \mathbf{a}(B^{-1}A^{-1}BA), \mathrm{Fix}^-(B^{-1}A^{-1}B)\rfloor\big). \tag{2.62}$$

Proof. It is a special case of Lemma 2.31, since

$$l(A)/2 = l(B^{-1}A^{-1}B)/2 = l(x)/2$$

and

$$l(-B^{-1}A^{-1}BA)/2 = \cosh^{-1}(-\tau/2) = \nu.$$

□

Remark 2.36 It is important to note that the above gap value depends only on A in the given pair A, B such that $B^{-1}A^{-1}BA$ is kept fixed.

Geometric meaning of the function Ψ. Given $x, y, z \in \mathbf{C}$, set

$$\mu = x^2 + y^2 + z^2 - xyz. \tag{2.63}$$

and assume $\mu \neq 0, 4$. As explained in §4 of [8], there exit $A, B \in \mathrm{SL}(2, \mathbf{C})$, unique up to conjugation in $\mathrm{SL}(2, \mathbf{C})$, so that the traces of A, B and $C := (BA)^{-1}$ are respectively x, y, z, that is,

$$\mathrm{tr}A = x, \quad \mathrm{tr}B = y, \quad \mathrm{tr}C = z.$$

Note that $CBA = I$. Denote $[B^{-1}, A^{-1}] = B^{-1}A^{-1}BA = B^{-1}A^{-1}C^{-1}$ and let $\tau := \mathrm{tr}\,[B^{-1}, A^{-1}] = \mathrm{tr}\,B^{-1}A^{-1}BA$. Then by the Fricke trace identity in $\mathrm{SL}(2, \mathbf{C})$, we have $\tau = \mu - 2 \neq \pm 2$. Let $\nu = \cosh^{-1}(-\tau/2) = \cosh^{-1}(1 - \mu/2)$.

Then there are matrices $Q, R, P \in \mathrm{SL}(2, \mathbf{C})$ such that $Q^2 = R^2 = P^2 = -I$ and $A = -RQ, B = -PR, C = -QP$. Hence $[B^{-1}, A^{-1}] = -(RPQ)^2$. Here matrices in $\mathrm{SL}(2, \mathbf{C})$ are considered as orientation-preserving isometries of $\mathbb{H}^3$, the upper half-space model of the hyperbolic 3-space. In fact, Q, R, P are involutions representing appropriately oriented lines in $\mathbb{H}^3$ which are common normals to $\mathbf{a}(C)$ and $\mathbf{a}(A)$, to $\mathbf{a}(A)$ and $\mathbf{a}(B)$, and to $\mathbf{a}(B)$ and $\mathbf{a}(C)$, respectively, as explained in §2.1. Note that the negative signs in this paragraph are important for later use in this section of Fenchel's cosine and sine rules for right angled hexagons.

Note that conjugation $A \mapsto KAK^{-1}$ by a matrix $K \in \mathrm{SL}(2, \mathbf{C})$ preserves the orientations of axes in the sense that for $A \in \mathrm{SL}(2, \mathbf{C})$ we have $\mathbf{a}(KAK^{-1}) = K\,\mathbf{a}(A)K^{-1}$, where $\mathbf{a}(KAK^{-1})$ and $\mathbf{a}(A)$ are regarded as the normalized line matrices representing the corresponding axes. In particular, when K is a normalized line matrix, that is $K^2 = -I$, the conjugation is given by $A \leftrightarrow -KAK$.

Now consider the schematic figure as illustrated in Figure 2.4. It is easy to check that $RPQ \leftrightarrow QRP \leftrightarrow PQR \leftrightarrow RPQ$ by conjugation by Q, P, R respectively. Hence $\mathbf{a}(RPQ) \leftrightarrow \mathbf{a}(QRP) \leftrightarrow \mathbf{a}(PQR) \leftrightarrow \mathbf{a}(RPQ)$ by conjugation by Q, P, R respectively.

Definition 2.37 (i) For each ordered pair of oriented lines $\mathbf{l}$ and $\mathbf{m}$ in $\mathbb{H}^3$, let $\lfloor \mathbf{l}, \mathbf{m} \rfloor$ denote a definitely chosen oriented common normal to them, so that $\lfloor \mathbf{l}, \mathbf{m} \rfloor$

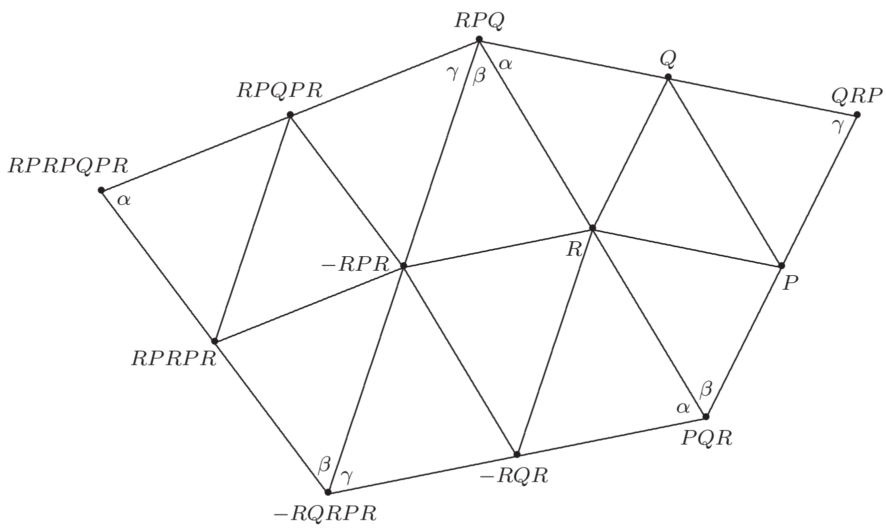

Figure 2.4:

and $\lfloor \mathbf{m}, \mathbf{l} \rfloor$ always have opposite directions. For example, we may assume $\lfloor \mathbf{l}, \mathbf{m} \rfloor$ is directed from $\mathbf{l}$ to $\mathbf{m}$ when $\mathbf{l}$ and $\mathbf{m}$ are disjoint.

(ii) Given oriented lines $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, let $\mathbf{H}(\mathbf{l}, \mathbf{m}, \mathbf{n})$ denote the oriented right-angled hexagon in $\mathbb{H}^3$ formed by the oriented lines $\mathbf{l}$; $\lfloor \mathbf{l}, \mathbf{m} \rfloor$; $\mathbf{m}$; $\lfloor \mathbf{m}, \mathbf{n} \rfloor$; $\mathbf{n}$; $\lfloor \mathbf{n}, \mathbf{l} \rfloor$ in this cyclic order.

Hence the oriented lines

$$\mathbf{a}(RPQ); \lfloor \mathbf{a}(RPQ), \mathbf{a}(PQR) \rfloor;$$
$$\mathbf{a}(PQR); \lfloor \mathbf{a}(PQR), \mathbf{a}(QRP) \rfloor;$$
$$\mathbf{a}(QRP); \lfloor \mathbf{a}(QRP), \mathbf{a}(RPQ) \rfloor,$$

in this cyclic order, form the oriented right-angled hexagon $\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$. Since, as we observed above, $\mathbf{a}(RPQ) \leftrightarrow \mathbf{a}(QRP) \leftrightarrow \mathbf{a}(PQR) \leftrightarrow \mathbf{a}(RPQ)$ by conjugation by Q, P, R respectively, the oriented right-angled hexagon $\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$ has the oriented lines $\mathbf{a}(R)$, $\mathbf{a}(P)$, $\mathbf{a}(Q)$ as the "midpoints" of its three sides, that is,

$$\Delta_{\lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor}\big(\mathbf{a}(RPQ),\mathbf{a}(R)\big) = \Delta_{\lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor}\big(\mathbf{a}(R),\mathbf{a}(PQR)\big) =: \tilde{c},$$
$$\Delta_{\lfloor \mathbf{a}(PQR),\mathbf{a}(QRP)\rfloor}\big(\mathbf{a}(PQR),\mathbf{a}(P)\big) = \Delta_{\lfloor \mathbf{a}(PQR),\mathbf{a}(QRP)\rfloor}\big(\mathbf{a}(P),\mathbf{a}(QRP)\big) =: \tilde{a},$$
$$\Delta_{\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor}\big(\mathbf{a}(QRP),\mathbf{a}(Q)\big) = \Delta_{\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor}\big(\mathbf{a}(Q),\mathbf{a}(RPQ)\big) =: \tilde{b}.$$

Let the other three side-lengths of the oriented right-angled hexagon

$$\mathbf{H}(\mathbf{a}(RPQ),\mathbf{a}(PQR),\mathbf{a}(QRP))$$

be denoted as

$$\Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor\big) =: \alpha, \tag{2.64}$$
$$\Delta_{\mathbf{a}(PQR)}\big(\lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor, \lfloor \mathbf{a}(PQR),\mathbf{a}(QRP)\rfloor\big) =: \beta, \tag{2.65}$$
$$\Delta_{\mathbf{a}(QRP)}\big(\lfloor \mathbf{a}(PQR),\mathbf{a}(QRP)\rfloor, \lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor\big) =: \gamma. \tag{2.66}$$

Also let

$$\Delta_{\mathbf{a}(RQ)}\big(\mathbf{a}(Q),\mathbf{a}(R)\big) =: \tilde{p}, \tag{2.67}$$
$$\Delta_{\mathbf{a}(PR)}\big(\mathbf{a}(R),\mathbf{a}(P)\big) =: \tilde{q}, \tag{2.68}$$
$$\Delta_{\mathbf{a}(QP)}\big(\mathbf{a}(P),\mathbf{a}(Q)\big) =: \tilde{r}. \tag{2.69}$$

Then

$$\cosh\tilde{p} = (1/2)\,\mathrm{tr}(-RQ) = (1/2)\,\mathrm{tr}A = x/2, \tag{2.70}$$
$$\cosh\tilde{q} = (1/2)\,\mathrm{tr}(-PR) = (1/2)\,\mathrm{tr}B = y/2, \tag{2.71}$$
$$\cosh\tilde{r} = (1/2)\,\mathrm{tr}(-QP) = (1/2)\,\mathrm{tr}C = z/2. \tag{2.72}$$

Applying Fenchel's cosine rule (2.8) to the oriented right-angled hexagons

$$\mathbf{H}(\mathbf{a}(Q),\mathbf{a}(P),\mathbf{a}(QRP)) \quad \text{and} \quad \mathbf{H}(\mathbf{a}(RPQ),\mathbf{a}(PQR),\mathbf{a}(QRP)),$$

we have respectively (noticing that $\Delta_{\mathbf{a}(PQ)}\big(\mathbf{a}(Q),\mathbf{a}(P)\big) = \tilde{r}$)

$$\cosh\tilde{r} = \cosh\tilde{a}\,\cosh\tilde{b} + \sinh\tilde{a}\,\sinh\tilde{b}\,\cosh\gamma, \tag{2.73}$$

$$\cosh 2\tilde{c} = \cosh 2\tilde{a}\, \cosh 2\tilde{b} + \sinh 2\tilde{a}\, \sinh 2\tilde{b}\, \cosh \gamma. \tag{2.74}$$

Claim.

$$\frac{\cosh \tilde{a}}{\cosh \tilde{p}} = \frac{\cosh \tilde{b}}{\cosh \tilde{q}} = \frac{\cosh \tilde{c}}{\cosh \tilde{r}} = \kappa, \tag{2.75}$$

where $\kappa^2 = 4/\mu$.

Proof. Multiplying $4\cosh\tilde{a}\cosh\tilde{b}$ to both sides of (2.73) gives

$$4\cosh\tilde{a}\, \cosh\tilde{b}\, \cosh\tilde{r} = 4\cosh^2\tilde{a}\, \cosh^2\tilde{b} + \sinh 2\tilde{a}\, \sinh 2\tilde{b}\, \cosh\gamma.$$

Comparing with (2.74) gives

$$4\cosh\tilde{a}\, \cosh\tilde{b}\, \cosh\tilde{r} - \cosh 2\tilde{c} = 4\cosh^2\tilde{a}\, \cosh^2\tilde{b} - \cosh 2\tilde{a}\, \cosh 2\tilde{b}.$$

After simplification we have

$$(2\cosh\tilde{a})(2\cosh\tilde{b})(2\cosh\tilde{r}) = (2\cosh\tilde{a})^2 + (2\cosh\tilde{b})^2 + (2\cosh\tilde{c})^2 - 4. \tag{2.76}$$

Similarly we have

$$(2\cosh\tilde{a})(2\cosh\tilde{q})(2\cosh\tilde{c}) = (2\cosh\tilde{a})^2 + (2\cosh\tilde{b})^2 + (2\cosh\tilde{c})^2 - 4, \tag{2.77}$$

$$(2\cosh\tilde{p})(2\cosh\tilde{b})(2\cosh\tilde{c}) = (2\cosh\tilde{a})^2 + (2\cosh\tilde{b})^2 + (2\cosh\tilde{c})^2 - 4. \tag{2.78}$$

Hence

$$\frac{\cosh \tilde{a}}{\cosh \tilde{p}} = \frac{\cosh \tilde{b}}{\cosh \tilde{q}} = \frac{\cosh \tilde{c}}{\cosh \tilde{r}}. \tag{2.79}$$

Let the common value be denoted κ. Then

$$2\cosh\tilde{a} = \kappa\, 2\cosh\tilde{p} = \kappa x, \tag{2.80}$$

$$2\cosh\tilde{b} = \kappa\, 2\cosh\tilde{q} = \kappa y, \tag{2.81}$$

$$2\cosh\tilde{c} = \kappa\, 2\cosh\tilde{r} = \kappa z. \tag{2.82}$$

Now by (2.76) we have

$$\kappa^2 xyz = \kappa^2(x^2 + y^2 + z^2) - 4, \tag{2.83}$$

and hence (recalling that $x^2 + y^2 + z^2 - xyz = \mu$)

$$\kappa^2 = 4/\mu. \tag{2.84}$$

This proves the claim. □

Alternative proof. The above claim can also be proved by direct calculations as follows. Here we prove $\cosh\tilde{c} = (2/\mu^{1/2})\cosh\tilde{r}$. We first find the normalized line matrix $\mathbf{k}$ representing the axis $\mathbf{a}(RPQ)$. This is given by $\mathbf{k} = K/(\det K)^{1/2}$ where $K = RPQ - (RPQ)^{-1} = RPQ + QPR$. Note that

$$\det K = -\mathrm{tr}^2(RPQ) + 4 = \mathrm{tr}[-(RPQ)^2] + 2 = \tau + 2 = \mu. \tag{2.85}$$

Hence we have

$$\begin{aligned}\cosh\tilde{c} &= (1/2)\,\mathrm{tr}(R\,\mathbf{k}) = (1/2)\,\mathrm{tr}[R(RPQ + QPR)]/\mu^{1/2} \\ &= (1/2)\,\mathrm{tr}(-PQ + RQPR)/\mu^{1/2} = \mathrm{tr}(-PQ)/\mu^{1/2} \\ &= (2/\mu^{1/2})\cosh\tilde{r}.\end{aligned}$$

This proves the claim alternatively. □

It follows from the first proof of the above claim that

Lemma 2.38

$$\cosh\gamma = -\frac{xy/\mu - z/2}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}, \tag{2.86}$$

where the square roots do not necessarily have nonnegative real parts.

Proof. From (2.73) we have

$$\begin{aligned}\cosh\gamma &= \frac{\cosh\tilde{r} - \cosh\tilde{a}\cosh\tilde{b}}{\sinh\tilde{a}\sinh\tilde{b}} \\ &= \frac{z/2 - \kappa(x/2)\kappa(y/2)}{[(\kappa x/2)^2 - 1]^{1/2}[(\kappa y/2)^2 - 1]^{1/2}} \\ &= \frac{z/2 - xy/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}\end{aligned}$$

since $\kappa^2 = 4/\mu$. □

As an immediate corollary, we have the following

Corollary 2.39

$$\sinh\gamma = \pm\frac{(\sinh\nu)z}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}. \tag{2.87}$$

The rest of this section is devoted to the determination of the sign in (2.87). Geometrically, we have

Lemma 2.40 $(\alpha + \pi i) + (\beta + \pi i) + (\gamma + \pi i) = \nu \mod 2\pi i.$

Proof. We refer to Figure 2.4. Consider the following right angled hexagons

$$\mathcal{H}_1 := \mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP)),$$
$$\mathcal{H}_2 := \mathbf{H}(\mathbf{a}(PQR), \mathbf{a}(RPQ), \mathbf{a}(-RQRPR)),$$
$$\mathcal{H}_3 := \mathbf{H}(\mathbf{a}(RPRPQPR), \mathbf{a}(-RQRPR), \mathbf{a}(RPQ)).$$

It is easy to check that the conjugation by R maps $\mathcal{H}_1$ to $\mathcal{H}_2$, and the conjugation by RPR maps $\mathcal{H}_2$ to $\mathcal{H}_3$; thus the conjugation by PR maps $\mathcal{H}_1$ to $\mathcal{H}_3$. Since conjugations preserve the relevant complex lengths, we have

$$\Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(PQR), \mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ), \mathbf{a}(-RQRPR)\rfloor\big) = \beta,$$
$$\Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(-RQRPR), \mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ), \mathbf{a}(RPRPQPR)\rfloor\big) = \gamma.$$

Thus

$$\begin{aligned}
& (\alpha+\pi i)+(\beta+\pi i)+(\gamma+\pi i) \\
= \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor\big) \\
+ \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(PQR),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(-RQRPR)\rfloor\big) \\
+ \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(-RQRPR),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(RPRPQPR)\rfloor\big)+\pi i \\
= \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor\big) \\
+ \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(RPQ),\mathbf{a}(PQR)\rfloor, \lfloor \mathbf{a}(-RQRPR),\mathbf{a}(RPQ)\rfloor\big) \\
+ \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(-RQRPR),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(RPRPQPR)\rfloor\big)+\pi i \\
= \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(RPQPRPR)\rfloor\big)+\pi i \\
= \;& \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor, \lfloor \mathbf{a}(RPRPQPR),\mathbf{a}(RPQ)\rfloor\big) \\
= \;& \nu \mod 2\pi i.
\end{aligned}$$

The last equality follows from Lemma 2.17 and Lemma 2.41 below, since it is easy to know that $\nu = \cosh^{-1}(-\tau/2)$ is the complex translation length of RPQ (recall that $[B^{-1},A^{-1}] = -(RPQ)^2$ and $\tau = \mathrm{tr}[B^{-1},A^{-1}]$) and that the conjugation by RPQ maps $\lfloor \mathbf{a}(QRP),\mathbf{a}(RPQ)\rfloor$ to $\lfloor \mathbf{a}(RPRPQPR),\mathbf{a}(RPQ)\rfloor$. □

Lemma 2.41 *The complex translation length of $K \in \mathrm{SL}(2,\mathbf{C})$ is given by*

$$l(K) = \cosh^{-1}\left(\tfrac{1}{2}\,\mathrm{tr}(K^2)\right). \tag{2.88}$$

Proof. Recall that $l(K)/2 = \cosh^{-1}\left(\frac{1}{2}\,\mathrm{tr}K\right)$. Hence

$$\begin{aligned}
\cosh l(K) &= 2\cosh^2(l(K)/2)-1 \\
&= 2\left(\tfrac{1}{2}\,\mathrm{tr}K\right)^2-1 \\
&= \tfrac{1}{2}(\mathrm{tr}^2K-2) \\
&= \tfrac{1}{2}\,\mathrm{tr}(K^2),
\end{aligned}$$

from which (2.88) follows since $\Re\, l(K) \geq 0$. □

Now we can determine the signs in the expressions (2.87) etc as follows.

Lemma 2.42

$$\sinh\gamma = -\frac{(\sinh\nu)\, z}{(x^2/\mu-1)^{1/2}(y^2/\mu-1)^{1/2}}. \tag{2.89}$$

Proof. Let $\Psi(x,y,z) \in \mathbf{C}$ be defined as in §2.6 by:

$$\Psi(x,y,z) = \log\frac{[xy+(e^\nu-1)z]/\mu}{(x^2/\mu-1)^{1/2}(y^2/\mu-1)^{1/2}}, \tag{2.90}$$

or equivalently, by the following two equations:

$$\cosh\Psi(x,y,z) = \frac{[xy-(\mu/2)z]/\mu}{(x^2/\mu-1)^{1/2}(y^2/\mu-1)^{1/2}}, \tag{2.91}$$

$$\sinh\Psi(x,y,z) = \frac{(\sinh\nu)z/\mu}{(x^2/\mu-1)^{1/2}(y^2/\mu-1)^{1/2}}. \tag{2.92}$$

Similarly for $\Psi(y,z,x), \Psi(z,x,y) \in \mathbf{C}$. We have from Proposition 2.34(i) that

$$\Psi(y,z,x)+\Psi(z,x,y)+\Psi(x,y,z) = \nu \mod 2\pi i.$$

On the other hand, it follows from Lemma 2.38 that $-\cosh\gamma = \cosh\Psi(x,y,z)$ etc and hence, modulo $2\pi i$,

$$\alpha+\pi i = \pm\Psi(y,z,x), \quad \beta+\pi i = \pm\Psi(z,x,y), \quad \gamma+\pi i = \pm\Psi(x,y,z). \tag{2.93}$$

Now applying Fenchel's sine rule as in §VI.2 of [18] to the right angled hexagon

$$\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$$

gives

$$\frac{\sinh\alpha}{\sinh 2\tilde{a}} = \frac{\sinh\beta}{\sinh 2\tilde{b}} = \frac{\sinh\gamma}{\sinh 2\tilde{c}}.$$

It follows from (2.87) together with $\kappa\, z = 2\cosh\tilde{c}$ that the above common value is given by

$$\begin{aligned}\frac{\sinh\gamma}{\sinh 2\tilde{c}} &= \pm\frac{(\sinh\nu)\,\kappa\, z}{\kappa\,\sinh\tilde{a}\,\sinh\tilde{b}\,\sinh 2\tilde{c}} \\ &= \pm\frac{\sinh\nu}{\kappa\,\sinh\tilde{a}\,\sinh\tilde{b}\,\sinh\tilde{c}},\end{aligned} \tag{2.94}$$

(note that in (2.94) the $\pm$ is the same as that in (2.87) etc) and hence the $\pm$ in expressions (2.87) etc of $\sinh\gamma, \sinh\alpha, \sinh\beta$ are constant. So are the signs in (2.93), that is, we have either, modulo $2\pi i$,

$$\alpha+\pi i=\Psi(y,z,x), \quad \beta+\pi i=\Psi(z,x,y), \quad \gamma+\pi i=\Psi(x,y,z) \tag{2.95}$$

or, modulo $2\pi i$,

$$\alpha+\pi i=-\Psi(y,z,x), \quad \beta+\pi i=-\Psi(z,x,y), \quad \gamma+\pi i=-\Psi(x,y,z). \tag{2.96}$$

Since we also have $(\alpha+\pi i)+(\beta+\pi i)+(\gamma+\pi i)=\nu \mod 2\pi i$ (Lemma 2.40) and $\nu\neq 0$ we may conclude that (2.95), and hence (2.89), must hold. This proves Lemma 2.42. □

Observing that

$$\begin{aligned}\alpha+\pi i &= \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(RPQ),\mathbf{a}(Q)\rfloor, \lfloor \mathbf{a}(R),\mathbf{a}(RPQ)\rfloor\big)+\pi i\\ &= \Delta_{\mathbf{a}(RPQ)}\big(\lfloor \mathbf{a}(RPQ),\mathbf{a}(Q)\rfloor, \lfloor \mathbf{a}(RPQ),\mathbf{a}(R)\rfloor\big)\end{aligned}$$

and $\mathbf{a}(RPQ)=\mathbf{a}(-B^{-1}A^{-1}C^{-1})$, the above argument then gives us the following geometric interpretation of $\Psi(y,z,x)$ etc.

Lemma 2.43 *Given $x,y,z\in\mathbf{C}$ so that $\mu:=x^2+y^2+z^2-xyz\neq 0,4$, there exit line matrices $Q,R,P\in\mathrm{SL}(2,\mathbf{C})$ (that is, $Q^2=R^2=P^2=-I$) such that $A:=-RQ$, $B:=-PR$, $C:=-QP\in\mathrm{SL}(2,\mathbf{C})$ satisfy $\operatorname{tr}A=x$, $\operatorname{tr}B=y$ and $\operatorname{tr}C=z$. Let $A':=[B^{-1},A^{-1}]$, $B':=[C^{-1},B^{-1}]$, $C':=[A^{-1},C^{-1}]$. Then*

$$\Psi(y,z,x)=\Delta_{\mathbf{a}(-A')}\big(\lfloor \mathbf{a}(-A'),\mathbf{a}(Q)\rfloor, \lfloor \mathbf{a}(-A'),\mathbf{a}(R)\rfloor\big) \mod 2\pi i, \tag{2.97}$$

$$\Psi(z,x,y)=\Delta_{\mathbf{a}(-B')}\big(\lfloor \mathbf{a}(-B'),\mathbf{a}(R)\rfloor, \lfloor \mathbf{a}(-B'),\mathbf{a}(P)\rfloor\big) \mod 2\pi i, \tag{2.98}$$

$$\Psi(x,y,z)=\Delta_{\mathbf{a}(-C')}\big(\lfloor \mathbf{a}(-C'),\mathbf{a}(P)\rfloor, \lfloor \mathbf{a}(-C'),\mathbf{a}(Q)\rfloor\big) \mod 2\pi i, \tag{2.99}$$

where $\Psi(y,z,x)$, $\Psi(z,x,y)$, $\Psi(x,y,z)\in\mathbf{C}$ are defined by (2.53). □

Chapter 3

Hyperbolic Cone-Surfaces and McShane's Identity

3.1 Introduction

In this chapter we generalize McShane's identity as in Theorem 1.2 to the cases of hyperbolic cone-surfaces possibly with cusps and/or geodesic boundary, where all cone points are assumed to have cone angles $\leq \pi$, except for the case of one-cone torus where we allow the cone angle up to 2π. (See [14] and [51] for basic facts on cone-manifolds.) In particular, by applying the generalized identity to the orbifolds which are the quotients of the hyperbolic one-holed torus by its elliptic involution and of the hyperbolic genus two surface by its hyper-elliptic involution, we obtain generalizations of McShane's Weierstrass identities for the one-holed torus, and identities for the genus two surface which are also obtained by McShane using different methods in [31], [33] and [32]. We also give an interpretation of the identity in terms of complex lengths.

The ideas are related in spirit to those in [3] while the method of proof follows closely that of McShane's in [30]. The key points are that (i) the assumption that all cone angles are $\leq \pi$ implies that all non-trivial non-peripheral simple closed curves are essentially realizable as simple geodesics in their free (relative) homotopy classes; and that (ii) the Birman-Series Theorem in [5] on the sparsity of simple geodesics carries over to this case, in particular to simple geodesic rays emanating normally from a fixed boundary component.

It should be noted that our derivation shows that the assumption of discreteness of the holonomy group is unnecessary, and that it gives identities for all hyperbolic orbifold surfaces. We also show how the result can be formulated in terms of complex lengths (see Theorem 3.17) even though the situation we consider here is real. This is particularly useful and will be explored further in Chapters 6 and 4, where we show how this approach allows us to generalize McShane's identity to marked classical Schottky groups, and how the Markoff maps approach adopted by Bowditch in [6] can be generalized as well to give generalizations of McShane's identity for representations in $\mathrm{SL}(2, \mathbf{C})$ of the once-punctured torus group which satisfy certain conditions. This also leads to generalizations of Bowditch's variations in [7] of McShane's identity for complete hyperbolic 3-manifolds which are once-punctured torus bundles over the circle to identities for hyperbolic 3-manifolds obtained by performing hyperbolic Dehn surgery on such bundles with incomplete hyperbolic structures.

To state the most general form of our generalized McShane's identities (see Theorem 3.5), we need to introduce some new terminology. However, to let the reader get the flavor of the generalized identities, we first state the corresponding generalizations of Theorems 1.1 and 1.2.

Theorem 3.1 *Let T be either a hyperbolic one-cone torus where the single cone point has cone angle $\theta \in (0, 2\pi)$ or a hyperbolic one-hole torus where the single boundary geodesic has length $l > 0$. Then we have respectively*

$$\sum_{\gamma} 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp|\gamma|}\right) = \frac{\theta}{2}, \tag{3.1}$$

$$\sum_{\gamma} 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp|\gamma|}\right)=\frac{l}{2}, \tag{3.2}$$

where the sum in either case extends over all simple closed geodesics on T.

Theorem 3.2 *Let M be a compact hyperbolic cone-surface with a single cone point of cone angle $\theta \in (0,\pi]$ and without boundary or let M be a compact hyperbolic surface with a single boundary geodesic having length $l > 0$. Then we have respectively*

$$\sum 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right)=\frac{\theta}{2}, \tag{3.3}$$

$$\sum 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right)=\frac{l}{2}, \tag{3.4}$$

where the sum in either case extends over all unordered pairs of simple closed geodesics on M which bound with the cone point (respectively, the boundary geodesic) an embedded pair of pants.

Definition 3.3 By a **compact hyperbolic cone-surface** M we mean a compact topological surface M with hyperbolic cone structure where each boundary component is a smooth simple closed geodesic and there are a finite number of interior points which form all the cone points and cusps. The **geometric boundary**, ΔM, of M is the union of all cusps, cone points and geodesic boundary components. (Note that ΔM is different from the usual topological boundary ∂M when there are cusps or cone points.) Thus a **geometric boundary component** is either a cusp, a cone point, or a boundary geodesic. The **geometric interior** of M is $M - \Delta M$.

We consider a compact hyperbolic cone-surface $M = M(\Delta_0; k, \Theta, L)$ with k cusps $C_1, C_2, \cdots, C_k$, with m cone points $P_1, P_2, \cdots, P_m$, where the cone angle of P_i is $\theta_i \in (0, \pi]$, $i = 1, 2, \cdots, m$, and with n geodesic boundary components $B_1, B_2, \cdots, B_n$, where the length of B_i is $l_i > 0$, $i = 1, 2, \cdots, n$, together with an extra **distinguished** geometric boundary component Δ_0. Thus Δ_0 is either a cusp C_0 or a cone point P_0 of cone angle $\theta_0 \in (0, \pi]$ or a geodesic boundary component B_0 of length $l_0 > 0$. Note that in the above notation $\Theta = (\theta_1, \theta_2, \cdots, \theta_m)$ and $L = (l_1, l_2, \cdots, l_n)$. We exclude the case where M is a geometric pair of pants for we have only trivial identities in that case.

We allow that some, even all, of the cone angles θ_i are equal to π, $i = 0, 1, \cdots, m$. These are often the cases of particular interest. However, for clarity of exposition, quite often in proofs/statements of lemmas/theorems we shall first consider the case where all the cone angles are less than π and then point out the addenda that should be made when there are angle π cone points. The advantage of this assumption of the strict inequality is that every non-trivial, non-peripheral simple closed curve on such M can be realized as a smooth simple closed geodesic in its free homotopy class in the geometric interior of M under the given hyperbolic cone-structure (see §3.3 for the proof of this statement). We call a simple closed curve on M **peripheral** if it is freely homotopic on M to a geometric boundary component of M.

Definition 3.4 By a **generalized simple closed geodesic** on M we shall mean either (i) an *interior simple closed geodesic*, that is, a simple closed geodesic in the geometric interior of M; or (ii) a *degenerate interior simple closed geodesic*, that is, the double of a simple geodesic arc in the geometric interior of M connecting two angle π cone points; or (iii) a *geometric boundary component*, that is, a cusp or a cone point or a boundary geodesic. In particular, generalized simple closed geodesics of the first two kinds are called **interior generalized simple closed geodesics**.

For each pair of generalized simple closed geodesics α, β which bound with

Δ_0 an embedded geometric pair of pants we shall define in §3.2 a **gap function** $\mathrm{Gap}(\Delta_0;\alpha,\beta)$ when Δ_0 is a cone point or a boundary geodesic as well as a **normalized gap function** $\mathrm{Gap}'(\Delta_0;\alpha,\beta)$ when Δ_0 is a cusp.

Now we are in a position to state the most general (real) form of our generalization of McShane's identity to hyperbolic cone-surfaces.

Theorem 3.5 *Let M be a compact hyperbolic cone-surface with all cone angles in $(0,\pi]$. Then one has either*

$$\sum \mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{\theta_0}{2}, \tag{3.5}$$

when Δ_0 is a cone point of cone angle θ_0; or

$$\sum \mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{l_0}{2}, \tag{3.6}$$

when Δ_0 is a boundary geodesic of length l_0; or

$$\sum \mathrm{Gap}'(\Delta_0;\alpha,\beta)=\frac{1}{2}, \tag{3.7}$$

when Δ_0 is a cusp; where in each case the sum is taken over all pairs of generalized simple closed geodesics α,β on M which bound with Δ_0 an embedded pair of pants.

Remark 3.6 Note that Maryam Mirzakhani in [34] had generalized McShane's identities to hyperbolic surfaces with geodesic boundary and used it to calculate the Weil-Petersson volumes of the corresponding moduli spaces. There is obviously an overlap of her results with ours, in particular, the identities she obtains are equivalent to ours in the case of hyperbolic surfaces with boundary (see §3.8 for further explanations). In fact, her expressions in terms of the log function seems particularly well suited to her purpose of calculating the Weil-Petersson volumes. It also seems (as already observed by her in [34]) that her methods should extend fairly easily to cover the case of volumes of the moduli spaces of compact hyperbolic cone-surfaces with all cone angles bounded above by π, as defined and used in our context, and that the formulas she exhibited for the volumes should hold in this case as well, using the convention that a cone point of angle θ corresponds to a geometric boundary component with purely imaginary length θi.

Remark 3.7 (i) In the case of the hyperbolic one-cone torus, the theorem holds for $\theta_0 \in (0, 2\pi)$.

(ii) In the special cases where the geometric boundary ΔM is a single cone point or a single boundary geodesic Theorem 3.5 gives all the previously stated generalized identities in Theorems 3.1 and 3.2.

(iii) The cusp case (that is, Δ_0 is a cusp) is the limit case of the other cases as the cone angle θ_0 or the boundary geodesic length l_0 approaches 0, and the identity in the cusp case can indeed be derived from the first order infinitesimal of the identities of the other cases.

It is also interesting to note that McShane's Weierstrass identities can be obtained as special cases of our general Theorem 3.5 applied to the quotient of the once punctured torus by its elliptic involution and then lifting back to the torus. Thus we have the following generalized Weierstrass identities.

Corollary 3.8 *Let T be either a hyperbolic one-cone torus where the single cone point has cone angle $\theta \in (0, 2\pi)$ or a hyperbolic one-hole torus where the single boundary geodesic has length $l > 0$. Then we have respectively*

$$\sum_{\gamma \in \mathcal{A}} \tan^{-1}\left(\frac{\cos \frac{\theta}{4}}{\sinh \frac{|\gamma|}{2}}\right) = \frac{\pi}{2}, \tag{3.8}$$

$$\sum_{\gamma \in \mathcal{A}} \tan^{-1}\left(\frac{\cosh \frac{l}{4}}{\sinh \frac{|\gamma|}{2}}\right) = \frac{\pi}{2}, \tag{3.9}$$

where the sum in either case is taken over all the simple closed geodesics γ in a Weierstrass class $\mathcal{A}$.

Remark 3.9 McShane's original Weierstrass identity (1.3) then corresponds to the case $\theta = 0$ or $l = 0$ in the above two identities, noticing that

$$\tan^{-1}\left(\frac{1}{\sinh\frac{|\gamma|}{2}}\right) = \sin^{-1}\left(\frac{1}{\cosh\frac{|\gamma|}{2}}\right).$$

As further corollaries, there are the following weaker but neater identities, each of which is obtained by summing the three McShane's Weierstrass identities in the corresponding case.

Corollary 3.10 *Let T be a hyperbolic torus whose geometric boundary is either a single cusp, a single cone point of cone angle $\theta \in (0, 2\pi)$, or a single boundary geodesic of length $l > 0$. Then we have respectively*

$$\sum_{\gamma} \tan^{-1}\left(\frac{1}{\sinh\frac{|\gamma|}{2}}\right) = \frac{3\pi}{2}, \tag{3.10}$$

$$\sum_{\gamma} \tan^{-1}\left(\frac{\cos\frac{\theta}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{3\pi}{2}, \tag{3.11}$$

$$\sum_{\gamma} \tan^{-1}\left(\frac{\cosh\frac{l}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{3\pi}{2}, \tag{3.12}$$

where the sum in each case is taken over all the simple closed geodesics γ on T.

Remark 3.11 The above identity (3.12) was also obtained by McShane [32] using Wolpert's variation of length method. It seems likely his method can be extended to prove some of the other identities as well.

Genus two surface. Similarly, for a genus two closed hyperbolic surface M, one can consider the six identities on the quotient surface M/η where η is the unique hyper-elliptic involution on M (note that M/η is a closed hyperbolic orbifold of genus 0 with six cone angle π points, and we may choose any one of these cone points to be the distinguished geometric boundary component) and re-interpret them as Weierstrass identities on the original surface M (see also McShane [33] where the Weierstrass identities were obtained directly). Combining all the six Weierstrass identities for M, we then have the following very neat identity.

Theorem 3.12 *Let M be a genus two closed hyperbolic surface. Then*

$$\sum \tan^{-1} \exp\left(- \frac{|\alpha|}{4} - \frac{|\beta|}{2} \right) = \frac{3\pi}{2}, \tag{3.13}$$

where the sum is taken over all ordered pairs (α, β) of disjoint simple closed geodesics on M such that α is separating and β is non-separating.

Remark 3.13 This is the only case that we know of where McShane's identity extends in a nice way to a closed hyperbolic surface.

We observe that the above identity (3.13) for closed genus two surface M also extends to the case of quasifuchsian representations of $\pi_1(M)$. Let γ be an essential simple closed curve on M. For a fuchsian representation $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{R})$, more precisely, the conjugacy class of ρ by elements of $\mathrm{PSL}(2, \mathbf{C})$, the length $l_\rho(\gamma) > 0$ of γ in the corresponding hyperbolic genus two surface $\mathbb{H}^2/\rho(\pi_1(M))$ is given by

$$l_\rho(\gamma) = 2\cosh^{-1}(|\mathrm{tr}\rho(\gamma)|/2) = \cosh^{-1}[(\mathrm{tr}^2\rho(\gamma) - 2)/2\,] \in \mathbf{R}_{>0}.$$

More generally, for a quasifuchsian representation $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$, or the conjugacy class of ρ by elements of $\mathrm{PSL}(2, \mathbf{C})$, let $l_\rho(\gamma) \in \mathbf{C}$ be defined by

$$\cosh l_\rho(\gamma) = (\mathrm{tr}^2\rho(\gamma) - 2)/2 \tag{3.14}$$

and by analytic continuation starting from those $l_\rho(\gamma)$ for fuchsian representations $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{R})$. Since the quasifuchsian space $\mathcal{QF}$ of M (or of $\pi_1(M)$) is simply connected, $l_\rho(\gamma) \in \mathbf{C}$ is well-defined. Recall that $\mathcal{QF} = \mathcal{QF}(M)$ is the space of quasifuchsian representations of $\pi_1(M)$ in $\mathrm{PSL}(2,\mathbf{C})$, modulo conjugation by elements in $\mathrm{PSL}(2,\mathbf{C})$. It is easy to see that $\Re\, l_\rho(\gamma) > 0$ since, for fixed γ, $\Re l_\rho(\gamma)$ is a nowhere vanishing continuous function of ρ on $\mathcal{QF}$ and is positive for ρ in the fuchsian subspace of $\mathcal{QF}$.

Addendum 3.14 *For a quasifuchsian representation* $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbf{C})$ *for the closed genus two surface* M*, we have*

$$\sum \tan^{-1} \exp\left(-\frac{l_\rho(\alpha)}{4} - \frac{l_\rho(\beta)}{2}\right) = \frac{3\pi}{2} \mod \pi, \tag{3.15}$$

where the sum is taken over all the ordered pairs $[\alpha],[\beta]$ *of free homotopy classes of disjoint unoriented essential simple closed curves* α,β *on* M *such that* α *is separating and* β *is non-separating.*

Remark 3.15 Here the inverse function $\tan^{-1} : \mathbf{C}\backslash\{\pm i\} \to \mathbf{C}$ is defined so that it has images with real parts in $(-\pi/2, \pi/2)$. Explicitly, we have

$$\tan^{-1} x = \frac{1}{2i} \log \frac{1+xi}{1-xi}, \tag{3.16}$$

where the log function takes its principal value, namely, for any nonzero $u \in \mathbf{C}$, $\log u$ has imaginary part in $(-\pi, \pi]$.

Complexified reformulation of Theorem 3.5. In the statement of Theorem 3.5 we did not write down the explicit expression for the gap functions due to their "case by case" nature as can be seen in §3.2. The cone points and boundary geodesics as geometric boundary components seem to have different roles in the series in the generalized identities, hence making the identities not in a unified form. This difference can, however, be removed by assigning purely imaginary length to a cone point as a geometric boundary component.

Definition 3.16 For each generalized simple closed geodesic δ, its **complex length** $|\delta|$ is defined as:

(i) $|\delta| = 0$ if δ is a cusp;

(ii) $|\delta| = \theta i$ if δ is a cone point of angle $\theta \in (0, \pi]$; and

(iii) $|\delta| = l$ if δ is a boundary geodesic or an interior generalized simple closed geodesic of length $l > 0$.

Then we can reformulate the generalized McShane's identities in Theorem 3.5 in terms of the complex lengths of the geometric boundary components of a hyperbolic cone-surface as follows.

Theorem 3.17 *Let M be a compact hyperbolic cone-surface with all cone angles in $(0, \pi]$, and let all its geometric boundary components be $\Delta_0, \Delta_1, \cdots, \Delta_N$ with complex lengths $L_0, L_1, \cdots, L_N$ respectively. Then*

$$\sum_{\alpha,\beta} 2\tanh^{-1}\left(\frac{\sinh\frac{L_0}{2}}{\cosh\frac{L_0}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right) + \sum_{j=1}^{N}\sum_{\beta}\tanh^{-1}\left(\frac{\sinh\frac{L_0}{2}\sinh\frac{L_j}{2}}{\cosh\frac{|\beta|}{2} + \cosh\frac{L_0}{2}\cosh\frac{L_j}{2}}\right) = \frac{L_0}{2}, \tag{3.17}$$

if Δ_0 is a cone point or a boundary geodesic; and

$$\sum_{\alpha,\beta}\frac{1}{1+\exp\frac{|\alpha|+|\beta|}{2}} + \sum_{j=1}^{N}\sum_{\beta}\frac{1}{2}\frac{\sinh\frac{L_j}{2}}{\cosh\frac{|\beta|}{2} + \cosh\frac{L_j}{2}} = \frac{1}{2}, \tag{3.18}$$

if Δ_0 is a cusp; where in either case the first sum is taken over all (unordered) pairs of generalized simple closed geodesics α, β on M which bound with Δ_0 an embedded pair of pants on M (note that one of α, β might be a geometric boundary component) and the sub-sum in the second sum is taken over all interior simple closed geodesics β which bounds with Δ_j and Δ_0 an embedded pair of pants on M.

Furthermore, the series in (3.17) and (3.18) converge absolutely.

Remark 3.18 Using the functions G and S defined in §2.2, the identity (3.17) can also be expressed as

$$\sum_{\alpha,\beta} G\left(\frac{L_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + \sum_{j=1}^{n}\sum_{\beta} S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right) = \frac{L_0}{2}. \tag{3.19}$$

The rest of this chapter is organized as follows. In §3.2 we define the gap functions used in Theorem 3.5 for the various cases. In §3.3 we deal with the problem of realization of simple closed curves by geodesics, and show that the assumption that all cone angles are less than or equal to π is essential. In §3.4 we analyze the so-called Δ_0-geodesics, that is, the geodesics emanating orthogonally from Δ_0, and determine all the gaps between all simple-normal Δ_0-geodesics. In §3.5 we calculate the gap function which is the width of a combined gap measured suitably. In §3.6 we generalize the Birman-Series Theorem (which states that the point set of all complete geodesics with bounded self intersection numbers on a compact hyperbolic surface has Hausdorff dimension 1) to the case of compact hyperbolic cone-surfaces with all cone angles less than or equal to π. We prove the theorems of this chapter in §3.7. Finally in §3.8 we interpret the geometric meanings of the summands in the complexified generalized McShane's identity.

3.2 Definition of the Gap functions

In this section, for a compact hyperbolic cone-surface $M = M(\Delta_0; k, \Theta, L)$ with all cone angles $\leq \pi$, we define

- the gap function $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ (when Δ_0 is a cone point or a boundary geodesic) and
- the normalized gap function $\mathrm{Gap}'(\Delta_0; \alpha, \beta)$ (when Δ_0 is a cusp)

where α, β are generalized simple closed geodesics on M which bound with Δ_0 a geometric pair of pants.

Definition 3.19 Throughout this chapter we use $|\alpha|$ to denote the length of α when α is an interior generalized simple closed geodesic or a boundary geodesic. In particular, **the length of a degenerate simple closed geodesic** α (that is,

α is the double cover of a simple geodesic arc α' which connects two angle π cone points) is defined as *twice* the length of α': $|\alpha| = 2|\alpha'|$.

Recall that an interior generalized simple closed geodesic is either a simple closed geodesic in the geometric interior of M or a degenerate simple closed geodesic on M which is the double cover of a simple geodesic arc which connects two angle π cone points.

Case 0. Δ_0 is a cusp.

Subcase 0.1. Both α and β are interior generalized simple closed geodesics.

In this case

$$\mathrm{Gap}'(\Delta_0; \alpha, \beta) = \frac{1}{1 + \exp \frac{1}{2}(|\alpha| + |\beta|)}. \tag{3.20}$$

Subcase 0.2. One of α, β, say α, is a boundary geodesic and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}'(\Delta_0; \alpha, \beta) = \frac{1}{2} - \frac{1}{2} \frac{\sinh \frac{|\beta|}{2}}{\cosh \frac{|\alpha|}{2} + \cosh \frac{|\beta|}{2}}. \tag{3.21}$$

Subcase 0.3. One of α, β, say α, is a cone point of cone angle $\varphi \in (0, \pi]$ and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}'(\Delta_0; \alpha, \beta) = \frac{1}{2} - \frac{1}{2} \frac{\sinh \frac{|\beta|}{2}}{\cos \frac{\varphi}{2} + \cosh \frac{|\beta|}{2}}. \tag{3.22}$$

Subcase 0.4. One of α, β, say α, is also a cusp and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}'(\Delta_0;\alpha,\beta)=\frac{1}{2}-\frac{1}{2}\frac{\sinh\frac{|\beta|}{2}}{1+\cosh\frac{|\beta|}{2}}=\frac{1}{1+\exp\frac{1}{2}|\beta|}, \tag{3.23}$$

which is the common value of $\mathrm{Gap}(\Delta_0;\alpha,\beta)$ in Subcases 0.1 through 0.3 when $|\alpha|=0$.

Case 1. Δ_0 is a cone point of cone angle $\theta\in(0,\pi]$.

Subcase 1.1. Both α and β are interior generalized simple closed geodesics.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right). \tag{3.24}$$

Subcase 1.2. One of α,β, say α, is a boundary geodesic and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{\theta}{2}-\tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.25}$$

Subcase 1.3. One of α,β, say α, is a cone point of cone angle $\varphi\in(0,\pi]$ and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{\theta}{2}-\tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2}+\cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.26}$$

Note that there is no gap when $\theta=\varphi=\pi$.

Subcase 1.4. One of α,β, say α, is a cusp and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\begin{aligned}\mathrm{Gap}(\Delta_0;\alpha,\beta) &= 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\exp\frac{|\beta|}{2}}\right) &(3.27)\\ &= \frac{\theta}{2}-\tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{1+\cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right), &(3.28)\end{aligned}$$

which is the common value of $\mathrm{Gap}(\Delta_0;\alpha,\beta)$ in Subcases 1.1 through 1.3 when $|\alpha| = 0$.

Case 2. Δ_0 is a boundary geodesic of length $l > 0$.

Subcase 2.1. Both α and β are interior generalized simple closed geodesics.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right). \tag{3.29}$$

Subcase 2.2. One of α, β, say α, is a boundary geodesic and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.30}$$

Subcase 2.3. One of α, β, say α, is a cone point of cone angle $\varphi \in (0, \pi]$ and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2}+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.31}$$

Subcase 2.4. One of α, β, say α, is a cusp and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\text{Gap}(\Delta_0;\alpha,\beta) = 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\beta|}{2}}\right) \tag{3.32}$$

$$= \frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{1+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right), \tag{3.33}$$

which is the common value of $\text{Gap}(\Delta_0;\alpha,\beta)$ in Subcases 2.1 through 2.3 when $|\alpha|=0$.

3.3 Realizing simple curves by geodesics

In this section we consider the problem of realizing essential simple curves in their free (relative) homotopy classes by geodesics on a compact hyperbolic cone-surface M with all cone angles smaller than π. We show that each essential simple closed curve in the geometric interior of M can be realized uniquely in its free homotopy class (where the homotopy takes place in the geometric interior of M) as either a geometric boundary component or a simple closed geodesic in the geometric interior of M. We also show that each essential simple arc which connects geometric boundary components of M can be realized uniquely in its free relative homotopy class (where the homotopy takes place in the geometric interior of M and the endpoints slide on the same geometric boundary components) as a simple geodesic arc which is normal to the geometric boundary components involved. We also make addenda for the cases when there are angle π cone points.

Theorem 3.20 *Let M be a compact hyperbolic cone-surface with all cone angles less than π.*

(i) *If c is an essential non-peripheral simple closed curve in the geometric interior of M, then there is a unique simple closed geodesic in the free homotopy class of c in the geometric interior of M.*

(ii) *If c is an essential simple arc which connects geometric boundary components, then there is a unique simple normal geodesic arc in the free relative homotopy class of c in the geometric interior of M with endpoints varying on the respective geometric boundary components.*

Addendum 3.21 *If in addition M has some cone angles equal to π, then*

(i) *in Theorem 3.20(i), if the simple closed curve c bounds with two angle π cone points an embedded pair of pants, then the geodesic realization for c is the double cover of the simple geodesic arc which connects these two angle π cone points and is homotopic (relative to boundary) to a simple arc lying wholly in the pair of pants;*

(ii) *in Theorem 3.20(ii), if the simple arc c connects a geometric boundary component Δ to itself and bounds together with Δ and an angle π cone point P an embedded cylinder then the geodesic realization for c is the double cover of the normal simple geodesic arc which connects Δ to P and is homotopic (relative to boundary) to a simple arc lying wholly in the cylinder.*

The simple geodesic in Theorem 3.20 and Addendum 3.21 is called the **geodesic realization** of the given simple curve in the respective homotopy class.

The proof is a well-known use of the Arzela-Ascoli Theorem as used in [11] with slight modifications.

Proof. (i) Suppose c is an essential non-peripheral simple closed curve in the geometric interior of M, parametrized on $[0,1]$ with constant speed. Let the length of c be $|c| > 0$. Then for each cusp C_i, there is an embedded neighborhood $N(C_i)$ of C_i on M, bounded by a horocycle, such that each non-peripheral simple closed curve c' in the geometric interior of M with length $\leq |c|$ cannot enter $N(C_i)$; for otherwise c' would be either peripheral or of infinite length. Now let M_0 be M with all the chosen horocycle neighborhoods $N(C_i)$ removed. Then M_0

is a compact metric subspace of M with the induced hyperbolic metric. Now choose a sequence of simple closed curves $\{c_k\}_1^\infty$, where each c_k is parametrized on $[0,1]$ with constant speed, in the free homotopy class of c (where the homotopy takes place in the geometric interior of M) such that their lengths $\leq |c|$ and are decreasing with limit the infimum of the lengths of the simple closed curves in the free homotopy class of c. Then by the Arzela-Ascoli Theorem (c.f. [11] Theorem A.19, page 429) there is a subsequence of $\{c_k\}_1^\infty$, assumed to be $\{c_k\}_1^\infty$ itself, such that it converges uniformly to a closed curve γ in M_0. It is clear that γ is a geodesic since it is locally minimizing. Note that γ is away from cusps by the choice of $\{c_k\}_1^\infty$. We claim that γ cannot pass through any cone point. For otherwise, suppose γ passes through a cone point P. Then for sufficiently large k, c_k can be modified in the free homotopy class of c to have length smaller than $|\gamma|$ (since the cone point has cone angle smaller than π), which is a contradiction. Thus γ must be a closed geodesic in the geometric interior of M. The uniqueness and simplicity of γ can be proved by an easy argument since there are no bi-gons in the hyperbolic plane.

(ii) For an essential simple arc c in the geometric interior of M which connects geometric boundary components, the proof of case (i) applies without modifications when none of the involved geometric boundary components is a cusp. Now suppose at least one of the involved geometric boundary components is a cusp. For definiteness let us assume that c connects cusps C_1 to C_2. Remove suitable horocycle neighborhoods $N(C_1)$ and $N(C_2)$ respectively for C_1 and C_2 where the two horocycles are H_1 and H_2 respectively. Choose a simple arc c_0 in $M - N(C_1) \cup N(C_2)$ which goes along c and connects H_1 to H_2. Let the length of c_0 be $|c_0| > 0$. Now for all other cusps C_i, there is a horocycle neighborhood $N(C_i)$ of C_i on M such that each non-peripheral simple closed curve c' in the geometric interior of M with length $\leq |c_0|$ cannot enter $N(C_i)$. Again let M_0 be M with all the chosen horocycle neighborhoods $N(C_i)$ removed. By the same argument as in

(i) we have a shortest simple geodesic realization γ_0 in the free relative homotopy class of c_0 in M_0 and c_0 does not pass through any cone point. Hence γ_0 must be perpendicular to both H_1 and H_2 at its endpoints. Thus γ_0 can be extended to a geodesic arc connecting C_1 to C_2. Again simplicity and uniqueness can be proved easily. □

The addendum can be verified easily since the realizations as degenerate simple geodesics in the respective cases are already known.

Remark 3.22 We make a remark that the following fact, whose proof is easy and hence omitted, is implicitly used through out this chapter: *On a hyperbolic cone-surface for each cone point P with angle less than π there is a cone region $N(P)$, bounded by a suitable circle centered at P, such that if a geodesic γ goes into $N(P)$ then either γ will go directly to the cone point P (hence perpendicular to all the circles centered at P) or γ will develop a self-intersection in $N(P)$.* The analogous fact for a cusp is used in [5], [21] and [30].

3.4 Gaps between simple-normal Δ_0-geodesics

We fix throughout this section a compact hyperbolic cone-surface M with a distinguished geometric boundary component Δ_0.

The Δ_0-geodesics. We make this terminology for the class of geodesics on M that we are interested in.

Definition 3.23 A **Δ_0-geodesic** on M is an *oriented* geodesic ray which starts from Δ_0 (and is perpendicular to it if Δ_0 is a boundary geodesic) and is fully developed, that is, it develops forever until it terminates at a geometric boundary component.

Definition 3.24 A Δ_0-geodesic is either non-simple or simple. It is regarded as **non-simple** if and only if it intersects itself transversely at an interior point (a cone point is not treated as an interior point) or at a point on a boundary geodesic. We shall see later that somewhat surprisingly, for our purposes, the set of non-simple Δ_0-geodesics is easier to analyze than the set of simple Δ_0-geodesics.

Definition 3.25 A simple Δ_0-geodesic is either normal or not-normal in the following sense: A simple Δ_0-geodesic is **normal** if when fully developed either it never intersects any boundary geodesic or it intersects (hence terminates at) a boundary geodesic perpendicularly. Note that a simple-normal Δ_0-geodesic may terminate at a cusp or a cone point. Thus a simple Δ_0-geodesic is **not-normal** if and only if it intersects a boundary geodesic (which might be Δ_0 itself) obliquely.

We shall analyze the structure of all non-simple and simple-not-normal Δ_0-geodesics and show that they form gaps between simple-normal Δ_0-geodesics. Furthermore, the naturally measured widths of the suitably combined gaps are given by the Gap functions defined before in §3.2.

Note that McShane [30] analyzes directly all simple Δ_0-geodesics (there are no simple-not-normal Δ_0-geodesics in his case since there are no geodesic boundary components). Our analysis of the structure of Δ_0-geodesics is a bit different from and actually simpler than that of McShane's. We shall analyze all non-simple and simple-not-normal Δ_0-geodesics and show that they arise in the nice ways we expect.

Parametrizing Δ_0-geodesics. First we parametrize all the Δ_0-geodesics and define the widths for gaps between simple-normal Δ_0-geodesics.

To parametrize Δ_0-geodesics note that the set of all Δ_0-geodesics can be identified naturally with the unit tangent cone of Δ_0 if Δ_0 is a cone point; with Δ_0 itself if Δ_0 is a boundary geodesic; and with a suitably small horocycle about Δ_0

when Δ_0 is a cusp. The set then has a natural measure and topology from this identification.

Definition 3.26 If Δ_0 is a cusp let $\mathcal{H}$ be a suitably chosen small horocycle as in McShane [30], see also [21]. If Δ_0 is a cone point let $\mathcal{H}$ be a suitably chosen small circle centered at Δ_0. Let $\mathcal{H}$ be Δ_0 itself if Δ_0 is a boundary geodesic.

Then each Δ_0-geodesic has a unique first intersection point with $\mathcal{H}$, which is the starting point when Δ_0 is a boundary geodesic. Note that the Δ_0-geodesics intersect $\mathcal{H}$ orthogonally at their first intersection points. Thus the set of all Δ_0-geodesics can be naturally identified with $\mathcal{H}$, with the induced topology and measure.

Definition 3.27 Let $\mathcal{H}_{\mathtt{ns}}$, $\mathcal{H}_{\mathtt{sn}}$, $\mathcal{H}_{\mathtt{snn}}$ be the point sets of the *first* intersections of $\mathcal{H}$ with respectively all non-simple, all simple-normal, all simple-not-normal Δ_0-geodesics.

Proposition 3.28 *The set $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$ is an open subset of $\mathcal{H}$ and hence $\mathcal{H}_{\mathtt{sn}}$ is a closed subset of $\mathcal{H}$.*

Proof. It is easy to see that the condition that either self-intersecting or ending obliquely at a boundary component is an open condition. □

For the open subset $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$ of $\mathcal{H}$, we determine its structure by determining its maximal open intervals (which are the gaps we are looking for). By a generalized Birman–Series Theorem (see §3.6), the subset $\mathcal{H}_{\mathtt{sn}}$ of $\mathcal{H}$ has Hausdorff dimension 0, and hence Lebesgue measure 0. Therefore the open subset $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$ of $\mathcal{H}$ has full measure, and our generalized McShane's identities (3.5)-(3.7) follow immediately.

Definition 3.29 A $[\Delta_0, \Delta_0]$-geodesic, γ, is an (oriented) Δ_0-geodesic which terminates at Δ_0 perpendicularly. (With the orientation one can refer to its starting point and ending point.) Hence the same geodesic with reversed orientation is also a $[\Delta_0, \Delta_0]$-geodesic, denoted by $-\gamma$.

Definition 3.30 We say that a $[\Delta_0, \Delta_0]$-geodesic γ is a **degenerate simple $[\Delta_0, \Delta_0]$-geodesic** if Δ_0 is not a π cone point, and γ is the double cover of a simple geodesic arc which connects Δ_0 to an angle π cone point, that is, γ reaches the angle π cone point along the simple geodesic arc and goes back to Δ_0 along the same arc. Note that in this case $\gamma = -\gamma$.

The main gaps. We show that each non-degenerate simple $[\Delta_0, \Delta_0]$-geodesic γ determines two maximal open intervals of $\mathcal{H}_{\tt ns} \cup \mathcal{H}_{\tt snn}$ as follows. (Their union is the *main gap*, defined later, determined by γ.)

Consider the configuration $\gamma \cup \mathcal{H}$. Assume γ is non-degenerate and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the two sub-arcs with endpoints inclusive that γ divides $\mathcal{H}$ into. Note that γ intersects $\mathcal{H}$ twice (if $\mathcal{H}$ is taken to be a suitably small circle about Δ_0 when Δ_0 is a cone point). Let γ_0 be the sub-arc of γ between the two intersection points. Thus we have two simple closed curves $\mathcal{H}_1 \cup \gamma_0$ and $\mathcal{H}_2 \cup \gamma_0$ on M. Their geodesic realizations are disjoint generalized simple closed geodesics, denoted α, β respectively (except when M is a hyperbolic torus with a single geometric boundary component, in which case $\alpha = \beta$). Note that α, β bound with Δ_0 an embedded geometric pair of pants, denoted $\mathcal{P}(\gamma)$, on M.

Let δ_α be the simple Δ_0-geodesic arc in $\mathcal{P}(\gamma)$ which terminates at α and is normal to α. Similarly, let δ_β be the simple Δ_0-geodesic arc in $\mathcal{P}(\gamma)$ which terminates at β and is normal to β. Let $[\alpha, \beta]$ be the simple geodesic arc in $\mathcal{P}(\gamma)$ which connects α and β and is normal to them. See Figure 3.1.

Cutting $\mathcal{P}(\gamma)$ along $\delta_\alpha, \delta_\beta$ and $[\alpha, \beta]$ one obtains two pieces; let the one which contains the initial part of γ be denoted $\mathcal{P}^+(\gamma)$. There are two simple Δ_0-geodesics, γ_α and γ_β, in $\mathcal{P}(\gamma)$ such that they are asymptotic to α and β re-

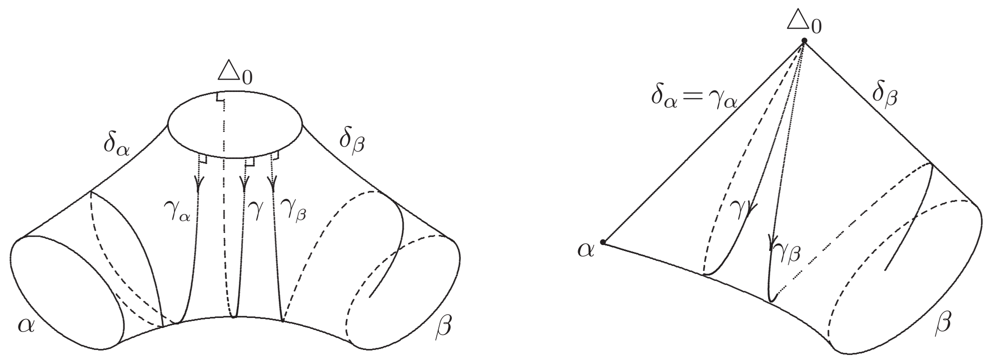

Figure 3.1:

spectively, and such that their initial parts are contained in $\mathcal{P}^+(\gamma)$. See Figure 3.1. Then we have

Lemma 3.31 *Each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and γ or between γ and γ_β is non-simple or simple-not-normal.*

The union of these two gaps between simple-normal Δ_0-geodesics formed by non-simple and simple-not-normal Δ_0-geodesics is called the **main gap** determined by γ.

Lemma 3.31 can be proved easily using a suitable model of the hyperbolic plane. The idea is that a Δ_0-geodesic ray whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and γ will not intersect γ_α or γ directly, so it must come back to first intersect either itself or Δ_0, hence is either non-simple or simple but not-normal (that is, intersecting Δ_0 obliquely). More precisely, if Δ_0 is a cusp or a cone point all the Δ_0-geodesics in the lemma are non-simple, while if Δ_0 is a boundary geodesic then there is a (critical) Δ_0-geodesic, ρ_γ, whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and γ such that ρ_γ is non-simple and its only self-intersection is at its starting point on Δ_0 (and hence terminates there) and it has the property that each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and ρ_γ is non-simple, whereas each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between ρ_γ and γ is simple-not-normal terminating at Δ_0. There is a similar dichotomy for the Δ_0-geodesics whose initial parts lie in $\mathcal{P}^+(\gamma)$ between γ and γ_β.

The extra gaps. Now suppose one of α, β, say α, is a boundary geodesic. Then there are two simple Δ_0-geodesics in $\mathcal{P}(\gamma)$ which are asymptotes to α. They are γ_α and $(-\gamma)_\alpha$.

The following lemma tells us that there is an **extra gap** determined by γ in $\mathcal{P}^+(\gamma)$ between simple-normal Δ_0-geodesics formed by simple-not-normal Δ_0-geodesics.

Lemma 3.32 *Each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between δ_α and γ_α is simple-not-normal.*

This is almost self-evident from the geometry of the pair of pants $P(\gamma)$, and the proof is similar to that of Lemma 3.31.

There is a similar and symmetric picture for the Δ_0-geodesics whose initial parts lie in $\mathcal{P}^-(\gamma) = \mathcal{P}^+(-\gamma)$.

Hence, for a non-degenerate $[\Delta_0, \Delta_0]$-geodesic γ, there are two main gaps determined respectively by γ and $-\gamma$ in the geometric pair of pants $\mathcal{P}(\gamma) = \mathcal{P}(-\gamma)$,. Furthermore, if (exactly) one of α, β is a boundary geodesic then there are two extra gaps determined by γ and $-\gamma$ respectively.

Remark 3.33 The case of a degenerate simple $[\Delta_0, \Delta_0]$-geodesic γ is handled in a similar way. Recall that γ is the double cover of a Δ_0-geodesic arc δ from Δ_0 to an angle π cone point α. Then there is a simple closed curve β', which is the boundary of a suitable regular neighborhood of $\Delta_0 \cup \delta$ on M, such that β' bounds with Δ_0 and α an embedded (topological) pair of pants. (i) If Δ_0 is not itself an angle π cone point, then β' can be realized as an interior generalized simple closed geodesic β which bounds with Δ_0 and α an embedded pair of pants $\mathcal{H}(\Delta_0, \alpha, \beta)$ on M and we can carry out the analysis as above with suitable modifications. In this case there are two main gaps, between γ and each of the two Δ_0-geodesics which are asymptotic to β in $\mathcal{H}(\Delta_0, \alpha, \beta)$. We say that one of the two main gaps is determined by γ and the other by $-\gamma$ although $\gamma = -\gamma$ in this case. (ii) If Δ_0 is itself an angle π cone point then in this case γ determines no gaps at all.

Definition 3.34 The **width** of an open subinterval $\mathcal{H}'$ of $\mathcal{H}$ is defined respectively as:

(i) Δ_0 is a cusp: the normalized parabolic measure, that is, the ratio of the Euclidean length of $\mathcal{H}'$ to the Euclidean length of $\mathcal{H}$;

(ii) Δ_0 is a cone point: the elliptic measure, that is, the angle (measured in radians) that $\mathcal{H}'$ subtends with respect to the cone point Δ_0;

(iii) Δ_0 is a boundary geodesic: the hyperbolic measure, that is, the hyperbolic length of $\mathcal{H}'$ (recall that in this case $\mathcal{H}$ is the same as the distinguished boundary geodesic Δ_0).

Definition 3.35 The **combined gap** between simple-normal Δ_0-geodesics determined by γ is the union of the main gap and the extra gap (if there is any) determined by γ. The **gap function** $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ where Δ_0 is a cone point or boundary geodesic or the **normalized gap function** $\mathrm{Gap}'(\Delta_0; \alpha, \beta)$ where Δ_0 is a cusp is defined as the total width of the combined gap determined by γ, which is the same as the total width of the combined gap determined by $-\gamma$.

The calculations of the explicit formulas of these gap functions will be carried out case by case in §3.5.

That's all. On the other hand, the following Lemma 3.36 shows that the non-simple and simple-not-normal Δ_0-geodesics obtained above are *all* the non-simple and simple-not-normal Δ_0-geodesics.

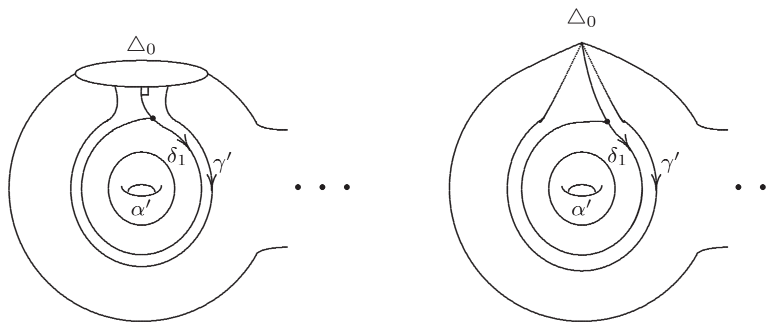

Figure 3.2:

Lemma 3.36 *Each non-simple or simple-not-normal Δ_0-geodesic lies in a main gap or an extra gap determined by some $[\Delta_0, \Delta_0]$-geodesic γ.*

Proof. First let δ be a non-simple Δ_0-geodesic, with its first self-intersection point Q, where Q lies in the geometric interior of M or in Δ_0 when Δ_0 is a boundary geodesic. Let δ_1 be the part of δ from starting point to Q; note that δ_1 has the shape of a lasso. Then in the boundary of a suitable regular neighborhood of δ_1 there is a simple arc γ' which connects Δ_0 to itself and is disjoint from δ_1 (except at Δ_0 when Δ_0 is a cone point); there is also a simple closed curve α' which is freely homotopic to the loop part of δ_1. See Figure 3.2. Let γ, α be the generalized simple closed geodesics on M which realize γ', α' in their respective free (relative) homotopy classes in the geometric interior of M. An easy geometric argument shows that α is disjoint from δ_1 and that γ is also disjoint from δ_1 except at Δ_0 when Δ_0 is a cone point or a cusp. Furthermore, γ and α cobound (together with Δ_0 when Δ_0 is a boundary geodesic) an embedded cylinder which contains δ_1. Hence the point in $\mathcal{H}$ which corresponds to the Δ_0-geodesic δ lies in the main gap determined by γ. See Figure 3.3.

Next let δ be a simple-not-normal Δ_0-geodesic which terminates at Δ_0 itself; in this case Δ_0 is a boundary geodesic and $\mathcal{H}$ is Δ_0 itself. Then the boundary of a suitably chosen regular neighborhood of $\delta \cup \mathcal{H}$ consist of two disjoint simple

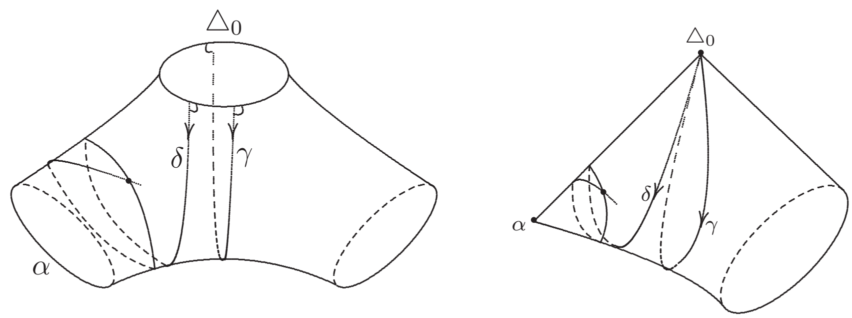

Figure 3.3:

closed curves in the geometric interior of M. Let their geodesic realizations be (disjoint) generalized simple closed geodesics α and β. Then α, β bound with Δ_0 an embedded pair of pants which contains δ in a main gap determined by the $[\Delta_0, \Delta_0]$-geodesic γ which is the geodesic realization of δ in its free relative homotopy class.

Finally let δ be a simple-not-normal Δ_0-geodesic which terminates at a boundary geodesic Δ_1 which is different from Δ_0. The boundary of a suitably chosen regular neighborhood of $\delta \cup \Delta_1$ on M is a simple arc connecting Δ_0 to itself and is disjoint from δ. Its geodesic realization is a $[\Delta_0, \Delta_0]$-geodesic, γ, which is disjoint from δ. Now Δ_1, γ bound with Δ_0 an embedded cylinder which contains δ. Hence δ lies in the extra gap determined by γ or $-\gamma$. □

3.5 Calculation of the gap functions

In this section we calculate the gap function $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ when Δ_0 is a cone point or a boundary geodesic; it is by definition the width of the combined gap determined by a simple $[\Delta_0, \Delta_0]$-geodesic γ on M.

Recall that α, β are the generalized simple closed geodesics determined by γ

and $\mathcal{P}(\gamma)$ is the geometric pair of pants that α, β bound with Δ_0 on M.

Case 1. Δ_0 is a cone point of cone angle $\theta \in (0, \pi]$.

In this case the width of the main gap determined by γ is the angle between γ_α and γ_β.

Let x be the angle between δ_α and γ_α and let y be the angle between δ_β and γ_β.

Subcase 1.1. Both α and β are interior generalized simple closed curves.

In this case the width of the combined gap determined by γ is the angle between γ_α and γ_β and is equal to $\frac{\theta}{2} - (x+y)$.

By a formula in Fenchel [18] VI.3.2 (line 10, page 87),

$$\sinh|\delta_\alpha| = \frac{\cosh\frac{|\beta|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\alpha|}{2}}{\sin\frac{\theta}{2}\sinh\frac{|\alpha|}{2}}, \tag{3.34}$$

$$\sinh|\delta_\beta| = \frac{\cosh\frac{|\alpha|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}. \tag{3.35}$$

Hence

$$\tan x = \frac{1}{\sinh|\delta_\alpha|} = \frac{\sin\frac{\theta}{2}\sinh\frac{|\alpha|}{2}}{\cosh\frac{|\beta|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\alpha|}{2}}, \tag{3.36}$$

$$\tan y = \frac{1}{\sinh|\delta_\beta|} = \frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}. \tag{3.37}$$

From these one can derive that

$$\tan(x+y) = \frac{\sin\frac{\theta}{2}\sinh\frac{|\alpha|+|\beta|}{2}}{1 + \cos\frac{\theta}{2}\cosh\frac{|\alpha|+|\beta|}{2}} \tag{3.38}$$

and hence that

$$\tan\frac{x+y}{2} = \tan\frac{\theta}{4}\tanh\frac{|\alpha|+|\beta|}{4}. \tag{3.39}$$

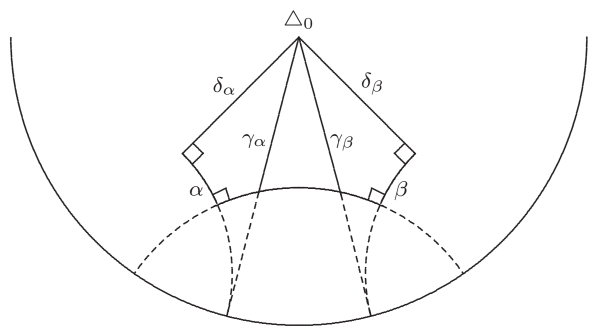

Figure 3.4: Subcases 1.1 and 1.2

Thus

$$\tan\left(\frac{\theta}{4} - \frac{x+y}{2}\right) = \frac{\tan\frac{\theta}{4}\left(1 - \tanh\frac{|\alpha|+|\beta|}{4}\right)}{1 + \tan^2\frac{\theta}{4}\tanh\frac{|\alpha|+|\beta|}{4}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp\frac{|\alpha|+|\beta|}{2}}. \tag{3.40}$$

Hence in this case we have

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = \frac{\theta}{2} - (x+y) = 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right).$$

Subcase 1.2. α is a boundary geodesic and β is an interior generalized simple closed geodesic.

In this case the width of the combined gap determined by γ is the angle between δ_α and γ_β and is equal to $\frac{\theta}{2} - y$. Hence by (3.37) we have

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = \frac{\theta}{2} - \tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.41}$$

Subcase 1.3. α is a cone point of cone angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic.

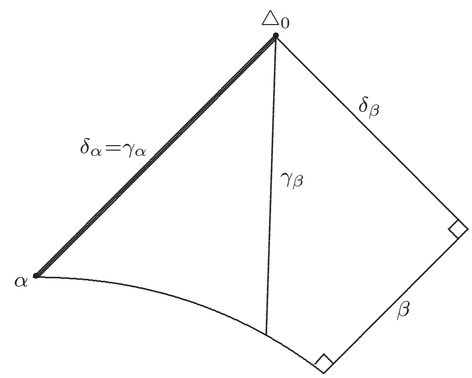

Figure 3.5: Subcase 1.3

Note that in this case γ_α coincides with δ_α and hence $x = 0$. Therefore the width of the combined gap determined by γ is the angle between δ_α and γ_β and is equal to $\frac{\theta}{2} - y$.

Now by a formula in Fenchel [18] VI.3.3 (line 13, page 88),

$$\sinh|\delta_\beta| = \frac{\cos\frac{\varphi}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}. \tag{3.42}$$

Hence

$$\tan y = \frac{1}{\sinh|\delta_\beta|} = \frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}. \tag{3.43}$$

Thus in this case we have

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{\theta}{2} - \tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.44}$$

Case 2. Δ_0 is a boundary geodesic of length $l > 0$.

In this case the width of the main gap determined by γ is the distance between γ_α and γ_β along Δ_0.

Let x be the distance between δ_α and γ_α along Δ_0 and let y be the distance between δ_β and γ_β along Δ_0.

We shall see that all calculations in this case are parallel to those in Case 1.

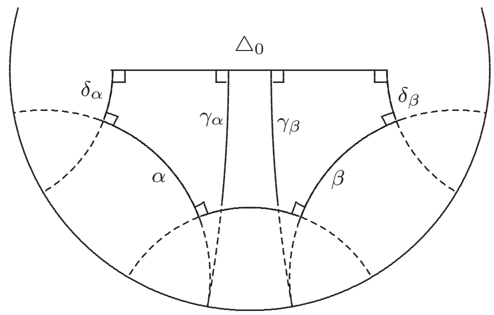

Figure 3.6: Subcases 2.1 and 2.2

Subcase 2.1. Both α and β are interior generalized simple closed curves.

In this case the width of the combined gap determined by γ is the distance between γ_α and γ_β along Δ_0 and is equal to $\frac{l}{2} - (x + y)$.

By the cosine rule for right angled hexagons on the hyperbolic plane (c.f. Fenchel [18] VI.3.1, page 86, or Beardon [4] Theorem 7.19.2, page 161),

$$\cosh|\delta_\alpha| = \frac{\cosh\frac{|\beta|}{2} + \cosh\frac{l}{2}\cosh\frac{|\alpha|}{2}}{\cosh\frac{l}{2}\sinh\frac{|\alpha|}{2}}, \tag{3.45}$$

$$\cosh|\delta_\beta| = \frac{\cosh\frac{|\alpha|}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}. \tag{3.46}$$

Hence

$$\tanh x = \frac{1}{\cosh|\delta_\alpha|} = \frac{\sinh\frac{l}{2}\sinh\frac{|\alpha|}{2}}{\cosh\frac{|\beta|}{2} + \cosh\frac{l}{2}\cosh\frac{|\alpha|}{2}}, \tag{3.47}$$

$$\tanh y = \frac{1}{\cosh|\delta_\beta|} = \frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}. \tag{3.48}$$

From these one can derive that

$$\tanh(x+y) = \frac{\sinh\frac{l}{2}\sinh\frac{|\alpha|+|\beta|}{2}}{1 + \cosh\frac{l}{2}\cosh\frac{|\alpha|+|\beta|}{2}} \tag{3.49}$$

and hence that

$$\tanh\frac{x+y}{2} = \tanh\frac{l}{4}\tanh\frac{|\alpha|+|\beta|}{4}. \tag{3.50}$$

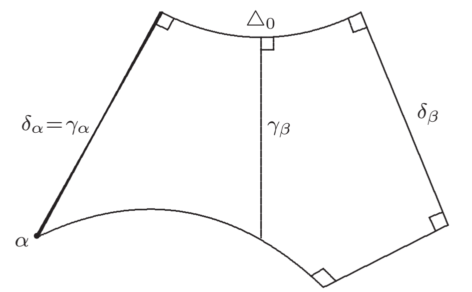

Figure 3.7: Subcase 2.3

Thus

$$\tanh\left(\frac{l}{4}-\frac{x+y}{2}\right) = \frac{\tanh\frac{l}{4}\left(1-\tanh\frac{|\alpha|+|\beta|}{4}\right)}{1-\tanh^2\frac{l}{4}\tanh\frac{|\alpha|+|\beta|}{4}} = \frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}. \tag{3.51}$$

Hence in this case we have

$$\begin{aligned}\mathrm{Gap}(\Delta_0;\alpha,\beta) &= \frac{l}{2}-(x+y)\\ &= 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right).\end{aligned}$$

Subcase 2.2. α is a boundary geodesic and β is an interior generalized simple closed geodesic.

In this case the width of the combined gap determined by γ is the distance between δ_α and γ_β along Δ_0 and is equal to $\frac{l}{2}-y$. Hence by (3.48) we have

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.52}$$

Subcase 2.3. α is a cone point of cone angle $\varphi\in(0,\pi]$ and β is an interior generalized simple closed geodesic.

Note that in this case γ_α coincides with δ_α and hence $x=0$. Hence the width of the combined gap determined by γ is the distance between δ_α and γ_β along Δ_0 and is equal to $\frac{l}{2} - y$.

Now by a formula in Fenchel [18] VI.3.2 (line 8, page 87),

$$\cosh|\delta_\beta| = \frac{\cos\frac{\varphi}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}. \tag{3.53}$$

Hence

$$\tanh y = \frac{1}{\cosh|\delta_\beta|} = \frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}. \tag{3.54}$$

Thus in this case we have

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{l}{2} - \tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{3.55}$$

Remark 3.37 We remark that the formulas in Case 0 for the normalized width $\mathrm{Gap}'(\Delta_0;\alpha,\beta)$ when Δ_0 is a cusp can be derived by similar (and simpler) calculations or by considering the first order infinitesimal terms of those formulas with respect to θ in Case 1 or with respect to l in Case 2. Hence all derivations in Case 0 are omitted.

3.6 Generalization of Birman–Series Theorem

The celebrated Birman–Series Theorem [5] in its simplest form states that complete simple geodesics on a closed hyperbolic surface are sparsely distributed.

More precisely, let M be a hyperbolic surface possibly with boundary such that M is either compact or obtained from a compact surface by removing a finite set of points which form the cusps and such that each boundary component of M is a simple closed geodesic. A geodesic on M is said to be *complete* if it is either closed and smooth, or open and of infinite length in both directions. Hence a complete geodesic never intersects ∂M. Let G_k be the family of complete geodesics on M which have at most k, counted with multiplicity, transversal self-intersections, $k \geq 0$. Then the main result in [5] is the following

Theorem 3.38 *For each $k \geq 0$, the point set S_k which is the union of all geodesics, as point sets, in G_k is nowhere dense and has Hausdorff dimension one.*

In this section we show that this theorem extends to the case when M is a compact hyperbolic cone-surface with geometric boundary where each cone point has cone angle in $(0, \pi]$, with complete geodesics replaced by complete-normal ones. This is the set of geodesics which are either complete, or intersect the boundary perpendicularly at each intersection.

Theorem 3.39 *Let M be a compact hyperbolic cone-surface with geometric boundary where each cone point has cone angle in $(0, \pi]$, and let G_k be the family of complete-normal geodesics on M which have at most k transversal self-intersections, $k \geq 0$. Then, for each $k \geq 0$, the point set S_k which is the union of all geodesics, as point sets, in G_k is nowhere dense and has Hausdorff dimension one.*

The proof of this generalization is is essentially the same as that of the original Birman–Series Theorem given in [5]. Hence for simplicity we shall only sketch the proof of the theorem for the case $k = 0$, that is, for simple complete-normal geodesics. The reader is referred to [5] for omitted details.

At the end of this section we point out that Birman-Series' arguments in [5] also give absolute convergence of the series appeared in various McShane's identities.

We only need to consider the case where M has *no* geodesic boundary components; for if M has nonempty geodesic boundary we can replace M by the double of M along its geodesic boundary. We also assume for clarity that each cone point of M has cone angle less than π. We decompose the set G_0 into finitely many subsets and prove the conclusion for each such subset. For the subset of simple complete geodesics on M, that is, the geodesics which never start from or terminate at cusps or cone points, the proof is the same as that in [5] with little modification (which can be seen from the sketch later in this section). For the subset of simple normal geodesics which connect a given cusp or cone point to another (possibly the same) given cusp or cone point, it is easy to see that in this subset each such geodesic is isolated in suitable neighborhoods of its endpoints and hence the conclusion follows. Thus it remains to prove the conclusion for the subset of simple complete-normal geodesics which starts from a given cusp or cone point P and never terminates at any geometric boundary component.

Since M is assumed to have no geodesic boundary components, one can cut M along normal geodesics connecting cusps or cone points to form a (convex) fundamental polygon R for M in the hyperbolic plane. Let $A = \{a_1, a_2, \cdots, a_m\}$ denote the ordered set of vertices and oriented sides of R with anti-clockwise ordering with some arbitrary but henceforth fixed initial element a_1.

Let J_0 be the set of oriented simple-normal geodesic arcs γ on M such that the initial point and the ending point of γ lie in ∂R. (Note that except at its initial point or ending point γ cannot pass through a vertex of R.) For $\gamma \in J_0$, we call the components of $\gamma \cap R$ the *segments* of γ and the points of $\gamma \cap \partial R$ the partition points of γ. We label the partition points $t_0, t_1, \cdots, t_n$ in the order in which they occur along γ and we denote by $t_i^+ \in \partial R$ as the initial point of the segment of γ from t_i to t_{i+1} and, similarly, $t_i^- \in \partial R$ the ending point of the segment from t_{i-1} to t_i. We set $\| \gamma \| = n$ as the combinatorial length of γ (with respect to R).

For $\gamma \in J_0$, the segments of γ give rise to a *simple diagram* on R which is a collection of finitely many pairwise disjoint (geodesic) arcs joining pairs of distinct elements of A. Two simple diagrams are regarded as being identical if they agree up to isotopy supported on each side of R. For $a_i, a_j \in A$, $i \neq j$, let n_{ij} denote the number of arcs joining a_i to a_j in the given simple diagram. The *length* of a simple diagram is $n = \sum n_{ij}$, $1 \le i < j \le m$.

The Birman–Series parametrization of elements of J_0 consists of two sets of data. The first is the ordered sequence $h_1(\gamma) = (n_{12}, n_{13}, \cdots, n_{m-1,m})$ which records for each ordered pair of distinct elements a_i, a_j of R the number n_{ij} of segments of γ which join a_i to a_j. The second set of data, $h_2(\gamma)$, records information about the position of the points $t_0^+, t_n^- \in \partial R$ of γ. Let $a(t_i^\pm)$ be the element of A containing $t_i^\pm$ and let $j(t_i^\pm) \in \mathbf{N}$ be the position of $t_i^\pm$ among the partition points of γ which lie along $a(t_i^\pm)$ counting in the anticlockwise direction round ∂R. Define $h_2(\gamma) = (a(t_0^+), j(t_0^+), a(t_n^-), j(t_n^-))$.

The following lemmas and their proofs in [5] still hold in our case.

Lemma 3.40 *Suppose that $\gamma, \gamma' \in J_0$ and that $h_1(\gamma) = h_1(\gamma'), h_2(\gamma) = h_2(\gamma')$. Let $t_0, t_1, \cdots, t_n$ and $t'_0, t'_1, \cdots, t'_n$ be the partition points of γ, γ' respectively. Then $a(t_i^\pm) = a(t_i'^\pm)$ for each $i = 0, 1, \cdots, n$.*

Lemma 3.41 *Let $J_0(n) = \{\gamma \in J_0 : \parallel \gamma \parallel = n\}$. Then there is a polynomial $P_0(n)$ such that the number of simple diagrams of length n*

$$\text{card}\{(h_1(\gamma), h_2(\gamma)) : \gamma \in J_0(n)\} \le P_0(n).$$

The main idea of the proof of Birman–Series Theorem in [5] is that (the central parts of) geodesic arcs in $J_0(n)$ (for sufficiently large n) with the same parametrization lie exponentially close in universal cover of M in the hyperbolic plane. It relies on the following key lemma which is Lemma 3.1 in [5].

Lemma 3.42 *There is a constant $\alpha > 0$ (depending only on the choice of the fundamental polygon R) so that*

$$|\gamma| \ge \alpha \parallel \gamma \parallel$$

for $\gamma \in J_0$ with $\parallel \gamma \parallel$ sufficiently large. Recall that here $|\gamma|$ denotes the hyperbolic length of γ.

Proof. There is a constant $\epsilon > 0$ so that any segment of γ, which does not connect two consecutive sides of R and does not intersect a suitably chosen disk neighborhood of each cusp or cone point, has hyperbolic length at least ϵ. Let q be the maximum number of sides of R, projected to M, which meet at any cusp or cone point of M. Then at most $q-1$ consecutive segments of γ can connect consecutive sides of R around the same cusp or cone point and intersect the chosen disk neighborhood of that cusp or cone point; for otherwise there will be a self-intersection on γ. Hence in any q consecutive segments of γ, at least one has hyperbolic length ϵ, which gives the result. □

The following two lemmas then apply respectively to the set of all complete simple geodesics which never intersect any cusp or cone point and to the set of simple geodesics which start from a fixed cusp or cone point and never terminates at any cusp or cone point. (Recall that we have assumed that M has no boundary geodesics.)

Lemma 3.43 *Let $\gamma, \gamma' \in J_0(2n+1)$ and suppose that $h_1(\gamma) = h_1(\gamma'), h_2(\gamma) = h_2(\gamma')$. Let $\delta \subset \gamma, \delta' \subset \gamma'$ denote the segments of γ, γ' lying between the partition points t_n, t_{n+1} and t'_n, t'_{n+1} respectively. Then $\delta' \subset B_{ce^{-\alpha n}}(\delta)$ where $c, \alpha > 0$ are universal constants and where $B_\epsilon(\delta)$ denotes the tubular neighborhood of δ of hyperbolic radius $\epsilon > 0$.*

Lemma 3.44 *Let $\gamma, \gamma' \in J_0(n+k)$ be such that they start at the same vertex of R and that $h_1(\gamma) = h_1(\gamma'), h_2(\gamma) = h_2(\gamma')$. Let $\delta \subset \gamma, \delta' \subset \gamma'$ denote the segments of γ, γ' lying between the partition points t_{i-1}, t_i and t'_{i-1}, t'_i respectively, where $1 \le i \le k$. Then $\delta' \subset B_{ce^{-\alpha n}}(\delta)$ where $c, \alpha > 0$ are universal constants and where $B_\epsilon(\delta)$ denotes the tubular neighborhood of δ of hyperbolic radius $\epsilon > 0$.*

Remark 3.45 Note that Lemma 3.43 above is Lemma 3.2 in [5]. The proof there works here in spite of the fact that the developing image of M in the hyperbolic plane is generally not embedded. What we really need in the proof is to develop M in the hyperbolic plane as dictated by the given simple diagram. The so developed image of M in the hyperbolic plane is embedded since the simple diagram comes from a simple complete geodesic in M. The image is convex since all cone points of M have cone angle bounded above by π. Hence if two such simple complete geodesics give the same simple diagram with large length n, then their central parts will have to lie exponentially close. Lemma 3.44 can be similarly proved.

From these we have the following proposition which is Proposition 4.1 in [5] and from which the conclusion of the Birman–Series Theorem for compact hyperbolic cone surfaces follows exactly as in the proofs in [5] §5.

Proposition 3.46 *There exist universal constants $L, c, \alpha > 0$ and a polynomial $P_0(\cdot)$ such that for each n there is a set F_n of simple geodesic arcs, each of length at most L, so that* $\mathrm{card}(F_n) \leq P_0(n)$ *and so that*

$$S_0 \subset \bigcup \{B_\epsilon(\gamma) \mid \gamma \in F_n\}, \ \epsilon = c\, e^{-\alpha n}.$$

Absolute convergence by Birman–Series' arguments. Finally we remark that the above Birman–Series' arguments will give rough estimates on the distribution of simple closed geodesics on a compact hyperbolic cone-surface M, which is enough for proving the absolute convergence of the series appearing in various generalized McShane's identities, as was observed and used in [2] for similar purposes in the case of complete hyperbolic surfaces.

Lemma 3.47 *Let M be a compact hyperbolic cone-surface with all cone angles in $(0, \pi]$. Then for any constant $c > 0$*

(i) *the series*

$$\sum_{\beta} \frac{1}{\exp c|\beta|} \tag{3.56}$$

converges absolutely, where the sum is taken over all generalized simple closed geodesics on M and all simple normal geodesic arcs connecting geometric boundary components of M;

(ii) *the series*

$$\sum_{\alpha,\,\beta} \frac{1}{\exp\, c(|\alpha| + |\beta|)} \tag{3.57}$$

converges absolutely, where the sum is taken over all pairs α, β of disjoint generalized simple closed geodesics on M and/or simple normal geodesic arcs connecting geometric boundary components of M. □

The idea of the proof is that, in the case of (i), each β determines and is determined by a unique complete simple diagram (in the case β is a simple closed geodesic, the diagram is *closed*, that is, we have $t_n = t_0$ using earlier notation). Hence the number of such β with combinatorial length $\|\beta\| = n$ is bounded by the polynomial value $P_0(n)$. Note that each β has hyperbolic length $|\beta|$ at least $\kappa\,\|\beta\|$ for some constant $\kappa > 0$ depending only the chosen fundamental region R; hence we have $c|\beta| \geq c\kappa\,\|\beta\|$. It follows that

$$\begin{aligned} \sum_{\beta} \frac{1}{\exp\, c|\beta|} &\leq \sum_{n=1}^{\infty} \sum_{\|\beta\|=n} \frac{1}{\exp\, c\kappa\|\beta\|} \\ &\leq \sum_{n=1}^{\infty} \frac{P_0(n)}{\exp(\,c\kappa\, n)} \end{aligned}$$

and hence the series in (3.56) converges. One can prove (ii) similarly, since each pair α, β can be determined by a suitable complete simple diagram.

3.7 Proof of the theorems

In this section we give proofs of Theorem 3.5, Corollary 3.8, Addendum 3.14 and Theorem 3.17.

Proof of Theorem 3.5 Now the proof is obvious from the previous discussions. Suppose Δ_0 is a cone point. Recall $\mathcal{H}$ is a suitably chosen small circle centered at Δ_0, and $\mathcal{H}_{\mathtt{ns}}$, $\mathcal{H}_{\mathtt{sn}}$, $\mathcal{H}_{\mathtt{snn}}$ are the point sets of the first intersections of $\mathcal{H}$ with respectively all non-simple, all simple-normal, all simple-not-normal Δ_0-geodesics. The elliptic measure of each of these subsets of $\mathcal{H}$ is the radian measure that it subtends to the cone point Δ_0. The generalized Birman–Series Theorem in §3.6 implies that the closed subset $\mathcal{H}_{\mathtt{sn}}$ has measure 0. Hence the open subset $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$ has full measure, that is, θ_0. Now the maximal open intervals of $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$, suitably combined, have measure $2\text{Gap}(\Delta_0; \alpha, \beta)$ for each unordered pair of generalized simple closed geodesics α, β on M which bound with Δ_0 an embedded pair of pants on M. Hence their sum is equal to θ_0 and the desired identity follows. The proof for the cases where Δ_0 is a boundary geodesic or a cusp is similar. □

Proof of Corollary 3.8 Consider the case where Δ_0 is a cone point. In this case T admits a unique elliptic involution η such that η maps each oriented simple closed geodesics on T onto itself with orientation reversed. Note that η fixes the cone point Δ_0 and three other interior points which are the so-called Weierstrass points of T. Each simple closed geodesics on T passes exactly two Weierstrass points; hence there are three Weierstrass classes of simple closed geodesics on T. Now the quotient of T under η is a sphere with three angle π cone points and a cone point with angle $\theta/2$. Then Theorem 3.5 applies to $M = T/\langle\eta\rangle$, with Δ_0 the angle π cone point whose inverse image under η is the Weierstrass point that the Weierstrass class $\mathcal{A}$ misses. Note that each generalized simple closed geodesic on $M = T/\langle\eta\rangle$ is either a geometric boundary component or degenerate simple closed geodesic which is the double cover of a simple geodesic arc which connects two Weierstrass points. Hence the set of all pairs of generalized simple closed geodesics which bound with Δ_0 an embedded pair of pants is exactly the set of pairs consisting of the angle $\theta/2$ cone point plus a degenerate simple closed geodesic γ' which is the double cover of the quotient simple geodesic arc of a simple closed geodesic γ on T in the given Weierstrass class $\mathcal{A}$ (note that by definition the length of γ' is the same as that of γ). Hence by (3.26) the summand in the summation is

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{\sin\frac{\pi}{2}\sinh\frac{|\gamma|}{2}}{\cos\frac{\theta}{4} + \cos\frac{\pi}{2}\cosh\frac{|\gamma|}{2}}\right) = \tan^{-1}\left(\frac{\cos\frac{\theta}{4}}{\sinh\frac{|\gamma|}{2}}\right).$$

The proof for the case where Δ_0 is a boundary geodesic is similar. □

Remark 3.48 Note that we can also choose Δ_0 to be the angle $\theta/2$ cone point on $T/\langle\eta\rangle$, then we obtain (3.1), the generalization of McShane's original identity to the cone-torus T. This is one way of seeing why we can allow the cone angle up to 2π in the one-cone torus case.

Proof of Theorem 3.12 It is well known that M admits a unique hyperelliptic involution η (see for example [22]) such that η maps each simple closed geodesic onto itself and preserves/reverses the orientations of separating/non-separating simple closed geodesics. Note that η leaves six points on M fixed, which are the six Weierstrass points on M. Consider the quotient $M' = M/\langle\eta\rangle$ which is a sphere with six angle π cone points. Each generalized simple closed geodesic on M' is either

(i) an angle π cone point; or

(ii) a degenerate simple closed geodesic β' which is the double cover of a simple geodesic arc c connecting two angle π cone points where the inverse image of c under η is a non-separating simple closed geodesic β on M; or

(iii) a separating (non-degenerate) simple closed geodesic α' whose inverse image under η is a separating simple closed geodesic α on M. In this case α' does not pass through any of the six angle π cone points and there are three of them on each side of α' on M'. Hence α passes none of six Weierstrass points and there are three of them on each side of α on M.

Now apply Theorem 3.5 to M' with Δ_0 one of the six angle π cone points. Then each pair of generalized simple closed geodesics on M' which bound with Δ_0 an embedded pair of pants $\mathcal{P}$ consists of a separating simple closed geodesic α' on M' and a degenerate simple closed geodesic β' on M' which lies on the same side of α' as Δ_0 and misses Δ_0. Let the inverse images of α', β' under η be α, β respectively. Then α is a separating simple closed geodesic on M and β is a non-separating simple closed geodesic on M. Furthermore, β and the Weierstrass point which is the inverse image of Δ_0 lie on the same side of α on M. Note that the hyperbolic lengths of α', β' are respectively $|\alpha|/2, |\beta|$. Hence by (3.24) in this case the summand in the resulting generalized McShane's Weierstrass identity for M' with the chosen Δ_0 is

$$2\tan^{-1}\left(\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}+\exp\frac{|\alpha|/2+|\beta|}{2}}\right) = 2\tan^{-1}\exp\left(-\frac{|\alpha|}{4}-\frac{|\beta|}{2}\right).$$

Note that each pair of disjoint simple closed geodesics (α, β) on M such that α is separating and β is non-separating arises as the inverse image of a unique pair of generalized simple closed geodesics on M' as described above, where the chosen Δ_0 is the angle π cone point which is the image under η of the Weierstrass point on M that lies on the same side of α as β and is missed by β.

Summing all the six resulting Weierstrass identities we then have

$$\sum 2\tan^{-1}\exp\left(-\frac{|\alpha|}{4}-\frac{|\beta|}{2}\right) = \frac{6\pi}{2},$$

where the sum is taken over all ordered pairs (α, β) of disjoint simple closed geodesics on M such that α is separating and β is non-separating. □

Proof of Addendum 3.14 We first prove that the series in (3.15) converges absolutely and uniformly on compact sets in the space $\mathcal{QF}$ of (conjugacy classes of) quasifuchsian representations of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbf{C})$ by the same argument as used in [2]. Then identity (3.15) will follow by analytic continuation since (a) each summand in it is an analytic function of the complex Fenchel–Nielsen coordinates for the quasifuchsian space $\mathcal{QF}(M)$ (see [44]), (b) the identity holds when all the coordinates take real values (by Theorem 3.12) and (c) the quasifuchsian space $\mathcal{QF}(M)$ is simply connected. Note that in the process of doing analytic continuation, the inverse function $\tan^{-1}$ also needs to be analytically continued. But since our result is just an identity modulo π, we may simple assume that $\tan^{-1}$ is defined by (3.16).

For each essential simple closed curve γ on M, we may define its combinatorial length $\|\gamma\|$ as the length of a cyclically reduced word representing γ in $\pi_1(M)$ with respect to some fixed set of canonical generators of $\pi_1(M)$. Then there is a constant $c > 0$ such that $l_{\rho_0}(\gamma) \geq c\|\gamma\|$ for every γ and for a fixed fuchsian representation of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbf{C})$.

As pointed out in [2] Lemma 5.2, for any compact subset $\mathcal{C}$ of $\mathcal{QF}(M)$, by [24] Lemma 3, there is a constant $k = k(\mathcal{C}) > 0$, such that

$$k\, l_{\rho_0}(\gamma) \leq \Re\, l_\rho(\gamma) \leq k^{-1} l_{\rho_0}(\gamma), \tag{3.58}$$

for any essential simple closed curve γ and for any $\rho \in \mathcal{C}$. Hence for each pair of disjoint, non-homotopic essential simple closed curves α, β on M and for each $\rho \in \mathcal{C}$ we have

$$\Re\Big(\frac{l_\rho(\alpha)}{4} + \frac{l_\rho(\beta)}{2}\Big) \geq ck\Big(\frac{\|\alpha\|}{4} + \frac{\|\beta\|}{2}\Big). \tag{3.59}$$

Thus the left hand side of (3.59) is sufficiently large except for a finite number of pairs of α, β and for each $\rho \in \mathcal{C}$.

Note that $|\tan^{-1}(x)| \leq 2|x|$ if $|x| \leq 1/2$ is sufficiently small, say, $|x| \leq \sqrt{2}/2$. Thus we have the following estimate for all but a finite number of pairs of (free homotopy classes of) disjoint essential simple closed curves α, β on M such that α is separating and β is non-separating and for each $\rho \in \mathcal{C}$:

$$\begin{aligned}\Big|\tan^{-1}\exp\Big(-\frac{l_\rho(\alpha)}{4} - \frac{l_\rho(\beta)}{2}\Big)\Big| &\leq 2\Big|\exp\Big(-\frac{l_\rho(\alpha)}{4} - \frac{l_\rho(\beta)}{2}\Big)\Big| \\ &= 2\exp\Big(-\frac{\Re\, l_\rho(\alpha)}{4} - \frac{\Re\, l_\rho(\beta)}{2}\Big) \\ &\leq 2\exp\Big(-k\Big(\frac{l_{\rho_0}(\alpha)}{4} + \frac{l_{\rho_0}(\beta)}{2}\Big)\Big).\end{aligned}$$

Thus the series in (3.15) converges absolutely and uniformly on the compact set $\mathcal{C}$ of $\mathcal{QF}$ since the series

$$\sum \exp\Big(-k\Big(\frac{l_{\rho_0}(\alpha)}{4}+\frac{l_{\rho_0}(\beta)}{2}\Big)\Big)$$

converges by Lemma 3.47. $\square$

Finally we prove the unified version (3.19) of our generalized McShane's identity using complex arguments and interpret it geometrically. We shall omit the proof of Theorem 3.17 in the case where Δ_0 is a cusp, for as remarked before, in the cusp case the identity (3.18) can either be proved similarly or be derived by considering the first order infinitesimal terms of the corresponding identity (3.17) in other cases.

Proof of Theorem 3.17. We first show that our generalized McShane's identities (3.5) and (3.6) can be reformulated as (3.19) modulo convergence.

First suppose that Δ_0 is a boundary geodesic of hyperbolic length $l_0 > 0$.

For a pair of interior generalized simple closed geodesics α, β which bound with Δ_0 an embedded pair of pants on M, we have directly by definition that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = G\Big(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\Big).$$

For a pair of generalized simple closed geodesics α, β such that α is a boundary geodesic and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by definition and the geometric meanings of G, S that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = G\Big(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\Big) + S\Big(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\Big).$$

For a pair of generalized simple closed geodesics α, β such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by (2.16) with $x = l_0/2, y = \varphi/2, z = |\beta|/2$ that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = G\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right).$$

Next suppose that Δ_0 is a cone point of angle $\theta_0 \in (0, \pi]$.

For a pair of interior generalized simple closed geodesics α, β which bound with Δ_0 an embedded pair of pants on M, we have by definition that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta)\, i = G\left(\frac{\theta_0 i}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right).$$

For a pair of generalized simple closed geodesics α, β such that α is a boundary geodesic and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by the analysis in §3.5 that

$$\begin{aligned} & \mathrm{Gap}(\Delta_0; \alpha, \beta)\, i \\ = \;& 2\, i \tan^{-1}\left(\frac{\sin\frac{\theta_0}{2}}{\cos\frac{\theta_0}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right) + i \tan^{-1}\left(\frac{\sin\frac{\theta_0}{2}\sinh\frac{|\alpha|}{2}}{\cosh\frac{|\beta|}{2} + \cos\frac{\theta_0}{2}\cosh\frac{|\alpha|}{2}}\right) \\ = \;& G\left(\frac{\theta_0 i}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{\theta_0 i}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right). \end{aligned}$$

For a pair of generalized simple closed geodesics α, β such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by (2.17) with $x = \theta_0/2, y = \varphi/2, z = |\beta|/2$ that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta)\, i = G\left(\frac{\theta_0 i}{2}, \frac{\varphi\, i}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{\theta_0 i}{2}, \frac{\varphi\, i}{2}, \frac{|\beta|}{2}\right).$$

Finally we prove the absolute convergence of the series in (3.19). It is not hard to see that we only need to prove, for each $j = 1, \cdots, n$, the absolute convergence of the series

$$\sum_{\beta} S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right),$$

where the sum is taken over all interior generalized simple closed geodesics β which bounds with Δ_j and Δ_0 an embedded pair of pants on M. The desired absolute convergence follows from Lemma 3.47 since

$$S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right) \sim \frac{\sinh \frac{L_0}{2} \sinh \frac{L_j}{2}}{\cosh \frac{|\beta|}{2}} \sim \text{const.} \exp\left(-\frac{|\beta|}{2}\right)$$

as $|\beta| \to \infty$. This completes the proof of Theorem 3.17. $\square$

3.8 Geometric interpretation of the complexified reformulation

In this section we would like to explore the geometric meanings of the summands in the complexified formula (3.19).

In the case that M has no cone points, all its geometric boundary components (here cusps are treated as boundary geodesics of length 0) $\Delta_0, \Delta_1, \cdots, \Delta_n$ are boundary geodesics with hyperbolic lengths $L_0, L_1, \cdots, L_n$ respectively. Assume Δ_0 is not a cusp, that is, $L_0 > 0$. Then as explained in §3.5, in the first sum the summand is the width of one of the main gaps in the pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ bounded by Δ_0 and α, β; while in the second sum the sub-summand is the width of one of the two extra gaps associated to Δ_j in the pair of pants $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ bounded by Δ_0, Δ_j and β. We would like to think of the union of the two extra gaps in $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ as the orthogonal projection of Δ_i onto Δ_0 along the common perpendicular δ of Δ_j and Δ_0 in $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ and think of its width as the *direct visual measure* of Δ_j at Δ_0 along δ. Hence the second part of the left hand side of (3.19) can be thought of as the total direct visual measure of all the non-distinguished geometric boundary components $\Delta_1, \cdots, \Delta_n$ at Δ_0.

In the case that Δ_0 is a cone point of angle $\theta_0 \in (0, \pi]$ (hence $L_0 = \theta_0 i$) and all other geometric boundary components of M are boundary geodesics (here cusps treated as boundary geodesics of length 0), for each pair of generalized simple closed geodesics α, β which bound with Δ_0 an embedded pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ on M, each of α, β has a direct visual angle at the cone point Δ_0; and the summand in the first sum is i times the angle measure of one of the two gaps at Δ_0 between the two Δ_0-geodesic rays asymptotic to α^+, β^-(respectively α^-, β^+). The sub-summand in the second sum is i times half the visual angle measure of Δ_j at Δ_0 in the pair of pants $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ on M.

When M has cone points other than Δ_0, the similar formulations of the generalized McShane's identities (3.5)–(3.7) in terms of $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ will not be as neat as in the above two special cases. The problem lies in that a cone point (other than Δ_0) *seems* to have direct visual measure zero at Δ_0, causing the formulas to be non-uniform. However, this non-uniformity is caused by the (wrong) point of view that we treat a cone point as only a point. The correct point of view is (perhaps) that a cone point (as a geometric boundary component) should be a geodesic perpendicular to the surface at the very cone point when the surface is "imagined" as lying in the hyperbolic 3-space and hence one should use purely complex length instead of real one for a cone point. (The point of view of using complex translation length for an isometry of the hyperbolic 3-space is well discussed in details in [18] and [41].)

First assume that Δ_0 is boundary geodesic of length $l_0 > 0$ and consider a pair of generalized simple closed geodesics α, β on M such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ on M.

Let the (unoriented) geodesic arc in $\mathcal{P}(\Delta_0, \alpha, \beta)$ which is perpendicular to Δ_0 and α (respectively, α and β, β and Δ_0) be denoted $[\Delta_0, \alpha]$ (respectively, $[\alpha, \beta]$, $[\beta, \Delta_0]$). We cut $\mathcal{P}(\Delta_0, \alpha, \beta)$ open along $[\Delta_0, \alpha]$, $[\alpha, \beta]$, $[\beta, \Delta_0]$ to obtain two congruent pentagons; lift one of them to a pentagon $\mathbf{P}(\Delta_0, \alpha, \beta)$ in the hyperbolic plane H^2. Then by Fenchel [18] $\mathbf{P}(\Delta_0, \alpha, \beta)$ can be regarded as a right angled hexagon $\mathbf{H}(\Delta_0, \tilde{\alpha}, \beta)$ spanned by straight lines $\Delta_0, \tilde{\alpha}, \beta$ in a hyperbolic 3-space H^3 containing the hyperbolic plane H^2. See Figure 3.8 for an illustration. Here $\tilde{\alpha}$ is the straight line in H^3 which passes through the cone point α in H^2 and is perpendicular to H^2. Let the common perpendiculars in H^3 between pairs of $\Delta_0, \tilde{\alpha}, \beta$ be $[\Delta_0, \tilde{\alpha}]$, $[\tilde{\alpha}, \beta]$, $[\beta, \Delta_0]$, where, as straight lines $[\Delta_0, \tilde{\alpha}]$, $[\tilde{\alpha}, \beta]$ are the same as $[\Delta_0, \alpha]$, $[\alpha, \beta]$ respectively.

We orient the six straight lines in the cyclic order $\Delta_0, [\Delta_0, \tilde{\alpha}], \tilde{\alpha}, [\tilde{\alpha}, \beta], \beta, [\beta, \Delta_0]$ as Fenchel did in [18]; see Figure 3.8. Then the three oriented sides $\Delta_0, \tilde{\alpha}, \beta$ of the right angled hexagon $\mathbf{H}(\Delta_0, \tilde{\alpha}, \beta)$ have complex lengths $\frac{l_0}{2} + \pi i, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i$ respectively.

Let the ideal points which are the starting and ending endpoints of an oriented straight line $\mathbf{l}$ in H^3 be denoted $\mathbf{l}^-, \mathbf{l}^+$ respectively. Then we have in H^3 an oriented straight line $[\Delta_0, \tilde{\alpha}^+]$ which intersects Δ_0 perpendicularly and has $\tilde{\alpha}^+$ as its ending ideal point, and similarly an oriented straight line $[\Delta_0, \beta^-]$ which intersects Δ_0 perpendicularly and has β^- as its ending ideal point.

Then it can be verified that

$$G\left(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right) = G\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$

is the complex length from $[\Delta_0, \beta^-]$ to $[\Delta_0, \tilde{\alpha}^+]$ measured along Δ_0 and

$$S\left(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right) = S\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$

is the the complex length from $[\Delta_0, \tilde{\alpha}^+]$ to $[\Delta_0, \tilde{\alpha}]$ measured along Δ_0.

Note that $S\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$ is purely imaginary, which is obvious from its geometric meaning.

Remark 3.49 We remark that it is important that in $G\left(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right)$ and $S\left(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right)$ the value used for Δ_0 is $\frac{l_0}{2}$ instead of $\frac{l_0}{2} + \pi i$. See §2.5 for an explanation.

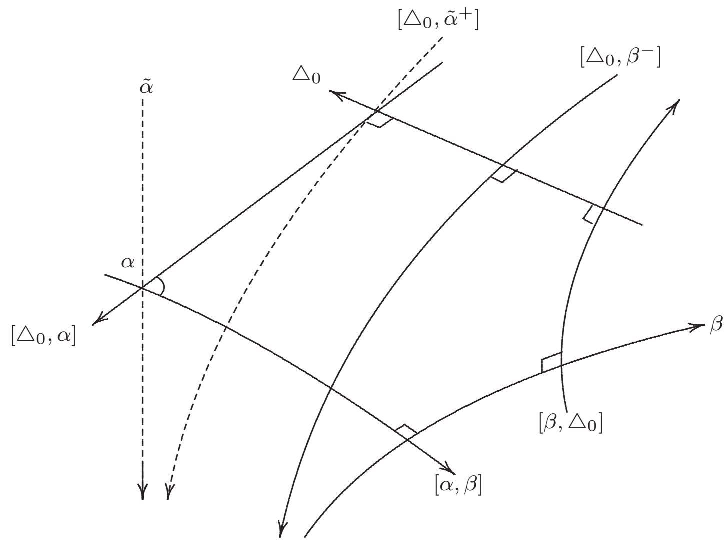

Figure 3.8:

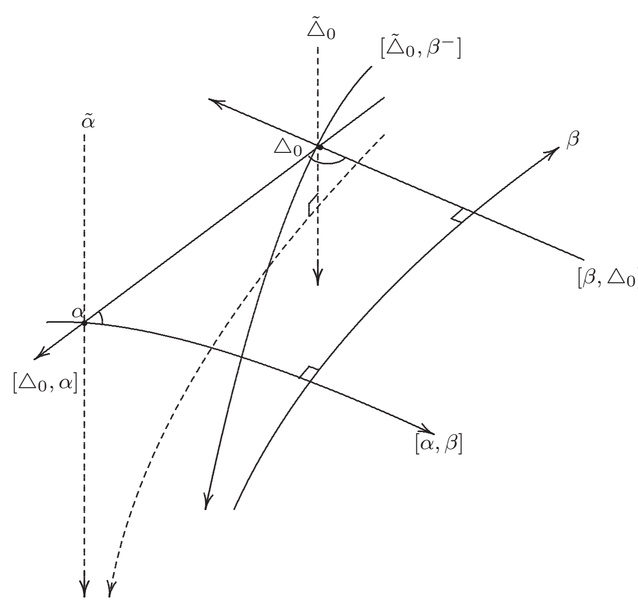

Figure 3.9:

Next assume Δ_0 is a cone point of angle $\theta_0 \in (0, \pi]$ and consider a pair of generalized simple closed geodesics α, β on M such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ on M.

In this case we cut $\mathcal{P}(\Delta_0, \alpha, \beta)$ open along $[\Delta_0, \alpha]$, $[\alpha, \beta]$, $[\beta, \Delta_0]$ to obtain two congruent quadrilaterals and lift one of them to a quadrilateral $\mathbf{Q}(\Delta_0, \alpha, \beta)$ in the hyperbolic plane H^2. As before, let $\tilde{\alpha}$ be the straight line in H^3 which passes the cone point α in H^2 and is perpendicular to H^2. Similarly for $\tilde{\Delta}_0$. Then we obtain a right angled hexagon $\mathbf{H}(\tilde{\Delta}_0, \tilde{\alpha}, \beta)$ in H^3. Let the six sides of $\mathbf{H}(\tilde{\Delta}_0, \tilde{\alpha}, \beta)$ be oriented as illustrated in Figure 3.9. Then the three oriented sides $\tilde{\Delta}_0, \tilde{\alpha}, \beta$ of the right angled hexagon $\mathbf{H}(\tilde{\Delta}_0, \tilde{\alpha}, \beta)$ have complex lengths $\frac{\theta_0 i}{2} + \pi i$, $\frac{\varphi}{2} i + \pi i$, $\frac{|\beta|}{2} + \pi i$ respectively.

Similarly, we have in H^3 an oriented straight line $[\tilde{\Delta}_0, \tilde{\alpha}^+]$ which intersects $\tilde{\Delta}_0$ perpendicularly and has $\tilde{\alpha}^+$ as its ending ideal point, and another oriented straight line $[\tilde{\Delta}_0, \beta^-]$ which intersects $\tilde{\Delta}_0$ perpendicularly and has β^- as its ending ideal point.

Then
$$G\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right) = G\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$
is the complex length from $[\tilde{\Delta}_0, \beta^-]$ to $[\tilde{\Delta}_0, \tilde{\alpha}^+]$ measured along $\tilde{\Delta}_0$ and
$$S\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right) = S\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$
is the the complex length from $[\tilde{\Delta}_0, \tilde{\alpha}^+]$ to $[\tilde{\Delta}_0, \tilde{\alpha}]$ measured along $\tilde{\Delta}_0$.

Note that $S\big(\frac{\theta_0 i}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\big)$ is real, which is obvious from its geometric meaning.

Remark 3.50 Here it is important that in $G\big(\frac{\theta_0 i}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\big)$ and $S\big(\frac{\theta_0 i}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\big)$ the value used for Δ_0 is $\frac{\theta_0 i}{2}$ instead of $\frac{\theta_0 i}{2} + \pi i$. See §2.5 for an explanation.

Chapter 4

Generalized Markoff Maps and McShane's Identity

Abstract

Following Bowditch [8], we study representations in $\mathrm{SL}(2,\mathbf{C})$ of the free group Γ on two generators, and the connection with generalized Markoff maps. We show that Bowditch's Q-conditions for generalized Markoff maps are sufficient for the generalized McShane identity to hold for the corresponding representations.

4.1 Introduction

Brian Bowditch in [8] extended McShane's identity (1.1) via Markoff maps to type-preserving representations in $\mathrm{SL}(2,\mathbf{C})$ of the once-punctured torus group satisfying certain conditions which we call here BQ-conditions (Bowditch's Q-conditions). He also obtained in [7] a variation of (1.1) which applies to complete hyperbolic once-punctured torus bundles over the circle. Subsequently, Akiyoshi-Miyachi-Sakuma [1] [2] refined Bowditch's results in [7] and generalized them to those which apply to complete hyperbolic punctured surface bundles over the circle.

In this chapter we show that the generalized McShane identity (3.2) for hyperbolic one-hole tori obtained in Chapter 3 hold for general representations in $\mathrm{SL}(2,\mathbf{C})$ of the punctured torus group satisfying the BQ-conditions. A proper statement (see Theorem 4.3) requires a fair bit of notation and is deferred to the next section. We remark that there are many interesting examples of representations which satisfy these conditions arising from some basic and important geometric constructions. For example, the one-cone/one-hole hyperbolic torus, the hyperbolic three-holed sphere (often called pair of pants), and more generally, classical Schottky groups with two generators all give rise to such representations. However, the class of representations satisfying the BQ-conditions is in general much larger, and an interesting problem is to classify this class geometrically. In the case of type-preserving representations studied in [8], Bowditch has conjectured that the class coincides with the quasifuchsian representations (Conjecture A in [8]). The general case is less clear since the representations are in general not discrete.

The strategy for proving Theorem 4.3 follows very closely that used in [8]. The problem is reformulated in terms of generalized Markoff maps and it is shown that if the BQ-conditions are satisfied, then the generalized Markoff map has Fibonacci growth. This is sufficient to obtain the absolute convergence of the series on the left hand side of the identity. To obtain the actual value of the series, in the type-preserving case, Bowditch made clever use of the quantities $\frac{x}{yz}$, $\frac{y}{zx}$ and $\frac{z}{xy}$ associated to a Markoff triple (x, y, z). By exploring the geometric interpretation of these quantities, we were able to find analogous quantities, $\Psi(y, z, x)$, $\Psi(z, x, y)$ and $\Psi(x, y, z)$ (for explicit expressions see §2.6 or §4.3), associated to a generalized Markoff triple (x, y, z), and follow through the proof of Bowditch to complete the proof of Theorem 4.3.

We remark that besides these connections with hyperbolic geometry, generalized Markoff maps are closely related to dynamical systems (see [20], [10]), algebraic number theory (see [16], and [43]) and mathematical physics (see [38]). See also [20] for related work on real generalized Markoff triples.

The rest of this chapter is organized as follows. In §4.2 we give the definitions and state the results in terms of representations. In §4.3 we reformulate the results in terms of generalized Markoff maps. We examine Bowditch's arguments in [8] and make them work for generalized Markoff maps to give the proof of our main result in this chapter, Theorem 4.10. Finally in §4.4 we explain how to draw the gaps in the extended complex plane to visualize the generalized McShane's identity (4.8).

4.2 Notation and statements of results

In this section, we set some basic notation and definitions and give precise statements of our results. The aim is to establish the basic correspondences between representations of the once-punctured torus group, Γ, into $\mathrm{SL}(2, \mathbf{C})$ and generalized Markoff maps, define the BQ-conditions, and state the main results of this chapter.

As much of this chapter is influenced by [8], we will borrow the notation and definitions from it as much as possible to avoid confusion.

The punctured torus group Γ. Let $\mathbb{T}$ be a (topological) once-punctured punctured torus and let Γ be its fundamental group. Note that algebraically, $\Gamma \cong \mathbf{Z} * \mathbf{Z}$, the free group on two generators.

We define an equivalence relation, $\sim$, on Γ such that $g \sim h$ if and only if g is conjugate to h or h^{-1}. Note that $\Gamma / \sim$ can be identified with the set of free homotopy classes of unoriented closed curves on $\mathbb{T}$.

Simple closed curves on the torus. Let $\mathscr{C}$ be the set of free homotopy classes of non-trivial, non-peripheral simple closed curves on $\mathbb{T}$ and let $\hat{\Omega} \subset \Gamma/\sim$ be the subset corresponding to $\mathscr{C}$.

Note that $\hat{\Omega}$ can be identified with $\mathbf{Q} \cup \{\infty\}$ by considering the “slope” of $[g] \in \hat{\Omega}$ as follows. Fix a pair of generators a and b of Γ. Then each class $[g]$ of $\hat{\Omega}$ has a representative $g = W(a^{\pm 1}, b)$ which is a cyclically reduced word in $\{a^{\pm 1}, b\}$, and the exponents of a in each word is either all positive or all negative. The word is unique up to cyclic permutations, and the slope is the quotient of the sum of the exponents of a with the sum of the exponents of b in the word; see [42] for details. Hence, a is identified with $\infty := \frac{1}{0}$, b with $0 := \frac{0}{1}$, ab with $1 := \frac{1}{1}$, and so on. Note that $\hat{\Omega}$ inherits a cyclic ordering from the cyclic ordering of $\mathbf{Q} \cup \{\infty\}$ induced from the standard embedding into $\mathbf{R} \cup \{\infty\} \cong S^1$.

The τ-representations. A representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ is said to be a *τ-representation*, where $\tau \in \mathbf{C}$, if for some (hence every) pair of free generators $a, b \in \Gamma$, $\mathrm{tr}\rho([a, b]) = \tau$, where $[a, b] = aba^{-1}b^{-1}$.

Note that we can also talk about τ-representations for representations ρ into $\mathrm{PSL}(2, \mathbf{C})$, since the trace of the commutator is well-defined. Most geometric constructions give rise to representations into $\mathrm{PSL}(2, \mathbf{C})$. It will, however, be more convenient for us to work with representations into $\mathrm{SL}(2, \mathbf{C})$, as the statement of the results is neater in this case. This will not affect the validity of our results since the results stated will be independent of the lift chosen, and the results could have been stated in terms of representations into $\mathrm{PSL}(2, \mathbf{C})$ as well.

It is well known that the image of the representation is non-elementary if and only if $\tau \neq 2$, so we will only be interested in τ-representations with $\tau \neq 2$. The case where $\tau = -2$ is particularly interesting. In this case, the representation ρ is said to be *type-preserving*, and except when $\mathrm{tr}\rho(a) = \mathrm{tr}\rho(b) = \mathrm{tr}\rho(ab) = 0$, $\rho([a, b])$ is always a parabolic element for any pair of generating elements a, b in Γ. This is a much studied case and is also the case extensively studied in [8].

The BQ-conditions. A representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ (or $\mathrm{PSL}(2, \mathbf{C})$) is said to satisfy the *BQ-conditions* if

(BQ1) $\mathrm{tr}\rho(g) \notin [-2, 2]$ for all $[g] \in \hat{\Omega}$; and

(BQ2) $|\mathrm{tr}\rho(g)| \le 2$ for only finitely many (possibly none) $[g] \in \hat{\Omega}$.

We also call such a representation ρ a *BQ-representation*.

Note that $\mathrm{tr}\rho(g_1) = \mathrm{tr}\rho(g_2)$ if $[g_1] = [g_2]$ (since g_1 is conjugate to g_2 or its inverse by definition); so the conditions (BQ1) and (BQ2) make sense.

Bowditch's extension of McShane identity. With the above notation and definitions, Bowditch's extension and reformulation of McShane's identity (1.1) can be stated as follows.

Theorem 4.1 (Theorem 3 in [8]) *Let $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ be a type-preserving representation which satisfies the BQ-conditions. Then*

$$\sum_{[g] \in \hat{\Omega}} h\big(\mathrm{tr}\, \rho(g)\big) = \frac{1}{2}, \tag{4.1}$$

where the function h is defined in §2.3. Moreover, the sum converges absolutely.

Remark 4.2 The identity (4.1) is also true for BQ-representations into $\mathrm{PSL}(2, \mathbf{C})$ noticing that $h(x)$ is an even function and the trace of an element of $\mathrm{PSL}(2, \mathbf{C})$ is well-defined up to sign.

Our main result of this chapter. Our generalization of above Theorem 4.1 of Bowditch, and Theorem 1.4 of [45], is as follows.

Theorem 4.3 *Let $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ be a τ-representation ($\tau \neq \pm 2$) satisfying the BQ-conditions. Let $\nu = \cosh^{-1}(-\tau/2)$. Then*

$$\sum_{[g] \in \hat{\Omega}} \mathfrak{h}\big(\mathrm{tr}\rho(g)\big) = \nu \mod 2\pi i, \tag{4.2}$$

where the function $\mathfrak{h}$ is defined in §2.3. Moreover, the sum converges absolutely.

Remark 4.4 (i) Here ν is a specific choice of *half* the complex translation length of the commutator $[\rho(a), \rho(b)]$.

(ii) Since $\mathfrak{h}$ is an even function, Theorem 4.3 also holds for representations in $\mathrm{PSL}(2, \mathbf{C})$ satisfying the BQ-conditions.

(iii) Note that for $x \neq 0$, when $\tau \to -2$, or equivalently, $\nu \to 0$, we have

$$\mathfrak{h}(x) = \log\left(1 + \frac{2(\sinh\nu)h(x)}{1 + (e^{-\nu} - 1)h(x)}\right) \sim 2\nu h(x). \tag{4.3}$$

Hence the identity (4.1) can be "obtained" by considering the first order infinitesimal terms of (4.2).

(iv) Theorem 4.3 has an equivalent formulation as Theorem 4.10 in terms of generalized Markoff maps.

(v) Bowditch [7] studied type-preserving representations in $\mathrm{PSL}(2, \mathbf{C})$ of Γ which are stabilized by a cyclic subgroup of the mapping class group of $\mathbb{T}$ generated by a hyperbolic element and proved a variation of the McShane's identity for such representations. This result can be generalized for τ-representations as well. We shall discuss it in Chapter 5 with applications to once-punctured torus bundles.

4.3 Generalized Markoff maps

In this section we study the basic geometry of generalized Markoff maps, showing that Bowditch's arguments for Markoff maps extend to this case as well, with the help of function $\Psi(x, y, z)$ defined in §2.6, to give the Fibonacci growth of generalized Markoff maps which satisfy Bowditch's Q-conditions. We shall follow the notation and proofs of Bowditch [8] whenever possible.

The binary tree Σ. Abstractly, Σ is a countably infinite simplicial tree all of whose vertices have degree 3. Geometrically, Σ is properly embedded in the hyperbolic plane and is dual to a tessellation by ideal triangles, that is, each ideal triangle in the tessellation gives rise to a vertex of Σ, which is the center of the ideal triangle, while each edge of Σ is the geodesic arc connecting a pair of adjacent ideal triangles in the tessellation.

A *complementary region* (or simply, a *region*) of Σ is the closure of a connected component of the complement. Denote by $V(\Sigma)$, $E(\Sigma)$ and $\Omega = \Omega(\Sigma)$ the set of vertices, edges, and complementary regions of Σ respectively. We also use the notation $e \leftrightarrow (X, Y; Z, W)$ to indicate that $e = X \cap Y$ and $e \cap Z$ and $e \cap W$ are the endpoints of e; see Figure 4.2 for an illustration, ignoring the direction of e there.

It will be convenient and extremely useful, although not necessary, to fix a concrete realization of the above concepts by thinking of Σ as the dual to the Farey tessellation of the hyperbolic plane in the upper half-plane model by ideal triangles. Recall that the Farey triangulation consists of edges which are complete hyperbolic geodesics joining all pairs $\{\xi, \eta\} \subset \mathbf{Q} \cup \{\infty\}$ which are Farey neighbors, and that $\xi = p/q$ and $\eta = r/s$ (where we always assume that $p, q, r, s \in \mathbf{Z}$ and $(p, q) = (r, s) = 1$) are Farey neighbors if $ps - rq = \pm 1$. See Figure 4.1. In this way, there is a natural action of $\mathrm{PSL}(2, \mathbf{Z})$ on Σ, and there is a natural correspondence of $\Omega(\Sigma)$ with $\mathbf{Q} \cup \{\infty\}$, together with the induced cyclic ordering. We use the letters $X, Y, Z, W, \ldots$ to denote the elements of Ω, and also introduce the notation $X(p/q)$ to indicate that $X \in \Omega$ corresponds to $p/q \in \mathbf{Q} \cup \{\infty\}$.

μ-Markoff triples. For a complex number $\mu \in \mathbf{C}$, a *μ-Markoff triple* is an ordered triple (x, y, z) of complex numbers satisfying the μ-Markoff equation:

$$x^2 + y^2 + z^2 - xyz = \mu. \tag{4.4}$$

Thus Markoff triples are just 0-Markoff triples.

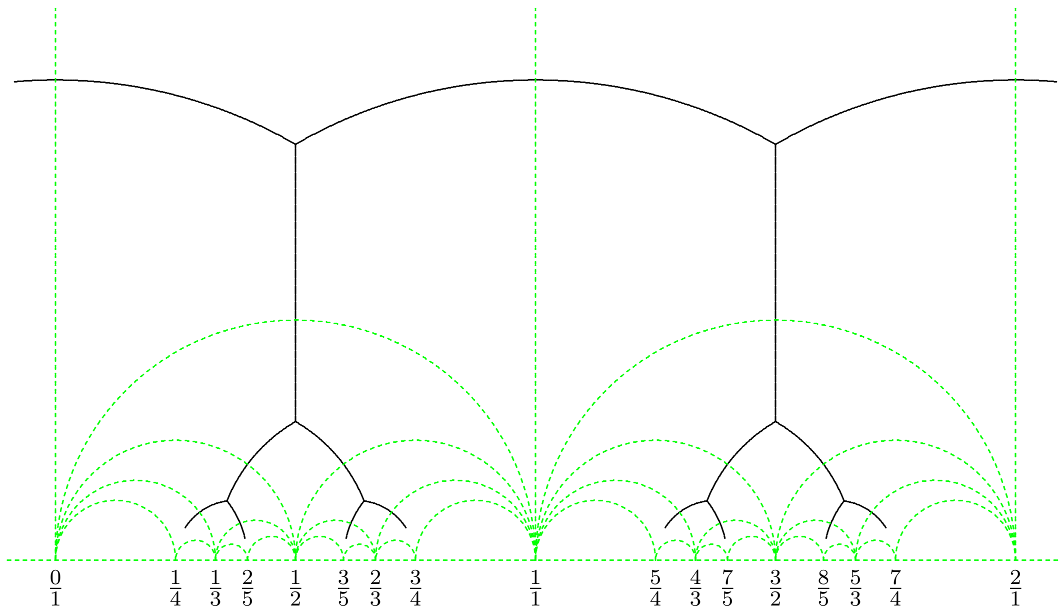

Figure 4.1: Farey tessellation and the binary tree Σ

It is easily verified that if (x, y, z) is a μ-Markoff triple, so are $(x, y, xy - z)$, $(x, xz - y, z)$, $(yz - x, y, z)$ and the permutation triples of each of them.

μ-Markoff maps. A *μ-Markoff map* is a function $\phi : \Omega \to \mathbf{C}$ such that

(i) for every vertex $v \in V(\Sigma)$, the triple $(\phi(X), \phi(Y), \phi(Z))$ is a μ-Markoff triple, where $X, Y, Z \in \Omega$ are the three regions meeting v; and

(ii) for every edge $e \in E(\Sigma)$ such that $e \leftrightarrow (X, Y; Z, W)$, we have

$$xy = z + w, \tag{4.5}$$

where $x = \phi(X), y = \phi(Y)$ and $z = \phi(Z)$.

We shall use $\mathbf{\Phi}_\mu$ to denote the set of all μ-Markoff maps. Recall that in [8] the set of all Markoff maps is denoted $\mathbf{\Phi}$; while here it is denoted $\mathbf{\Phi}_0$ in our notation.

As in the case of Markoff maps, if the edge relation (4.5) is satisfied along all edges, then it suffices that the vertex relation (4.4) be satisfied at a single vertex.

In fact one may establish a bijective correspondence between μ-Markoff maps and μ-Markoff triples, by fixing three regions X, Y, Z which meet at some vertex v_0, say $X(\infty), Y(0)$ and $Z(1)$. This process may be inverted by constructing a tree of μ-Markoff triples as Bowditch did in [8] for Markoff triples: given a triple (x, y, z) set $\phi(X) = x, \phi(Y) = y, \phi(Z) = z$, and extend over Ω as dictated by the edge relations. In this way one obtains an identification of $\mathbf{\Phi}_\mu$ with the variety in $\mathbf{C}^3$ given by the μ-Markoff equation. In particular, $\mathbf{\Phi}_\mu$ gets a nice topology as a subset of $\mathbf{C}^3$.

The natural action of $\mathrm{PSL}(2, \mathbf{Z})$ on Σ induces an action on $\mathbf{\Phi}_\mu$. There is also an action of the Klein-four group, $\mathbf{Z}_2^2$, on $\mathbf{\Phi}_\mu$ obtained by changing two of the signs in a μ-Markoff triple, for example, $(x, y, z) \mapsto (-x, -y, z)$. (We get the same action, up to automorphisms of $\mathbf{Z}_2^2$, no matter at which vertex we choose to perform this operation.) This action is free and properly discontinuous on $\mathbf{\Phi}_\mu \backslash \{(\pm\sqrt{\mu}, 0, 0), (0, \pm\sqrt{\mu}, 0), (0, 0, \pm\sqrt{\mu})\}$.

Natural correspondence: $\mathcal{X}_{\mu-2} \equiv \mathbf{\Phi}_\mu$. There is also a natural correspondence between $(\mu - 2)$-representations ρ and μ-Markoff maps, obtained by fixing a generating pair a, b for Γ. First note that by fixing a and b, we get a correspondence between Ω and $\hat{\Omega}$ via the correspondence with $\mathbf{Q} \cup \{\infty\}$ and that $X(p/q), Y(r/s) \in \Omega$ share an edge if and only if p/q and r/s are Farey neighbors, if and only if the corresponding elements $[g_1], [g_2] \in \hat{\Omega}$ correspond to simple closed curves on $\mathbb{T}$ which have geometric intersection number one, or equivalently, have representatives g_1 and g_2 which are a pair of free generators for Γ.

A $(\mu - 2)$-representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ naturally gives rise to a μ-Markoff map $\phi \in \mathbf{\Phi}_\mu$ via the correspondence between Ω and $\hat{\Omega}$ by taking the traces of $[g] \in \hat{\Omega}$, that is, by $\phi(X) = \mathrm{tr}\rho(g)$ where $[g] \in \hat{\Omega}$ represents the simple closed curve corresponding to $X \in \Omega$. The edge and vertex relations follow from the trace identities in $\mathrm{SL}(2,\mathbf{C})$:

$$\mathrm{tr}A\,\mathrm{tr}B = \mathrm{tr}AB + \mathrm{tr}AB^{-1}, \tag{4.6}$$

$$2 + \mathrm{tr}[A,B] = (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - \mathrm{tr}A\,\mathrm{tr}B\,\mathrm{tr}AB. \tag{4.7}$$

Representations conjugate in $\mathrm{SL}(2,\mathbf{C})$ give rise to the same μ-Markoff map.

Conversely, given any μ-Markoff map ϕ, we can recover the $(\mu-2)$-representation $\rho : \Gamma \to \mathrm{SL}(2,\mathbf{C})$ up to conjugacy as follows. Consider the three regions $X(\infty)$, $Y(0)$ and $Z(1)$ which meet at the vertex v_0 and consider the μ-Markoff triple (x,y,z), where $x = \phi(X)$, $y = \phi(Y)$ and $z = \phi(Z)$. Then one can find $A, B \in \mathrm{SL}(2,\mathbf{C})$, unique up to simultaneous conjugacy, such that $\mathrm{tr}A = x$, $\mathrm{tr}B = y$ and $\mathrm{tr}AB = z$; a specific choice is given by Bowditch in §4 of [8], see also [20]. This gives a $(\mu-2)$-representation ρ with $\rho(a) = A$, $\rho(b) = B$ and hence $\mathrm{tr}\rho(a) = x$, $\mathrm{tr}\rho(b) = y$, $\mathrm{tr}\rho(ab) = z$.

In this way we can identify $\mathbf{\Phi}_\mu$ naturally with the set of $(\mu-2)$-representations of Γ into $\mathrm{SL}(2,\mathbf{C})$.

Definition 4.5 Given $\phi \in \mathbf{\Phi}_\mu$ and $k \geq 0$, the set $\Omega(k) = \Omega_\phi(k) \subseteq \Omega$ of regions is defined by

$$\Omega_\phi(k) = \{X \in \Omega \mid |\phi(X)| \leq k\}.$$

Then Bowditch's Q-conditions for μ-Markoff maps can be stated as follows.

BQ-conditions for μ-Markoff maps. A generalized Markoff map $\phi \in \mathbf{\Phi}_\mu$ is said to satisfy the *BQ-conditions* if

(BQ1) $\phi^{-1}([-2,2]) = \emptyset$; and

(BQ2) $\Omega_\phi(2)$ is finite (possibly empty).

We denote by $(\mathbf{\Phi}_\mu)_Q$ the set of all generalized μ-Markoff maps which satisfy the BQ-conditions.

In the rest of this section we shall examine the results and arguments given in §1–§4 of [8] about Markoff maps and point out those which still hold for generalized Markoff maps.

Results for generalized Markoff maps. Let us first state the results for generalized Markoff maps corresponding to those in [8] for Markoff maps.

Theorem 4.6 *If $\phi \in \mathbf{\Phi}_\mu$ then*

(1) $\Omega_\phi(m)$ *is non-empty for some constant $m = m(\mu) > 0$ depending on μ but not on ϕ; and*

(2) *for any $k \geq 2$, the union $\bigcup \Omega_\phi(k) := \bigcup_{X \in \Omega_\phi(k)} X$ is connected (as a subset of the hyperbolic plane). In particular, $\bigcup \Omega_\phi(2)$ is connected.*

Note that it gives a well-defined topology on $\mathbf{\Phi}_\mu$ by identifying $\mathbf{\Phi}_\mu$ as the complex variety $\{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 - xyz = \mu\}$ with the subspace topology of $\mathbf{C}^3$ under the correspondence $\phi \leftrightarrow (\phi(X), \phi(Y), \phi(Z))$ for any three regions X, Y, Z meeting at a vertex.

Theorem 4.7 *The set $(\mathbf{\Phi}_\mu)_Q$ is an open subset of $\mathbf{\Phi}_\mu$.*

We shall give, in a later part of this section, the definition of Bowditch's term "$\log^+|\phi|$ has Fibonacci growth", where $\log^+ : (0, \infty) \to [0, \infty)$ is defined by $\log^+(x) = \max\{0, \log(x)\}$. It implies that $\sum_{X \in \Omega} |\phi(X)|^{-t}$ converges absolutely for any $t > 0$; in particular, $\Omega_\phi(k)$ is finite for all k and so $\phi(\Omega) \subseteq \mathbf{C}$ is discrete.

Theorem 4.8 *If $\phi \in (\mathbf{\Phi}_\mu)_Q$ $(\mu \neq 4)$, then the function $\log^+|\phi| : \Omega \to [0, \infty)$ has Fibonacci growth.*

The proof will be the same as that of Theorem 2 in [8] with minor modifications which will be pointed out in a later part of this section.

Exactly as in [8], the argument of proving Theorem 4.8 can be applied to a single branch of the tree Σ. Let us first give the notation to state Proposition 4.9 below.

For $\vec{e} \in \vec{E}(\Sigma)$ let $\Sigma^+(\vec{e}), \Sigma^-(\vec{e})$ be the two disjoint subtrees of Σ obtained by removing the interior of e from Σ, such that $e \cap \Sigma^+(\vec{e})$ is the head of $\vec{e}$ and $e \cap \Sigma^-(\vec{e})$ is its tail. Let $\Omega^+(\vec{e}), \Omega^-(\vec{e}) \subseteq \Omega$ be the sets of regions whose boundaries lie in $\Sigma^+(\vec{e})$ and $\Sigma^-(\vec{e})$ respectively, and set $\Omega^0(e) = \{X_1, X_2\}$ where $e = X_1 \cap X_2$. Then Ω can be written as a disjoint union: $\Omega = \Omega^0(e) \sqcup \Omega^+(\vec{e}) \sqcup \Omega^-(\vec{e})$.

Let $\Omega^{0+}(\vec{e}) = \Omega^0(e) \cup \Omega^+(\vec{e})$ and $\Omega^{0-}(\vec{e}) = \Omega^0(e) \cup \Omega^-(\vec{e})$.

Proposition 4.9 (Proposition 3.9 in [8]) *Suppose $\phi \in \mathbf{\Phi}_\mu$ and $\vec{e}$ is a directed edge of Σ such that $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2)$ is finite and $\Omega^{0-}(\vec{e}) \cap \phi^{-1}[-2,2] = \emptyset$. Then $\log^+|\phi|$ has a lower Fibonacci bound on $\Omega^{0-}(\vec{e})$.* □

For $\phi \in (\mathbf{\Phi}_\mu)_Q$, we have the following version of the generalized McShane's identity, which is a reformulation of Theorem 4.3 in terms of generalized Markoff maps.

Theorem 4.10 *If $\phi \in (\mathbf{\Phi}_\mu)_Q$ $(\mu \neq 4)$ then*

$$\sum_{X \in \Omega} \mathfrak{h}(\phi(X)) = \nu \mod 2\pi i, \tag{4.8}$$

where $\nu = \cosh^{-1}(1-\mu/2)$, $\mathfrak{h} = \mathfrak{h}_\tau$ (with $\tau = \mu - 2$) is defined as in §2.3, and the sum converges absolutely.

There is also a version, Proposition 4.11 below, of Theorem 4.10 applicable to a single branch of the binary tree Σ, on which ϕ satisfies the BQ-conditions. For this we need to define the ϕ-edge weight $\psi(\vec{e}) = \psi_\phi(\vec{e})$ and a half function $\hat{\mathfrak{h}}$ of $\mathfrak{h}$ as follows.

The ϕ-edge weight $\psi(\vec{e})$. We denote a directed edge $\vec{e}$ by $\vec{e} = (X, Y; \to Z)$ if $e = X \cap Y$ and $\vec{e}$ points towards Z, as shown in Figure 4.2.

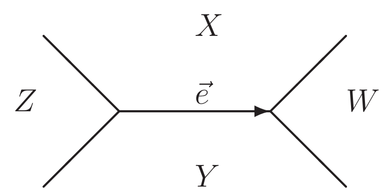

Figure 4.2: The directed edge $\vec{e} = (X, Y; Z \to W)$

For a given $\phi \in \mathbf{\Phi}_\mu$, we define the **$\phi$-edge weight** $\psi(\vec{e})$ of a directed edge $\vec{e} = (X, Y; \to Z)$ by

$$\psi(\vec{e}) = \log \frac{1 + (e^\nu - 1)(z/xy)}{\sqrt{1-\mu/x^2}\sqrt{1-\mu/y^2}} \in \mathbf{C}. \tag{4.9}$$

Note that $\psi(\vec{e}) = \Psi(x, y, z)$ where the function Ψ is defined in §2.6, with specific choices of the square roots involved.

The half gap function $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_\tau$. For given $\tau \in \mathbf{C}$ with $\tau \neq 0, 2$, we define the half gap function $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_\tau : \mathbf{C} \backslash \{0, \tau + 2\} \to \mathbf{C}$ by

$$\hat{\mathfrak{h}}(x) = \log \frac{1 + (e^\nu - 1)h(x)}{\sqrt{1 - \mu/x^2}} \in \mathbf{C}. \tag{4.10}$$

Then it is easy to check that for all $x \in \mathbf{C} \backslash \{0, \tau + 2\}$,

$$2\,\hat{\mathfrak{h}}(x) = \mathfrak{h}(x) \mod 2\pi i. \tag{4.11}$$

Proposition 4.11 *Suppose $\phi \in \mathbf{\Phi}_\mu\,(\mu \neq 0, 4)$ and $\vec{e}$ is a directed edge of Σ such that $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2)$ is finite and that $\Omega^{0-}(\vec{e}) \cap \phi^{-1}[-2, 2] = \emptyset$. Then*

$$\psi(\vec{e}) = \sum_{X \in \Omega^0(e)} \hat{\mathfrak{h}}(\phi(X)) + \sum_{X \in \Omega^-(\vec{e})} \mathfrak{h}(\phi(X)) \mod 2\pi i, \tag{4.12}$$

where the infinite sum converges absolutely.

Examination of Bowditch's arguments. For the rest of this section, let us fix on one μ-Markoff map, ϕ, where $\mu \neq 4$.

We make the **assumption** that $\phi^{-1}(0) = \emptyset$, which is not essential, to simplify the exposition a bit. Note that the assumption is true for $\phi \in (\mathbf{\Phi}_\mu)_Q$.

We shall also adopt the following **convention** of Bowditch [8]: We use upper case latin letters for elements of Ω, and the corresponding lower case letters for the values assigned to them by ϕ; that is, $x = \phi(X)$, $y = \phi(Y)$ etc.

The above convention and assumption will allow us to write the edge relation, (4.5), in the following convenient form:

$$\frac{z}{xy} + \frac{w}{xy} = 1. \tag{4.13}$$

Denote by $\vec{E}(\Sigma)$ the set of directed edges of Σ where the direction is always taken to be from the tail to the head (as in the direction of the arrow), and for a directed edge $\vec{e} \in \vec{E}(\Sigma)$, we use $\vec{e} = (X, Y; W \to Z)$ to indicate that $\vec{e}$ is the directed edge from W to Z. See Figure 4.2. We shall always use e to indicate the underlying undirected edge corresponding to the directed edge $\vec{e}$.

Arrows assigned by a μ-Markoff map. As Bowditch did in [8], we may use $\phi \in \mathbf{\Phi}_\mu$ to assign to each undirected edge, e, a particular directed edge, $\vec{e} = \alpha_\phi(e)$, with underlying edge e. Suppose $e \leftrightarrow (X, Y; Z, W)$. If $|z| > |w|$ then the arrow on e points towards W; in other words, $\vec{e} = (X, Y; Z \to W)$. Note that the statement is equivalent to $\Re\left(\frac{z}{xy}\right) > \frac{1}{2}$. In particular, it implies that $2|z| > |xy|$. If $|z| < |w|$, we put an arrow on e pointing towards Z, that is, $\vec{e} = (X, Y; W \to Z)$. If it happens that $|z| = |w|$ then we choose $\alpha_\phi(e)$ arbitrarily.

Then the following lemma, which is Lemma 3.2(3) of [8], holds with the same one-sentence proof there.

Lemma 4.12 (Lemma 3.2(3) in [8]) *Suppose $X, Y, Z \in \Omega$ meet at a vertex $v \in V(\Sigma)$, and that the arrows on the edges $X \cap Y$ and $X \cap Z$ both point away from v. Then $|x| \le 2$.* $\square$

Proof of Theorem 4.6. The proof of Theorem 1(2) in [8] applies word by word here to give the proof of Theorem 4.6. We sketch the idea here. If $\Omega(k)$ (where $k \geq 2$) were not connected then there is a finite path in Σ connecting two components of $\Omega(k)$ so that all regions adjacent to the path are not in $\Omega(k)$. The two (directed) edges at the end of the path, with arrows assigned by ϕ, will point away from the path. Hence either the path consists of a single edge or there is a vertex v on the path such that the two edges on the path and incident to v both point away from v. In the latter case the region adjacent to these two edges has norm no greater than 2 by Lemma 4.12. In the former case one of the regions adjacent to the single edge must have norm no greater than 2 by the edge relation (4.5). In each case there is a contradiction. □

Note that Lemma 3.2(1) in [8] (which states that there are no sources for $\phi \in \mathbf{\Phi}_0 \backslash \{\mathbf{0}\}$) will *no longer* hold for general generalized Markoff maps. For example, if $x \in [-2, 2]$, then the generalized Markoff map which corresponds to the triple (x, x, x) at a fixed vertex v has a source at v. On the other hand, Lemma 3.2(2) in [8] can be modified as follows.

Lemma 4.13 (Lemma 3.2(2) in [8]) *There is a constant $m = m(\mu) > 0$, not depending on ϕ, such that if three regions X, Y, Z meet at a vertex which is a sink with respect to ϕ, then*

$$\min\{|x|, |y|, |z|\} \leq m. \tag{4.14}$$

Proof. We show that if $|x|, |y|, |z|$ are all sufficiently large then the vertex, v, which the regions X, Y, Z meet, cannot be a sink. We may assume $x, y, z \neq 0$ and $|x| \leq |y| \leq |z|$. We can thus rewrite (4.4) as:

$$\frac{z}{xy} + \frac{x}{yz} + \frac{y}{zx} = 1 + \frac{\mu}{xyz}, \tag{4.15}$$

hence

$$\Re\Big(\frac{z}{xy}\Big) + \Re\Big(\frac{x}{yz}\Big) + \Re\Big(\frac{y}{zx}\Big) = 1 + \Re\Big(\frac{\mu}{xyz}\Big). \tag{4.16}$$

Since $|x|, |y|, |z|$ are all sufficiently large and μ is fixed, $\left|\frac{x}{yz}\right|, \left|\frac{y}{zx}\right|$ and $\left|\frac{\mu}{xyz}\right|$ are all sufficiently small and so are $\left|\Re\left(\frac{x}{yz}\right)\right|, \left|\Re\left(\frac{y}{zx}\right)\right|$ and $\left|\Re\left(\frac{\mu}{xyz}\right)\right|$. It follows that $\Re\left(\frac{z}{xy}\right) > \frac{1}{2}$. Hence the arrow on the edge $X \cap Y$ is directed away from v and therefore v is not a sink. □

It seems not easy to determine the optimal $m(\mu)$ for general μ, although it is shown by Lemma 4.13, [8] that the optimal $m(0) = 3$.

Neighbors around a region. For each $X \in \Omega$, its boundary ∂X is a bi-infinite path consisting of a sequence of edges of the form $X \cap Y_n$, where $(Y_n)_{n \in \mathbf{Z}}$ is a bi-infinite sequence of complimentary regions. Let $x = \lambda + \lambda^{-1}$ where $|\lambda| \geq 1$. If $x = 2$, then the vertex relation tells us that $y_{n+1} = y_n \pm \sqrt{\mu - 4}$, and the edge relation tells us that $y_{n+1} - y_n = y_n - y_{n-1}$, hence the $\pm$ sign is constant in n. Similarly, if $x = -2$, then $y_{n+1} = -y_n \pm \sqrt{\mu - 4}$, but this time, $y_{n+1} - y_n = -(y_n - y_{n-1})$, hence the $\pm$ sign alternates in n. If $x = \pm\sqrt{\mu}$, then $y_{n+1} = (1/2)(x \pm \sqrt{\mu - 4})y_n$ where the $\pm$ sign is constant in n. If $x \notin \{\pm 2, \pm\sqrt{\mu}\}$ then there are constants $A, B \in \mathbf{C} \backslash \{0\}$ with $AB = (x^2 - \mu)/(x^2 - 4)$ such that $y_n = A\lambda^n + B\lambda^{-n}$. Note that $|\lambda| = 1$ if and only if $x \in [-2, 2] \subseteq \mathbf{R}$. Hence we deduce that the following holds. (This is Corollary 3.3 in [8] in the case $\mu = 0$.)

Lemma 4.14 (1) *If $x \notin [-2, 2] \cup \{\pm\sqrt{\mu}\}$, then $|y_n|$ grows exponentially as $n \to \infty$ and as $n \to -\infty$.*

(2) *If $x \in (-2, 2)$, then $|y_n|$ remains bounded.*

(3) *If $x = 2$, then either $y_n = y_0 + n\sqrt{\mu - 4}$ for all n, or $y_n = y_0 - n\sqrt{\mu - 4}$ for all n.*

(4) *If $x = -2$, then either $y_n = (-1)^n(y_0 + n\sqrt{\mu - 4})$ for all n, or $y_n = (-1)^n(y_0 - n\sqrt{\mu - 4})$ for all n.*

(5) *If $x = \pm\sqrt{\mu}$, then either $y_n = y_0[(x + \sqrt{\mu - 4})/2]^n$ for all n, or $y_n = y_0[(x - \sqrt{\mu - 4})/2]^n$ for all n.* □

The following lemma tells us that if $\phi \in (\mathbf{\Phi}_\mu)_Q$ then $\phi(X) \neq \pm\sqrt{\mu}$ for all regions $X \in \Omega$.

Lemma 4.15 *If $\phi \in \mathbf{\Phi}_\mu$ and $\phi(X) = \pm\sqrt{\mu}$ for some $X \in \Omega$, then $\phi \notin (\mathbf{\Phi}_\mu)_Q$.*

Proof. Without loss of generality, we may assume that $y_n = y_0[(x + \sqrt{\mu-4})/2]^n$ for all n. If $y_0 = 0$ then $\psi \notin (\mathbf{\Phi}_\mu)_Q$. Suppose $y_0 \neq 0$ and let $\lambda = (x + \sqrt{\mu-4})/2$. Then it is easy to see that $\lambda \neq 0$. If $|\lambda| = 1$, then $\lambda^{-1} = (x - \sqrt{\mu-4})/2$. Hence $x = \lambda + \lambda^{-1} \in [-2, 2]$, again $\phi \notin (\mathbf{\Phi}_\mu)_Q$. Now suppose $|\lambda| \neq 1$. If $|\lambda| < 1$ then $|y_n| \to 0$ as $n \to \infty$, whereas if $|\lambda| > 1$ then $|y_n| \to 0$ as $n \to -\infty$. Hence $\Omega_\phi(2)$ is infinite and $\phi \notin (\mathbf{\Phi}_\mu)_Q$. □

The following lemma, together with Lemma 4.12, will imply that if $\phi \in (\mathbf{\Phi}_\mu)_Q$ then the arrows assigned by ϕ on the edges of Σ which are far away from a fixed subtree $T \subset \Sigma$ will have to point towards T.

Lemma 4.16 (Lemma 3.4 in [8]) *Suppose β is an infinite ray in Σ consisting of a sequence, $(e_n)_{n\in\mathbf{N}}$, of edges of Σ such that the arrow on each e_n assigned by ϕ is directed towards e_{n+1}. Then β meets at least one region X with $|\phi(X)| < 2$.*

Proof. This is the proof of Lemma 3.4 of [8] there with slight refinement for its last part as follows. Suppose that all regions X which are incident to β have $|\phi(X)| \geq 2$. Then by the argument there, for n sufficiently large, there are directed edge $\alpha_\phi(e_n) \leftrightarrow (X, Y; Z, W)$ in β with $Y \cap W \subseteq \beta$ and $X \cap Z \subseteq \beta$ so that (where $\simeq$ means 'arbitrarily close to') $z/xy \simeq 1/2, x^2 \simeq 4$ and $y^2 \simeq 4$. Hence $z^2 \simeq 4$ and $xyz \simeq 2z^2 \simeq 8$. Thus $\mu = x^2 + y^2 + z^2 - xyz \simeq 4$ which is impossible since we have assumed that $\mu \neq 4$. □

Fibonacci functions. Now let us review the definition given in [8] that a function $f : \Omega \to [0, \infty)$ has an upper/lower Fibonacci bound.

Given a directed edge $\vec{e}$, let $-\vec{e}$ be the same edge e pointing in the opposite direction. Note that $\Omega^+(\vec{e}) = \Omega^-(-\vec{e})$.

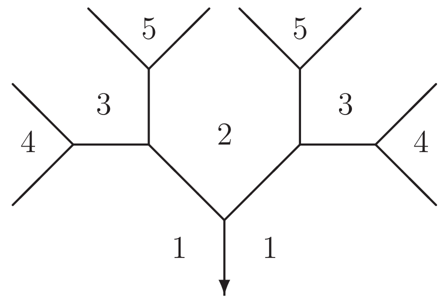

Figure 4.3: A few values of the Fibonacci function F_e

For $X \in \Omega^{0-}(e)$ we define $d(X)$ to be the number of edges in the shortest path joining the head of $\vec{e}$ to X. Given any $Z \in \Omega^-$, there are precisely two regions $X, Y \in \Omega^{0-}$ meeting Z and satisfying $d(X) < d(Z)$ and $d(Y) < d(Z)$. Note that X, Y, Z all meet in a vertex.

Now we can define the **Fibonacci function** $F_e : \Omega \to \mathbf{N}$ with respect to an edge e as follows. We orient e arbitrarily as $\vec{e}$ and define $F_{\vec{e}} : \Omega^{0-}(\vec{e}) \to \mathbf{N}$ by $F_{\vec{e}}(W) = 1$ for $W \in \Omega^0(e)$ and $F_{\vec{e}}(Z) = F_{\vec{e}}(X) + F_{\vec{e}}(Y)$ for $Z \in \Omega^-(\vec{e})$ where $X, Y \in \Omega^{0-}(\vec{e})$ are the two regions meeting Z and satisfying $d(X) < d(Z)$ and $d(Y) < d(Z)$. Now we define $F_e(X) = F_{\vec{e}}(X)$ for $X \in \Omega^{0-}(\vec{e})$ and $F_e(X) = F_{-\vec{e}}(X)$ for $X \in \Omega^+(\vec{e})$. See Figure 4.3 with $F_e(X)$ printed for some $X \in \Omega$.

The functions F_e provide a means for measuring the growth rates of functions defined on subsets of Ω.

Lemma 4.17 (Lemma 2.1.1 in [8]) *Suppose* $f : \Omega^{0-}(\vec{e}) \to [0, \infty)$ *where* $\Omega^0(e) = \{X_1, X_2\}$.

(1) *If f satisfies $f(Z) \leq f(X) + f(Y) + c$ for some fixed constant c and arbitrary $X, Y, Z \in \Omega^{0-}(\vec{e})$ meeting at a vertex and satisfying $d(X) < d(Z)$ and $d(Y) < d(Z)$, then $f(X) \leq (K + c)F_e(X) - c$ for all $X \in \Omega^{0-}(\vec{e})$, where* $K = \max\{f(X_1), f(X_2)\}$.

(2) *If f satisfies $f(Z) \geq f(X) + f(Y) - c$ for some fixed constant c where* $0 < c < k = \min\{f(X_1), f(X_2)\}$ *and arbitrary X, Y, Z as in part* (1)*, then* $f(X) \geq$

$(k-c)F_e(X)+c$ for all $X \in \Omega^{0-}(\vec{e})$.

Corollary 4.18 (Corollary 2.1.2 in [8]) *Suppose $f : \Omega \to [0,\infty)$ satisfies an inequality of the form $f(Z) \leq f(X)+f(Y)+c$ for some fixed constant c, whenever $X, Y, Z \in \Omega$ meet at a vertex. Then for any given edge $e \in E(\Sigma)$, there is a constant $K > 0$, such that $f(X) \leq KF_e(X)$ for all $X \in \Omega$.*

Note that for any edge $e' \in E(\Sigma)$, $f = F_{e'}$ satisfies the hypotheses of Corollary 4.18, with $c = 0$. Thus for any two edgs $e, e' \in E(\Sigma)$, there is some $K > 0$ such that

$$K^{-1}F_e(X) \leq F_{e'}(X) \leq KF_e(X)$$

for all $X \in \Omega$. Hence the properties in the following definitions are independent of the choices the edge $e \in E(\Sigma)$.

Fibonacci bounds. Suppose $f : \Omega \to [0,\infty)$, and $\Omega' \subseteq \Omega$. We say that

- f has an **upper Fibonacci bound on** Ω' if there is some constant $\kappa > 0$ such that $f(X) \leq \kappa F_e(X)$ for all $X \in \Omega'$.
- f has an **lower Fibonacci bound on** Ω' if there is some constant $\kappa > 0$ such that $f(X) \geq \kappa F_e(X)$ for all but finitely many $X \in \Omega'$.
- f has **Fibonacci growth on** Ω' if it has both upper and lower Fibonacci bounds on Ω'.
- f has **Fibonacci growth** if f has Fibonacci growth on *all* of Ω.

Remark 4.19 Note that if Ω' is the union of a finite set of subsets $\Omega_1, \cdots, \Omega_m \subseteq \Omega$, then f has an upper (lower) Fibonacci bound if and only if it has an upper (lower) Fibonacci bound on each of Ω_i.

Upper Fibonacci bounds. The following lemma tells us that for an arbitrary μ-Markoff map ϕ, the function $\log^+|\phi|$ always has an upper Fibonacci bound on Ω. Hence we only need to consider criteria for it to have a lower Fibonacci bound on certain branches of Σ.

Lemma 4.20 *For every $\phi \in \Phi_\mu$, $\log^+|\phi|$ has an upper Fibonacci bound on Ω.*

Proof. By Corollary 4.18 we only need to show that for an arbitrary μ-Markoff triple (x, y, z)

$$\log^+|z| \le \log 4 + \log^+|\mu| + \log^+|x| + \log^+|y|. \tag{4.17}$$

If $|z| \le 2|x|$ or $|z| \le 2|y|$ then (4.17) holds already. So we suppose $|z| \ge 2|x|$ and $|z| \ge 2|y|$. Then from $\mu + xyz = x^2 + y^2 + z^2$ we have

$$\begin{aligned} |\mu| + |xyz| &\ge |z|^2 - |x|^2 - |y|^2 \\ &= |z|^2/2 + (|z|^2/4 - |x|^2) + (|z|^2/4 - |y|^2) \\ &\ge |z|^2/2. \end{aligned}$$

Hence $|z|^2 \le 4|xyz|$ or $|z|^2 \le 4|\mu|$ according to whether $|\mu| \le |xyz|$ or $|xyz| \le |\mu|$. Thus

$$|z| \le 4|xy| \quad \text{or} \quad |z| \le |z|^2 \le 4|\mu|$$

(since we may assume $|z| \ge 1$) from which (4.17) follows easily. □

Lower Fibonacci bounds. The lower Fibonacci bounds are more interesting since, as the following proposition tells, they will give the convergence of certain series, in particular, the absolute convergence of series on the left hand side of (4.8) since we have $|\mathfrak{h}(x)| = O(|x|^{-2})$ as $|x| \to \infty$.

Proposition 4.21 (Proposition 2.1.4 in [8]) *If $f : \Omega \to [0, \infty)$ has a lower Fibonacci bound, then $\sum_{X\in\Omega} f(X)^{-s}$ converges for all $s > 2$ (after excluding a finite subset of Ω on which f takes the value 0).* □

Corollary 4.22 *If $\phi \in (\Phi_\mu)_Q$ then for any $t > 0$, the series $\sum_{X\in\Omega} |\phi(X)|^t$ converges absolutely.* □

The following lemmas and corollaries of [8] hold with the same proofs there, giving criteria for $\log^+|\phi|$ to have a lower Fibonacci bound on certain branches of the binary tree Σ.

Lemma 4.23 (Lemma 3.5 in [8]) *Suppose $\vec{e} \in \vec{E}(\Sigma)$ is such that $\alpha_\phi(e) = \vec{e}$ and $\Omega^0(e) \cap \Omega(2) = \emptyset$. Then $\Omega^{0-}(\vec{e}) \cap \Omega(2) = \emptyset$. Moreover, the arrow on each edge of $\Sigma^-(\vec{e})$ is directed towards e.* □

Corollary 4.24 (Corollary 3.6 in [8]) *With the hypotheses of Lemma 4.23, write $f(X) = \log|\phi(X)|$ for any $X \in \Omega^{0-}(\vec{e})$. Let $m = \min\{f(X) \mid X \in \Omega^0(e)\} > \log 2$. Then $f(X) \geq (m - \log 2)F_e(X)$ for all $X \in \Omega^{0-}(\vec{e})$.* □

Corollary 4.25 (Corollary 3.7 in [8]) *If $\Omega(2) = \emptyset$, then there is a unique sink, and $\log^+|\phi|$ has a lower Fibonacci growth.* □

To prove Theorem 4.8 we expand somewhat on Lemma 4.23 and consider the case where $\Omega^-(\vec{e}) \cap \Omega_\phi(2) = \emptyset$ and exactly one of the two regions in $\Omega^0(e)$ has norm no greater than 2.

Lemma 4.26 (Lemma 3.8 in [8]) *Suppose $\vec{e} \in \vec{E}(\Sigma)$ is such that $\alpha_\phi(e) = \vec{e}$ and $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2) = \{X_0\}$ where $X_0 \in \Omega^0(\vec{e})$ with $x_0 \notin [-2, 2]$. Then $\log^+|\phi|$ has a lower Fibonacci bound on $\Omega^{0-}(\vec{e})$.*

Proof. The proof of Lemma 3.8 in [8] applies. We repeat it here. Let $(\vec{e}_n)_{n=0}^\infty$ be the sequence of directed edges lying in the boundary of X_0 and in $\Omega^{0-}(\vec{e})$ so that $\vec{e}_0 = \vec{e}$ and $\vec{e}_n$ is directed away from $\vec{e}_{n+1}$. For $n \geq 1$, let v_n be the vertex incident on both e_{n-1} and e_n, and let $\vec{\varepsilon}_n$ be the third edge (distinct from e_{n-1} and e_n) incident on v_n and directed towards v_n. For $n \geq 0$, let Y_n

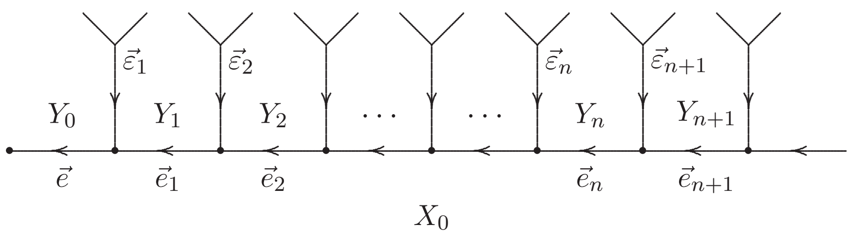

Figure 4.4: Regions and edges around the region X_0

be the region such that $Y_n \cap X_0 = e_n$. See Figure 4.4 for an illustration. Thus $\Omega^{0-}(\vec{e}) = \{X_0\} \cup \bigcup_{n=1}^{\infty} \Omega^{0-}(\vec{\varepsilon}_n)$.

By Lemma 4.14, $|y_n|$ grows exponentially as $n \to \infty$, and so $\log|y_n| \geq cn$ for some $c > 0$. Hence we have $\log^+|\phi(X)| \geq cnF_{\vec{\varepsilon}_n}(X)$ for all $n \geq 1$ and for all $X \in \Omega^{0-}(\vec{\varepsilon}_n)$. Thus it follows easily (using Lemma 4.17(2)) that $\log^+|\phi|$ has a lower Fibonacci bound on $\Omega^{0-}(\vec{e})$. $\square$

Proof of Theorem 4.8. The proof of Theorem 4.8 is then the same as that of Theorem 2 in [8]. We sketch it as follows. By Lemma 4.20, we only need to show that $\log^+|\phi|$ has a lower Fibonacci bound on Ω. If $\Omega_\phi(2)$ has at most one element, the conclusion follows easily by Corollary 4.25 and Lemma 4.26. Hence we suppose $\Omega_\phi(2)$ has at least two elements.

Recall that $\Omega_\phi(2) \subseteq \Omega$ is finite and $\bigcup \Omega_\phi(2)$ is connected. Let T be the (finite) subtree of Σ *spanned* by the set of edges e such that $\Omega^0(e) \subseteq \Omega_\phi(2)$. Let $C = C(T)$ be the circular set of directed edges given by T. Note that $\Omega = \bigcup_{\vec{e} \in C} \Omega^{0-}(\vec{e})$. Hence it suffices to show that $\log^+|\phi|$ has a lower Fibonacci bound on $\Omega^{0-}(\vec{e})$ for every $\vec{e} \in C$. Then the conclusion of Theorem 4.8 follows by the following claim, Lemma 4.23 and Lemma 4.26.

Claim. For each $\vec{e} \in C$, we have $\vec{e} = \alpha_\phi(e)$, $\Omega^-(\vec{e}) \cap \Omega_\phi(2) = \emptyset$ and $\Omega^0(e) \cap \Omega_\phi(2)$ has at most one element.

To prove the claim, let $\vec{e} = (X, Y; Z \to W) \in C(T)$. If one of X and Y, say

X, is in $\Omega_\phi(2)$ then $Y, Z \notin \Omega_\phi(2)$ and $W \in \Omega_\phi(2)$ by the definition of T. Hence in this case $\vec{e} = \alpha_\phi(e)$, $\Omega^-(\vec{e}) \cap \Omega_\phi(2) = \emptyset$ and $\Omega^0(e) \cap \Omega_\phi(2)$ has one element, X. Now suppose neither X nor Y is in $\Omega_\phi(2)$ then $W \in \Omega_\phi(2)$ and $Z \notin \Omega_\phi(2)$ since $\bigcup \Omega_\phi(2)$ is connected. Thus in this case $\vec{e} = \alpha_\phi(e)$, $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2) = \emptyset$. This proves the claim, and hence Theorem 4.8. □

The definition of $H(x)$. To prove Theorem 4.7, that is, $(\mathbf{\Phi}_\mu)_Q$ is an open subset of $\mathbf{\Phi}_\mu$, we introduce a continuous function $H : \mathbf{C}\backslash\big([-2,2] \cup \{\pm\sqrt{\mu}\}\big) \to \mathbf{R}_{>0}$ so that for any $\phi \in \mathbf{\Phi}_\mu$ and $X \in \Omega$, if $(Y_n)_{n\in\mathbf{Z}}$ is the bi-infinite sequence of regions meeting X, then there are integers $n_1 \le n_2$ such that $|y_n| \le H(x)$ if and only if $n_1 \le n \le n_2$ and that $|y_n|$ is monotonically decreasing for $n \in (-\infty, n_1]$ and is monotonically increasing for $n \in [n_2, \infty)$.

Lemma 4.27 *For $x \neq \pm\sqrt{\mu}$ and $x \notin [-2,2]$, $H(x)$ can be chosen as*

$$H(x) = \sqrt{\left|\frac{x^2-\mu}{x^2-4}\right|}\,\frac{2|\lambda(x)|^2}{|\lambda(x)|-1}, \tag{4.18}$$

where

$$\lambda(x) = \frac{x}{2}\left(1+\sqrt{1-\frac{4}{x^2}}\,\right) = e^{l(x)/2}. \tag{4.19}$$

Proof. Let us write $\lambda = \lambda(x)$. Then $|\lambda| > 1$ since $x \notin [-2,2]$. As explained before Lemma 4.14, there exist A, B with $AB = (x^2-\mu)/(x^2-4)$ such that $y_n = A\lambda^n + B\lambda^{-n}$. Note that since we can replace A, B respectively by $A\lambda^m, B\lambda^{-m}$ for $m \in \mathbf{Z}$, we may assume that $\sqrt{|AB|}|\lambda|^{-1} \le |A|, |B| \le \sqrt{|AB|}|\lambda|$, that is,

$$\sqrt{\left|\frac{x^2-\mu}{x^2-4}\right|}\,|\lambda|^{-1} \le |A|, |B| \le \sqrt{\left|\frac{x^2-\mu}{x^2-4}\right|}\,|\lambda|. \tag{4.20}$$

(Note that this argument fails for $x = \pm\sqrt{\mu}$ since in that case we have $AB = 0$.)

In the rest of the proof we assume that $n > 0$.

Claim. If

$$|\lambda|^{2n-1} \ge \frac{|\lambda|+1}{|\lambda|-1}, \tag{4.21}$$

then $|y_{n+1}| \geq |y_n|$ and $|y_{-n-1}| \geq |y_{-n}|$.

To prove the claim, note that

$$|y_{n+1}| \geq |A||\lambda|^{n+1} - |B||\lambda|^{-n-1} \text{ and } |y_n| \leq |A||\lambda|^n + |B||\lambda|^{-n}.$$

Thus $|y_{n+1}| \geq |y_n|$ if

$$|A||\lambda|^{n+1} - |B||\lambda|^{-n-1} \geq |A||\lambda|^n + |B||\lambda|^{-n}$$

or equivalently

$$|\lambda|^{2n+1} \geq \frac{|B|}{|A|}\frac{|\lambda|+1}{|\lambda|-1}.$$

Since $|B|/|A| \leq |\lambda|^2$, we have $|y_{n+1}| \geq |y_n|$ if $|\lambda|^{2n-1} \geq (|\lambda|+1)/(|\lambda|-1)$. The other inequality $|y_{-n-1}| \geq |y_{-n}|$ can be similarly proved.

We continue the proof of Lemma 4.27. Note that

$$\begin{aligned} |y_n| &\leq |A||\lambda|^n + |B||\lambda|^{-n} \\ &\leq \sqrt{\left|\frac{x^2-\mu}{x^2-4}\right|}\,|\lambda|\,(|\lambda|^n + |\lambda|^{-n}) \\ &\leq \sqrt{\left|\frac{x^2-\mu}{x^2-4}\right|}\,|\lambda|\,(|\lambda|^n + 1). \end{aligned}$$

Hence if

$$|y_n| \geq H(x) = \sqrt{\left|\frac{x^2-\mu}{x^2-4}\right|}\,|\lambda|\left(\frac{|\lambda|+1}{|\lambda|-1}+1\right)$$

then $|\lambda|^n \geq \frac{|\lambda|+1}{|\lambda|-1}$. Hence (4.21) holds. It follows from the claim that $|y_{n+1}| \geq |y_n|$. Similarly, if $|y_{-n}| \geq H(x)$ then $|y_{-n-1}| \geq |y_{-n}|$. □

Remark. We observe from the proof that we always have $|y_0| < H(x)$.

On the other hand, for $x \in [-2, 2]$ or $x = \pm\sqrt{\mu}$, we set $H(x) = \infty$. By replacing $H(x)$ by $\max\{H(x), 2\}$, we may assume that $H(x) \geq 2$. Hence H is continuous on $\mathbf{C}\backslash\{\pm\sqrt{\mu}\}$ (but is not continuous at $\pm\sqrt{\mu}$ if $\mu \notin [0, 4]$).

The definition of $T(t)$. Now for each $\phi \in \mathbf{\Phi}_\mu$ and $t \geq 0$, we define a subtree $T(t)$ of Σ as follows. First, for each $X \in \Omega$ with $x = \phi(X) \notin [-2, 2]$ and $x \neq \pm\sqrt{\mu}$, and for $r \geq H(x) \geq 2$, we set

$$J_r(X) = \bigcup \{X \cap Y_n \mid |y_n| \leq r\}.$$

It is an subarc of ∂X determined by ϕ with the property that if e is any edge in ∂X not lying in $J_r(X)$ then the arrow on e assigned by ϕ points towards $J_r(X)$. Note that $J_r(X)$ contains at least one edge by the observation we have just made after the proof of Lemma 4.27.

Now, for given $t \geq 0$, let $T(t) = T_\phi(t)$ be the union of all the arcs $J_{H(x)+t}(X)$ as X varies in $\Omega(2+t) = \Omega_\phi(2+t)$.

It can be shown (by the argument of Lemma 3.11, [8]) that $T(t)$ is connected, hence a subtree of Σ. It follows that $T(t)$ has the following nice property.

Lemma 4.28 *If $T(t) \neq \emptyset$, then the arrow on any edge not in the subtree $T(t)$ points towards $T(t)$.* □

Actually, by Theorem 4.6(1), we have

Lemma 4.29 *$T(t) \neq \emptyset$ for any $\phi \in \mathbf{\Phi}_\mu$ and for any $t \geq m(\mu)-2$, where $m(\mu) > 2$ is a constant such that $\Omega_\phi(m(\mu)) \neq \emptyset$ for all $\phi \in \mathbf{\Phi}_\mu$.* □

Alternatively, we can describe $T(t)$ directly in terms of its edges.

Lemma 4.30 *An edges $e = X \cap Y$ is an edge of $T(t)$ if and only if either $|x| \leq 2+t$ and $|y| \leq H(x)+t$ or $|y| \leq 2+t$ and $|x| \leq H(y)+t$.* □

Thus we have the following lemma which gives a finite criterion for recognizing that a given μ-Markoff map ϕ lies in $(\mathbf{\Phi}_\mu)_Q$. This is Lemma 3.15 in [8], and the proof there also works here.

Lemma 4.31 *For any fixed $t \geq 0$, $\phi \in (\mathbf{\Phi}_\mu)_Q$ if and only if $T_\phi(t)$ is finite.* □

Now we can give the proof of Theorem 4.7 by showing that this criterion is an open property.

Proof of Theorem 4.7. The proof is the same as that of Theorem 3.16, [8], which states that $(\mathbf{\Phi}_0)_Q$ is an open subset of $\mathbf{\Phi}_0$. Here we supply a bit more details.

Fix any $t_1 > m(\mu) - 2$. Suppose $\phi \in (\mathbf{\Phi}_\mu)_Q$ and write $T(t)$ for $T_\phi(t)$. By Lemma 4.31, $T(t_1)$ is a finite subtree of Σ. We may choose $t_2 > t_1$ large enough so that $T(t_2)$ contains $T(t_1)$ in its interior, that is, it contains $T(t_1)$, together with all the edges of the circular set $C(T(t_1))$. Note that $T(t_2)$ is also a finite subtree of Σ.

For any given $\phi' \in \mathbf{\Phi}_\mu$, we write $T'(t)$ for $T_{\phi'}(t)$.

Claim. If ϕ' is sufficiently close to ϕ, then $T'(t_1) \cap T(t_2) \subseteq T(t_1)$.

To prove the claim, choose $e = X \cap Y \in T(t_2) \backslash T(t_1)$. We may assume $|\phi(X)| \leq 2 + t_2$ and $|\phi(Y)| \leq H(x) + t_2$ since $e \in T(t_2)$. Then either $|\phi(X)| > 2 + t_1$ or $|\phi(Y)| > H(x) + t_1 \geq 2 + t_1$ since $e \notin T(t_1)$. Thus if ϕ' is sufficiently close to ϕ, we have either $|\phi(X)| > 2 + t_1$ or $|\phi(Y)| > 2 + t_1$, hence $e \notin T'(t_1)$. This proves the claim since there are only finitely many edges e in $T(t_2) \backslash T(t_1)$.

Since $T(m(\mu) - 2)$ is a non-empty subtree of $T(t_2)$ and $t_1 > m(\mu) - 2$, it follows that $T'(t_1) \cap T(t_2) \supseteq T(m(\mu) - 2) \neq \emptyset$, provided that ϕ' is sufficiently close to ϕ. Since $T'(t_1) \cap T(t_2) \subseteq T(t_1)$ and $T(t_1)$ is contained in the interior of $T(t_2)$, we know that $T(t_2)$ contains a connected component of $T'(t_1)$. Since $T'(t_1)$ is connected, we must have $T'(t_1) \subseteq T(t_2)$. Therefore $T'(t_1)$ is finite, and so $\phi' \in (\mathbf{\Phi}_\mu)_Q$. This proves Theorem 4.7. □

Now we proceed to prove Theorem 4.10, for given $\phi \in \mathbf{\Phi}_\mu$, using the ϕ-weight $\psi(\vec{e})$ for each directed edge $\vec{e} \in \vec{E}(\Sigma)$ defined in §2.6 and following through the proof of Theorem 2 in [8]. The geometric meaning of $\psi(\vec{e})$ is explored in §2.7.

First we need an estimate of $|2\Psi(x,y,z)-\mathfrak{h}(x)|$ and $|\Psi(x,y,z)-\hat{\mathfrak{h}}(x)|$.

Proposition 4.32 *If* $\phi \in (\mathbf{\Phi}_\mu)_Q$ $(\mu \neq 4)$ *then there is a constant* $C = C(\phi) > 0$ *such that*

$$|2\Psi(x,y,z)-\mathfrak{h}(x)| \leq C|y|^{-2}, \tag{4.22}$$

$$|\Psi(x,y,z)-\hat{\mathfrak{h}}(x)| \leq C|y|^{-2}, \tag{4.23}$$

for all $x=\phi(X)$, $y=\phi(Y)$, $z=\phi(Z)$ *where* $X,Y,Z \in \Omega$ *meet at a vertex such that* $|y|$ *is sufficiently large and*

$$z = \frac{xy}{2}\left(1-\sqrt{1-4\left(\frac{1}{x^2}+\frac{1}{y^2}-\frac{\mu}{x^2y^2}\right)}\,\right). \tag{4.24}$$

Proof. Here we prove (4.22) only; (4.23) can be proved similarly. It is shown in the proof of Proposition 2.34(iii) that

$$\mathfrak{h}(x) = \log\frac{[\,1+(e^\nu-1)h(x)\,]^2}{1-\mu/x^2}$$

and

$$2\Psi(x,y,z) = \log\frac{[\,1+(e^\nu-1)(z/xy)\,]^2}{(1-\mu/x^2)(1-\mu/y^2)}.$$

The estimate (4.22) then follows from the fact that

$$|\log(1+u)\,| \leq 2\,|u| \quad \text{for } u \in \mathbf{C} \quad \text{with } |u| \leq 1/2$$

and the following calculations:

$$\begin{aligned}
&2\Psi(x,y,z)-\mathfrak{h}(x) \\
=\;& \log\frac{x^2y^2[\,1+(e^\nu-1)(z/xy)\,]^2}{(x^2-\mu)(y^2-\mu)} - \log\frac{x^2[\,1+(e^\nu-1)h(x)\,]^2}{x^2-\mu} \\
=\;& \log\left(\frac{y^2}{y^2-\mu}\left[\frac{1+(e^\nu-1)(z/xy)}{1+(e^\nu-1)h(x)}\right]^2\right) \\
=\;& \log\left(1+\frac{\mu}{y^2-\mu}\right) + 2\log\left(1+\frac{(e^\nu-1)[\,z/xy-h(x)\,]}{1+(e^\nu-1)h(x)}\right),
\end{aligned}$$

(note that the above are true equalities in $\mathbf{C}$ without modulo $2\pi i$: the first one is because of (4.24) and the other two because each expression after the log symbol is sufficiently close to 1 provided that $|y|$ is sufficiently large)

$$\begin{aligned}\big|z/xy - h(x)\big| &= \Big|\sqrt{1-4/x^2} - \sqrt{1-4/x^2-4/y^2+4\mu/x^2y^2}\,\Big|/2 \\ &= \frac{\big|(4/y^2)(1-\mu/x^2)\big|}{2\Big|\sqrt{1-4/x^2}+\sqrt{1-4/x^2-4/y^2+4\mu/x^2y^2}\,\Big|} \\ &\le 2|y|^{-2}\big|1-\mu/x^2\big|/\sqrt{|1-4/x^2|},\end{aligned}$$

since by our convention the square roots here all have nonnegative real parts and for $\phi \in (\mathbf{\Phi}_\mu)_Q$ the image $\phi(\Omega)$ is discrete. □

Proof of Theorem 4.10. The proof is essentially the same as that of Theorem 3, [8], with $\psi(\vec{e}) = z/xy$ there now replaced by $\Psi(x, y, z) \in \mathbf{C}$. We write it out with details as follows, taking care of multiples of $2\pi i$. Note that this difficulty does not occur in [8].

Suppose $\phi \in (\mathbf{\Phi}_\mu)_Q$ where $\mu \neq 0, 4$. We write $\mathfrak{h} = \mathfrak{h}_\tau$ where $\tau = \mu - 2$. Then $\log^+|\phi|$ has Fibonacci growth and the sum in (4.8) converges absolutely since $\mathfrak{h}(x) = O(|x|^{-2})$ as $|x| \to \infty$ and since $\sum_{X\in\Omega} |\phi(X)|^{-2}$ converges absolutely by Corollary 4.22.

On the other hand, since $\Omega_\phi(2)$ is finite, as in the proof of Theorem 4.8, there is a finite subtree T of Σ with the property that for each edge not in T, its arrow assigned by ϕ points towards T.

Let $C_n, n \geq 0$ be the set of directed edges $\vec{e} = \alpha_\phi(e)$ at a distance n away from T (that is, $e \notin T$, and the minimal arc in Σ from the head of $\vec{e}$ to T consists of n edges). It is easy to see that for each $n \geq 0$, C_n is a circular set.

Now let $\Omega_n, n \geq 0$ be the set of regions X such that $X \in \Omega^0(e)$ for some edge $\vec{e} \in C_n$. (Actually, each such region X intersects C_n in exactly two edges.) Note that $\Omega_n \subset \Omega_{n+1}$ are nested with $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$, while all $C_n, n \geq 0$ are disjoint.

Note also that

$$\sum_{X\in\Omega}\mathfrak{h}(\phi(X)) = \lim_{n\to\infty}\sum_{X\in\Omega_n}\mathfrak{h}(\phi(X)).$$

For each $\vec{e} = (X, Y; \to Z) \in C_n$, we have defined

$$\psi(\vec{e}) = \Psi(x, y, z) = \log\frac{1 + (e^\nu - 1)(z/xy)}{\sqrt{1-\mu/x^2}\sqrt{1-\mu/y^2}} \in \mathbf{C}.$$

It is easy to see from (2.57) and (2.58) that $\sum_{\vec{e}\in C_n}\psi(\vec{e}) = \nu \mod 2\pi i$. Furthermore, we have

Claim. $\tilde{\nu} := \lim_{n\to\infty}\sum_{\vec{e}\in C_n}\psi(\vec{e}) \in \mathbf{C}$ exists.

To prove the claim, it suffices to show that $\sum_{\vec{e}\in C_n}\psi(\vec{e}) = \sum_{\vec{e}\in C_{n+1}}\psi(\vec{e})$ for sufficiently large n. Consider any $\vec{e} = (X, Y; W \to Z) \in C_n$. Let $\vec{e}_X = (Y, W; \to X)$ and $\vec{e}_Y = (X, W; \to Y)$. Then $\vec{e}_X, \vec{e}_Y \in C_{n+1}$. It is easy to see that when n is sufficiently large, $|w|$ and one of $|x|, |y|$, say $|x|$, are sufficiently large. It follows from the definition of the edge weight $\psi(\vec{e}_Y)$ that $|\psi(\vec{e}_Y)|$ is sufficiently small. Since $\psi(\vec{e}) = \psi(\vec{e}_X) + \psi(\vec{e}_Y) \mod 2\pi i$ by Proposition 2.34, we know that in fact $\psi(\vec{e}) = \psi(\vec{e}_X) + \psi(\vec{e}_Y)$ without modulo $2\pi i$, for each $\vec{e} \in C_n$. This proves the claim.

Now for each $\vec{e} \in C_{n+1}$, we write $\vec{e} = (X, Y; \to Z)$, where we assume $X \in \Omega_n$ and $Y \in \Omega_{n+1}\backslash\Omega_n$. Here $z = \phi(Z)$ satisfies (4.24) since $\vec{e}$ points towards Z. Note that as $\vec{e}$ ranges over C_{n+1}, each $X \in \Omega_n$ gets counted twice. Since $\phi \in (\mathbf{\Phi}_\mu)_Q$, $|y| = |\phi(Y)|$ will be sufficiently large if n is. Thus by (4.22) we have, as $n \to \infty$,

$$\begin{aligned}
2\Big|\sum_{X\in\Omega_n}\mathfrak{h}(\phi(X)) - \tilde{\nu}\Big| &= \Big|\sum_{X\in\Omega_n} 2\,\mathfrak{h}(x) - \sum_{\vec{e}\in C_{n+1}} 2\,\psi(\vec{e})\Big| \\
&= \Big|\sum_{\vec{e}\in C_{n+1}}\big(\mathfrak{h}(x) - 2\,\psi(\vec{e})\big)\Big| \\
&\quad (\text{where } \ \vec{e} = (X, Y; \to Z) \ \text{ with } \ X \in \Omega_n,\ Y \in \Omega_{n+1}\backslash\Omega_n\,) \\
&\le \sum_{\vec{e}\in C_{n+1}}\big|\,\mathfrak{h}(x) - 2\,\psi(\vec{e})\,\big| \\
&\le \sum_{Y\in\Omega_{n+1}\backslash\Omega_n} \text{constant}\cdot|\phi(Y)|^{-2} \ \longrightarrow\ 0,
\end{aligned}$$

since $\sum_{Y\in\Omega_{n+1}\setminus\Omega_n}|\phi(Y)|^{-2}\to 0$ (as $n\to\infty$) by the convergence of

$$\sum_{Y\in\Omega}|\phi(Y)|^{-2}=\lim_{n\to\infty}\sum_{Y\in\Omega_n}|\phi(Y)|^{-2}.$$

Hence

$$\sum_{X\in\Omega}\mathfrak{h}(\phi(X))=\lim_{n\to\infty}\sum_{X\in\Omega_n}\mathfrak{h}(\phi(X))=\tilde{\nu}. \tag{4.25}$$

This proves Theorem 4.10. □

Proposition 4.11 can be similarly proved using (4.23).

Proof of Proposition 4.11. By Proposition 4.9, we know that both $\sum_{X\in\Omega^-(\vec{e})}\mathfrak{h}(\phi(X))$ and $\sum_{X\in\Omega^-(\vec{e})}2\,\hat{\mathfrak{h}}(\phi(X))$ converge absolutely, and they have difference a multiple of $2\pi i$ since $\mathfrak{h}=2\,\hat{\mathfrak{h}} \mod 2\pi i$. Thus we only need to evaluate

$$\sum_{X\in\Omega^0(\vec{e})}\hat{\mathfrak{h}}(\phi(X))+\sum_{X\in\Omega^-(\vec{e})}2\,\hat{\mathfrak{h}}(\phi(X)),$$

and we may apply the arguments of the above proof of Theorem 4.10.

Let C_n' be the set of directed edges $\vec{e'}$ in $\Sigma^-(\vec{e})$ at a distance n from $\vec{e}$ (that is, $e'\subset\Sigma^-(\vec{e})$ and the minimal arc in Σ from the head of $\vec{e'}$ to the tail of $\vec{e}$ has exactly n edges). Then $C_n=C_n'\cup\{-\vec{e}\}$ is a circular set. By the assumption on $\vec{e}$, we know that, when n is sufficiently large, $\vec{e'}=\alpha_\phi(e')$ for every $\vec{e'}\in C_n'$.

Let $\Omega_n:=\Omega_n^{0-}(\vec{e})$ be the set of regions X in $\Omega^{0-}(\vec{e})$ such that $X\in\Omega^0(e')$ for some $\vec{e'}\in C_n'$. Then $\Omega_n\subset\Omega_{n+1}$ and $\Omega^{0-}(\vec{e})=\bigcup_{n=0}^{n}\Omega_n$. Thus

$$\sum_{X\in\Omega^{0-}(\vec{e})}\mathfrak{h}(\phi(X))=\lim_{n\to\infty}\sum_{X\in\Omega_n}\mathfrak{h}(\phi(X)).$$

By (2.57) and (2.58), we have $\psi(\vec{e})=\sum_{\vec{e'}\in C_n'}\psi(\vec{e'}) \mod 2\pi i$ for all $n\geq 0$. Similar to the claim in the proof of Theorem 4.10, we can prove that

Claim. $\tilde{\psi}(\vec{e}):=\lim_{n\to\infty}\sum_{\vec{e}\in C_n}\psi(\vec{e})\in\mathbf{C}$ exists. □

Thus for n sufficiently large we have from (4.23)

$$\begin{aligned}
&\Big|\tilde{\psi}(\vec{e}) - \sum_{X\in\Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) - \sum_{X\in\Omega^-(\vec{e})} 2\,\hat{\mathfrak{h}}(\phi(X))\Big| \\
=& \Big|\sum_{\vec{e'}\in C'_{n+1}} \psi(\vec{e'}) - \sum_{X\in\Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) - \sum_{X\in\Omega^-(\vec{e})} 2\,\hat{\mathfrak{h}}(\phi(X))\Big| \\
=& \Big|\sum_{\vec{e'}\in C'_{n+1}} \big[\psi(\vec{e'}) - \hat{\mathfrak{h}}(\phi(X))\big]\Big| \\
& (\text{where } \ \vec{e'} = (X, Y; \to Z) \ \text{ with } \ X \in \Omega_n \ \text{ and } \ Y \in \Omega_{n+1}\backslash\Omega_n\,) \\
\leq& \sum_{\vec{e'}\in C_{n+1}} \big|\,\psi(\vec{e'}) - \hat{\mathfrak{h}}(x)\,\big| \\
\leq& \sum_{Y\in\,\Omega_{n+1}\backslash\,\Omega_n} \text{constant}\cdot|\phi(Y)|^{-2} \ \longrightarrow\ 0,
\end{aligned}$$

since $\sum_{Y\in\Omega_{n+1}\backslash\,\Omega_n} |\phi(Y)|^{-2} \to 0$ (as $n\to\infty$) by the convergence of

$$\sum_{Y\in\Omega^{0-}(\vec{e})} |\phi(Y)|^{-2} = \lim_{n\to\infty} \sum_{Y\in\,\Omega_n} |\phi(Y)|^{-2}.$$

Hence

$$\begin{aligned}
\tilde{\psi}(\vec{e}) &= \sum_{X\in\Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) + \lim_{n\to\infty} \sum_{X\in\Omega_n} 2\,\hat{\mathfrak{h}}(\phi(X)) \\
&= \sum_{X\in\Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) + \sum_{X\in\Omega^-(\vec{e})} 2\,\hat{\mathfrak{h}}(\phi(X)). \qquad (4.26)
\end{aligned}$$

This proves Proposition 4.11 since $\tilde{\psi}(\vec{e}) = \psi(\vec{e}) \mod 2\pi i$. □

4.4 Drawing the gaps

In this section we explain, for a given representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$, how to draw correctly the gaps in the extended complex plane which is the ideal boundary of $\mathbb{H}^3$ in the upper half-space model.

First let us recall the geometric meaning of the gap function $\mathfrak{h}$ which we have discussed in §2.7.

Let $\phi \in \Phi_\mu$ and $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ be the corresponding $(\mu-2)$-representation. Let a, b be an arbitrary pair of generators of Γ and let $X, Y \in \Omega$ be the two regions which correspond to a, b respectively. Let $A = \rho(a), B = \rho(b)$. Then by our convention, $x = \phi(X) = \mathrm{tr}A$, $y = \phi(Y) = \mathrm{tr}B$.

Consider the commutator $C := [B^{-1}, A^{-1}] = (B^{-1}A^{-1}B)A$. If A is loxodromic, so is $B^{-1}A^{-1}B$. Let the attractive and repulsive fixed points of A be respectively denoted as $\mathrm{Fix}^+(A)$ and $\mathrm{Fix}^-(A)$. Similarly we have $\mathrm{Fix}^+(B^{-1}A^{-1}B)$ and $\mathrm{Fix}^-(B^{-1}A^{-1}B)$. Then as explained in §2.7, *the gap value* $\mathfrak{h}(x)$ *is the complex length from* $\lfloor \mathbf{a}(C), \mathrm{Fix}^+(A) \rfloor$ *to* $\lfloor \mathbf{a}(C), \mathrm{Fix}^-(B^{-1}A^{-1}B) \rfloor$ *along* $\mathbf{a}(C)$ where $\lfloor \mathbf{a}(C), \mathrm{Fix}^+(A) \rfloor$ is the oriented line in $\mathbb{H}^3$ which is normal to the axis $\mathbf{a}(\mathrm{C})$ of C and has $\mathrm{Fix}^+(A)$ as its ending ideal point and similarly for $\lfloor \mathbf{a}(C), \mathrm{Fix}^-(B^{-1}A^{-1}B) \rfloor$.

Remark 4.33 Here it is important to note that the gap value $\mathfrak{h}(x)$ is independent of the choice of b in the generating pair a, b of Γ. Note also that $\mathrm{Fix}^-(B^{-1}A^{-1}B) = \mathrm{Fix}^+(B^{-1}AB)$ and that $B^{-1}AB$ is a conjugate of A and $C = (B^{-1}AB)^{-1}A$.

Now let us fix on a pair of generators $a, b \in \Gamma$ and let (in this section only) $c = [b^{-1}, a^{-1}] = b^{-1}a^{-1}ba$. Then each element of Γ can be written uniquely as a reduced word in the letters a, b, a^{-1}, b^{-1}. We wish to draw, for each $g \in \Gamma$ such that $[g] \in \hat{\Omega}$, the gap corresponding to $\rho(g)$ with respect to the axis $\mathbf{a}(C)$ of the ρ-image $C = \rho(c)$ of the commutator c. Thus, as discussed above, we need to produce a word $g' \in \Gamma$ which is a conjugate of g so that $(g')^{-1}g = c$ in Γ. Hence we must have $g' = gc^{-1}$ after cancellation. We need to do this consistently so that all gaps are in their correct places viewed in the extended complex plane.

Constructing the 'universal cover' of $\hat{\Omega}$. Thus, to draw the gaps with respect to a fixed commutator, we need to construct the 'universal cover' of the rational projective lamination space $\hat{\Omega}$ as follows. We shall construct pairs of words $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$ in Γ, parametrized by $\frac{p}{q} \in \mathbf{Q}$, as dictated by the developing image in $\mathbb{H}^2$

of the once-punctured hyperbolic torus which has lifted holonomy representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ such that

$$A = \rho(a) = \begin{pmatrix} 0 & \sqrt{2}/2 \\ -\sqrt{2} & 2\sqrt{2} \end{pmatrix} \quad \text{and} \quad B = \rho(b) = \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}.$$

Note that $\rho(b^{-1}a^{-1}ba) = B^{-1}A^{-1}BA = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}$. See Figure 4.5, where we use $\bar{a}, \bar{b}, \bar{c}$ to denote a^{-1}, b^{-1}, c^{-1} respectively and we label the attractive fixed points of A, A^{-1} by $a, \bar{a}$ respectively, and similarly for others.

(I) First, we construct a bi-infinite sequence of pairs of reduced words (L_n, R_n), $n = \frac{n}{1} \in \mathbf{Z}$, inductively as follows.

(i) Set $R_0 = a$ and $L_1 = b$.

(ii) We require

$$(L_n)^{-1} R_n = c \tag{4.27}$$

for all $n \in \mathbf{Z}$. Hence, for example, $L_0 = b^{-1}ab = ac^{-1}$ and $R_1 = a^{-1}ba = bc$.

(iii) Define

$$L_{n+2} = (R_n)^{-1} \tag{4.28}$$

or equivalently

$$R_{n-2} = (L_n)^{-1} \tag{4.29}$$

for $n \in \mathbf{Z}$. Hence $R_{n+2} = (R_n)^{-1} c$.

Remark 4.34 It is easy to check that the following relations hold for all $n \in \mathbf{Z}$:

$$[(L_{n+1})^{-1}, (R_n)^{-1}] = c, \tag{4.30}$$

and

$$L_{n+4} = c^{-1}\, L_n\, c. \tag{4.31}$$

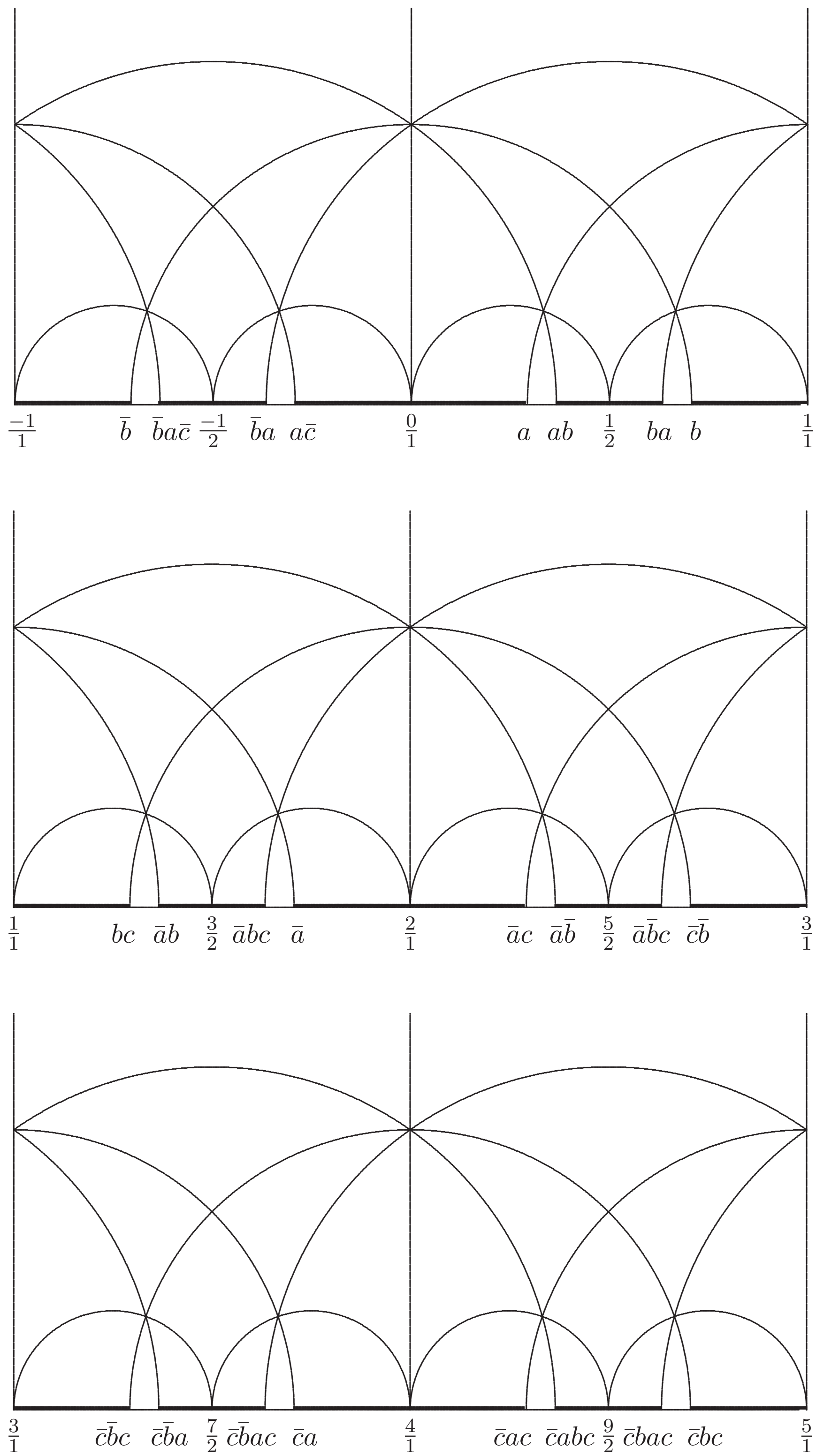

Figure 4.5: The **Q**-sequence of pairs of words

(II) Next, we generate the general $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$ for all $\frac{p}{q} \in \mathbf{Q}$ by the following rule:

(iv) For $\frac{p+r}{q+s} \in \mathbf{Q}$, constructed from Farey neighbors $\frac{p}{q}, \frac{r}{s} \in \mathbf{Q}$, where $\frac{p}{q} < \frac{r}{s}$, we define

$$L_{\frac{p+r}{q+s}} = R_{\frac{p}{q}} L_{\frac{r}{s}}, \quad R_{\frac{p+r}{q+s}} = L_{\frac{r}{s}} R_{\frac{p}{q}}. \tag{4.32}$$

Remark 4.35 Note that $\frac{p}{q}$ is not the slope of the free homotopy class $[L_{\frac{p}{q}}] = [R_{\frac{p}{q}}] \in \hat{\Omega}$.

We can prove by induction that

(a) for any Farey neighbors $\frac{p}{q}, \frac{r}{s} \in \mathbf{Q}$, where $\frac{p}{q} < \frac{r}{s}$, the following commutator relation holds:

$$[(L_{\frac{r}{s}})^{-1}, (R_{\frac{p}{q}})^{-1}] = c; \tag{4.33}$$

(b) for all $\frac{p}{q} \in \mathbf{Q}$, the following relations hold:

$$L_{\frac{p}{q}+2} = (R_{\frac{p}{q}})^{-1}, \tag{4.34}$$

$$L_{\frac{p}{q}+4} = c^{-1} L_{\frac{p}{q}} c. \tag{4.35}$$

Thus the pair of words $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$ we constructed above are indeed conjugate to each other, as the following lemma shows.

Lemma 4.36 *For any Farey neighbors $\frac{p}{q}, \frac{r}{s} \in \mathbf{Q}$, where $\frac{p}{q} < \frac{r}{s}$, we have*

$$L_{\frac{p}{q}} = (L_{\frac{r}{s}})^{-1} R_{\frac{p}{q}} L_{\frac{r}{s}}. \tag{4.36}$$

Proof. It follows from (4.32) by induction. □

Drawing the gaps. Now for each pair of words $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$, $\frac{p}{q} \in \mathbf{Q}$, constructed as above, we draw a line segment in the extended complex plane connecting the attractive fixed points of $\rho(L_{\frac{p}{q}})$ and $\rho(R_{\frac{p}{q}})$ to indicate the gap corresponding to the

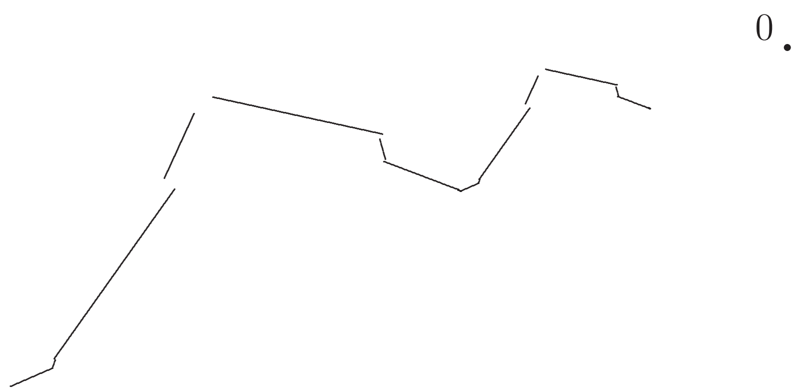

Figure 4.6: Gaps for a (−0.8)-Markoff map with $x = 6, y = 3 + i$

pair. The picture obtained for a particular generalized Markoff map $\phi \in (\mathbf{\Phi}_\mu)_Q$ is given in Figure 4.6, where we have normalized $A = \rho(a)$ and $B = \rho(b)$ by conjugation so that $C = \rho(c) = B^{-1}A^{-1}BA$ has 0 and ∞ as the fixed points on the extended complex plane, and we have drown some of the gaps for $\frac{p}{q}$ between 0 and 4. The complete picture for $\frac{p}{q} \in \mathbf{Q}$ can be obtained by applying the Möbius transformation $C = \rho(c)$ and its inverse on the shown picture repeatedly.

Chapter 5

Variations to once-punctured torus bundles

Abstract

In this chapter we generalize Bowditch's variations in [7] of McShane's identity. Bowditch obtained identities for complete, finite volume hyperbolic 3-manifolds which fiber over the circle with the fiber a punctured-torus. We now obtain similar identities for certain incomplete hyperbolic structures on such manifolds and hence identities for those closed hyperbolic 3-manifolds which are obtained by performing hyperbolic Dehn surgery on these incomplete manifolds.

5.1 Introduction

Bowditch's variations in [7] of McShane's identity are for complete, finite volume hyperbolic 3-manifolds fibering over the circle, with fiber the once-punctured torus. These complete structures can be deformed to incomplete structures, as shown by Thurston in [50], and in certain cases, one can perform hyperbolic Dehn surgery to obtain closed (complete) hyperbolic 3-manifolds. The main result of this chapter is that a further variation of the Bowditch's identities holds for these deformations, and hence for the closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on almost all such manifolds (Theorems 5.3, 5.4 and Corollary 5.5). Briefly, this is done as follows. Let M be a once-punctured torus bundle over the circle with non-trivial monodromy H which is an hyperbolic element in $\mathrm{SL}(2, \mathbf{Z})$ (where $\mathrm{SL}(2, \mathbf{Z})$ is identified with the mapping class group of the once-punctured torus). Starting with a complete, finite volume hyperbolic structure on M, and by considering the holonomy representation restricted to the fundamental group of the fiber, Bowditch [7] constructed a Markoff map which is stabilized by the cyclic subgroup $\langle H \rangle < \mathrm{SL}(2, \mathbf{Z})$. His variations of McShane's identity (Theorems 5.1 and 5.2) are obtained by studying the Markoff map modulo the action of $\langle H \rangle$. If we deform the complete hyperbolic structure on M to an incomplete one and look again at the holonomy representation restricted to the fiber, we get a generalized Markoff map, with the same stabilizer group $\langle H \rangle$. We then obtain further variations of the McShane-Bowditch identities (Theorems 5.3 and 5.4) for these deformations, which also hold for the closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on these manifolds (Corollary 5.5).

5.2 Bowditch's settings for torus bundles

In this section let us describe Bowditch's variations of McShane's identity to a once-punctured torus bundle over the circle, equipped with a complete, finite volume hyperbolic structure. The presentations here are borrowed from [7].

Let M be an orientable 3-manifold which fibers over the circle, with the fiber a once-punctured torus, $\mathbb{T}$. Suppose that M has a complete finite-volume hyperbolic structure.

Let $\mathscr{S}$ be the set of closed geodesics in M which are homotopic in M to simple closed curves in the fiber. Geometrically, to each $\sigma \in \mathscr{S}$, we associate its **complex length**, $l(\sigma) \in \mathbf{C}/2\pi i\mathbf{Z}$, where $\Re\, l(\sigma)$ is the (real) hyperbolic length of σ and $\Im\, l(\sigma)$ is the rotational component, i.e. the angle through which a normal vector turns when parallel transported once around the closed geodesic γ. In particular, $e^{l(\sigma)} \in \mathbf{C}$ is well-defined. Then Bowditch obtained

Theorem 5.1 (Theorem A in [7])

$$\sum_{\sigma \in \mathscr{S}} \frac{1}{1+e^{l(\sigma)}} = 0,$$

where the sum converges absolutely.

The curves in $\mathscr{S}$ fall naturally into two classes, $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\mathrm{R}}$, as follows. Recall that $\mathscr{C}$ is the set of free homotopy classes of non-trivial non-peripheral simple closed curves on $\mathbb{T}$ and can be thought of as the set of rational points in the projective lamination space, $\mathscr{P}$, of $\mathbb{T}$, which in this case is a circle. The mapping class group $\mathcal{MCG}$ of $\mathbb{T}$ acts on $\mathscr{P}$ preserving the set $\mathscr{C}$. The monodromy $H \in \mathcal{MCG}$ of M generates an infinite cyclic subgroup, $\langle H \rangle$, of the mapping class group. This subgroup, or H, has two fixed points in $\mathscr{P}$, namely the stable and unstable laminations, μ_{s} and μ_{u}, of the monodromy. These two points separate $\mathscr{P}$ into two open intervals. Since μ_{s} and μ_{u} are irrational points, this gives a natural partition of $\mathscr{C}$ into two subsets, $\mathscr{C}_{\mathrm{L}}$ and $\mathscr{C}_{\mathrm{R}}$, which in turn partitions $\mathscr{S}$ into two subsets, $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\mathrm{R}}$.

If we restrict the sum appearing in Theorem 5.1 to one or other of $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\mathrm{R}}$, we will get the same answer up to changing sign. Bowditch showed (see Theorem 5.2) that this number turns out essentially to be the modulus of the cusp of M.

More precisely, since M has a single parabolic cusp, M is homeomorphic to the interior of a compact manifold $M \cup \partial M$, with one toroidal boundary component, ∂M. Then ∂M carries a natural Euclidean structure, well-defined up to similarity, which arises from identifying ∂M with a horocycle. If M has positive monodromy, that is, if $H \in \mathcal{MCG} \cong \mathrm{SL}(2,\mathbf{Z})$ has positive trace, we may represent ∂M as the quotient of $\mathbf{C}$ with the Euclidean metric by the lattice $\mathbf{Z} \oplus \lambda\mathbf{Z}$, generated by the translations $\zeta \mapsto \zeta + 1$ and $\zeta \mapsto \zeta + \lambda$ corresponding to the meridian and longitude respectively, where the **longitude** is defined as the intersection of two leaves, one from each of the two foliations of M determined by the stable and unstable laminations. We call $\lambda = \lambda(M)$ the **modulus** of the cusp. We can suppose that $\Im\,\lambda(M) > 0$. If M has negative monodromy, H, we define $\lambda(M)$ as the modulus of the cusp of the sister of M (that is, the manifold with monodromy $-H$). In this setting Bowditch showed

Theorem 5.2 (Theorem B in [7])

$$\sum_{\sigma \in \mathscr{C}_{\mathrm{L}}} \frac{1}{1 + e^{l(\sigma)}} = \pm\lambda(M),$$

where the sign depends only on our conventions of orientation.

Natural ideal triangulations. Recall the setting of §4.3. Let us fix an ordered pair of free generators a, b of Γ so that the algebraic intersection number of the corresponding ordered pair of oriented simple closed curves C_a, C_b on $\mathbb{T}$ is equal to $+1$. Note that the commutator $[a,b] = aba^{-1}b^{-1}$ is peripheral.

The mapping class group $\mathcal{MCG}$ of $\mathbb{T}$ may be identified as $\mathrm{SL}(2,\mathbf{Z})$ which acts naturally on $\mathbb{H}^2, \Sigma, \Omega, \mathscr{P}$ and $\mathscr{C}$. In each case, the kernel is given by the hyperelliptic involution, so the induced action of $\mathrm{PSL}(2,\mathbf{Z})$ is faithful. An element $H \in \mathrm{SL}(2,\mathbf{Z})$ is *hyperbolic* if it has two fixed points μ_{u} and μ_{s} in $\mathscr{P}$, namely the stable and unstable laminations. The points μ_{u} and μ_{s} are joined by a bi-infinite arc $\beta \subseteq \Sigma$, which is translated by H in the direction of μ_{s}. The path β can be described combinatorially in terms of the "right-left" decomposition of the matrix H. Note that some conjugate of H can be written as a product of the matrices

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

which correspond to Dehn twists about C_b and C_a respectively. This decomposition is well-defined up to cyclic reordering, and the sequence of Ls and Rs is the same as the periodic sequence of left and right turns of β in Σ. For more details, see [9]. Note that β partitions Ω into two subsets, $\Omega_{\rm L}$ and $\Omega_{\rm R}$, which lie on, respectively, the left and right of β. These correspond to the subsets, $\mathscr{C}_{\rm L}$ and $\mathscr{C}_{\rm R}$ of $\mathscr{C}$ described earlier.

Now, if we take a homeomorphism, η, of $\mathbb{T}$ representing the mapping class H, we may form the mapping torus, M_H, which is given by $(\mathbb{T} \times [0,1])/\sim$, where $\sim$ identifies $(x,1)$ with $(\eta(x),0)$ for all $x \in \mathbb{T}$. The manifold M_H has a natural compactification by adjoining a toroidal boundary, ∂M_H. The compactified manifold, $M_H \cup \partial M_H$, as described in [19], has a natural ideal triangulation arising from the left-right decomposition of H. This can briefly be summarized as follows. Associated with each vertex of Σ is an ideal triangulation of $\mathbb{T}$ (the edges of which are dual to the simple closed curves corresponding to the three complementary regions incident to the vertex). Moving along an edge in the tree corresponds to performing a (dual) Whitehead move. If we traverse a period of the path β, we get a sequence of Whitehead moves which take us from a given triangulation to its image under η. Each Whitehead move gives rise to an ideal simplex in $\mathbb{T} \times [0,1]$, and so, after identifying $\mathbb{T} \times \{0\}$ with $\mathbb{T} \times \{1\}$ via the relation $\sim$, we obtain an ideal triangulation of M_H. This ideal triangulation gives us, in particular, a triangulation of the boundary, ∂M_H. As in [19], we may describe the combinatorial structure of this triangulation lifted to the universal cover, $\mathbf{R}^2$, of ∂M_H. First, we describe the case of positive monodromy. To do this, consider the bi-infinite sequence of vertices of Σ lying along the arc $\beta \subseteq \Sigma$. This sequence is dual to a sequence of ideal triangles in our tessellation of $\mathbb{H}^2$. The union of these triangles gives a bi-infinite strip invariant under the transformation H. We transfer this strip homeomorphically to the vertical strip $[0,1] \times \mathbf{R}$ in $\mathbf{R}^2$, so that the transformation H is conjugated to the map $[(x,y) \mapsto (x,y+1)]$. We extend this to a triangulation of $\mathbf{R}^2$ by a process of repeated reflection in the pair of vertical lines which form the boundary of this strip. The triangulation of ∂M_H is given by the quotient by the group generated by $[(x,y) \mapsto (x,y+1)]$ and $[(x,y) \mapsto (x+4,y)]$. It is not hard to see that these transformations describe, respectively, the longitude and meridian of ∂M_H. Note that the triangulation is in fact invariant under the map $[(x,y) \mapsto (x+2,y)]$. Up to this symmetry, there are two "vertical" lines in the triangulation. The bi-infinite sequence of vertices along one of these lines corresponds to sequence of regions of Ω which meet β and all lie either in Ω_{L}, or in Ω_{R}. Two vertices are joined by an edge in this triangulation if and only if the corresponding regions are adjacent. Thus the "vertical" edges correspond to regions meeting on the same side of β, whereas all the other edges correspond to regions meeting on opposite sides of β. The picture where the monodromy is negative is similar, except that in this case, ∂M_H is given as a quotient of $\mathbf{R}^2$ by the group generated by $[(x,y) \mapsto (x+2,y+1)]$ and $[(x,y) \mapsto (x+4,y)]$. The "longitude" might be thought of as a "half of" the curve given by $[(x,y) \mapsto (x,y+2)]$. Now, it follows from the work of Thurston [49], that $M = M_H$ admits a complete finite-volume hyperbolic structure. See also [35] for an alternative proof and exposition. This structure is unique by Mostow rigidity. Note that ∂M_H carries a euclidean structure, well defined up to similarity, obtained for example by identifying it with a horocycle in M_H. In this hyperbolic structure, we may realize each tetrahedron in our ideal triangulation of M as a hyperbolic ideal tetrahedron. In this way we get a "hyperbolic ideal triangulation" of M. This gives rise to a euclidean realization of the combinatorial triangulation of ∂M_H. Note that Parker [36] has prove that this hyperbolic ideal triangulation of M obtained above is indeed positively oriented, hence is a genuine hyperbolic ideal triangulation. Thus we have a resulting geometric triangulation of ∂M_H which will be used to prove the variations of McShane's identity.

Regarding $\mathbb{T}$ as a fibre of M, we get an identification of $\Gamma = \pi_1(\mathbb{T})$ as a normal subgroup of $\pi_1(M)$. In fact, $\pi_1(M)$ is an HNN extension of Γ with stable letter t so that $tgt^{-1} = H_*(g)$ for all $g \in \Gamma$, where H_* is the automorphism of Γ induced by the monodromy H. We also get an identification of $\mathscr{S}$ with the quotient, $\Omega/\langle H\rangle$, of Ω under the cyclic group, $\langle H\rangle$, generated by H. Clearly, H respects the partition of Ω as $\Omega_{\mathrm{L}} \sqcup \Omega_{\mathrm{R}}$, and we may identify $\mathscr{S}_{\mathrm{L}}$ with $\Omega_{\mathrm{L}}/\langle H\rangle$ and $\mathscr{S}_{\mathrm{R}}$ with $\Omega_{\mathrm{R}}/\langle H\rangle$. The hyperbolic structure on M may be described by a representation, $\hat{\rho} : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$. It follows from [15] that $\hat{\rho}$ lifts to a representation $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$.

Restricting our attention to the fibre subgroup $\Gamma < \pi_1(M)$, we define a Markoff map $\phi : \Omega \to \mathbf{C}$ by $\phi(X) = \operatorname{tr} \rho(g)$, where $g \in \Gamma$ represents the simple closed curve on $\mathbb{T}$ corresponding to the region $X \in \Omega$. Clearly, ϕ is invariant under the $\langle H\rangle$-action, and so gives rise to a well-defined map $\Omega/\langle H\rangle \to \mathbf{C}$ which we also denote by ϕ. We write $[X]$ for the orbit of X under $\langle H\rangle$. If $\sigma \in \mathscr{S}$ corresponds to $[X] \in \Omega/\langle H\rangle$, then the complex length, $l(\sigma)$, of σ is determined by the formula $l(\sigma) = l(\phi([X])) = 2\cosh^{-1}(\phi([X])/2)$. Thus $h(\phi([X])) = 1/(1 + e^{l(\sigma)})$, where $h : \mathbf{C}\backslash[-2, 2] \to \mathbf{C}$ is defined by $h(\zeta) = \big(1 - \sqrt{1 - 4/\zeta^2}\,\big)/2$ as in §2.3. By our convention, we take the square root with positive real part, corresponding to the fact that $\Re\, l(\sigma) > 0$.

In the above terms Bowditch's Theorems 5.1 and 5.2 can be respectively expressed as the identities:

$$\textstyle\sum_{[X]\in\Omega/\langle H\rangle} h(\phi([X])) = 0, \tag{5.1}$$

$$\textstyle\sum_{[X]\in\Omega_{\mathrm{L}}/\langle H\rangle} h(\phi([X])) = \lambda(\partial M). \tag{5.2}$$

5.3 Incomplete hyperbolic torus bundles

In this section we consider certain once-punctured torus bundles over the circle with incomplete hyperbolic structures.

Let M be a once-punctured torus bundle over the circle such that the holonomy $H \in \mathcal{MCG}$ is pseudo-Anosov. Then M can be given a complete, finite volume hyperbolic structure. Actually M can be decomposed into a collection of ideal hyperbolic tetrahedra as described in §5.2. Hence we may obtain M by gluing a collection of ideal hyperbolic tetrahedra with suitable edge invariants. Now we consider an incomplete hyperbolic structure on M obtained by changing the edge invariants while keeping the consistency conditions as described by Thurston [50]. We can also think of it being obtained by deforming the complete hyperbolic structures slightly in the representation space of $\pi_1(M)$.

The developing image of M, with this incomplete hyperbolic structure, in $\mathbb{H}^3$ misses some lines in $\mathbb{H}^3$. We may assume the z-axis $[0, \infty]$ is among them. Consider, for $\epsilon > 0$ sufficiently small, an ϵ-neighborhood N of $[0, \infty]$ in $\mathbb{H}^3$, i.e. a solid cone around the z-axis. Its boundary ∂N is the ϵ-equidistant surface with center the z-axis, hence has a similarity structure. Actually, we can identify this similarity surface ∂N with $\mathbf{C} \backslash \{0\}$, where the identification is given by orthogonal projection in $\mathbb{H}^3$ from $\mathbf{C} \backslash \{0\}$ onto the z-axis via lines normal to it. Note that $N \backslash [0, \infty]$ projects onto a neighborhood of ∂M in M, hence ∂N projects onto ∂M. In this way ∂M gets a similarity structure from the hyperbolic structure of M.

Recall the combinatorial triangulation of M described in §5.2. Now the triangulation of M is realized similarly by "hyperbolic ideal triangulation" and ∂M gets an induced conformal triangulation. This triangulation can be thought of as being deformed from the triangulation in the complete case when we deform the complete structure on M into an incomplete one.

The incomplete hyperbolic structure on M is given by a representation $\hat{\rho} : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$ which lifts to a representation $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$. Restricted to $\Gamma = \pi_1(\mathbb{T}) < \pi_1(M)$, we obtain a τ-representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ for some $\tau \in \mathbf{C}$ with $\tau \neq \pm 2$. Let $\nu = \cosh^{-1}(-\tau/2)$ as in Chapter 4.

Lifting the similarity structure on ∂M to $\mathbf{C} \backslash \{0\}$, we get a triangulation of

$\mathbf{C}\backslash\{0\}$ which is invariant under the transformations $\zeta \mapsto e^{\nu}\zeta$ and $\zeta \mapsto e^{\lambda}\zeta$ if M has positive monodromy. These two transformations give respectively the meridian and longitude of ∂M, similar to the complete case in §5.2. If M has negative monodromy then the two transformations are given by $\zeta \mapsto e^{\nu}\zeta$ and $\zeta \mapsto e^{\nu+\lambda}\zeta$.

Note that $\nu \in \mathbf{C}/2\pi i\mathbf{Z}$ is the half complex length of the ρ-image of a peripheral simple closed curved on the punctured torus $\mathbb{T}$, the fiber of M.

We assume the notation introduced in §5.2. In particular, for each closed geodesic $\sigma \in \mathscr{S}$, its complex length $l(\sigma) \in \mathbf{C}/2\pi i\mathbf{Z}$ is defined now under the given incomplete hyperbolic structure. Then we have the following results corresponding to Bowditch's Theorems 5.1 and 5.2.

We say that M *has discrete length spectrum on the punctured torus fiber* $\mathbb{T}$ if for each $\kappa > 0$ there are only finitely many free homotopy classes of simple closed curves on $\mathbb{T}$ such that their geodesic realizations in M have length $\leq \kappa$.

It can be shown (see Remark 5.8 in next section) that M will have this property if it is obtained from the one with the unique complete finite volume hyperbolic structure by a slight deformation.

Theorem 5.3 *Suppose M has discrete length spectrum on its torus fiber. Then*

$$\sum_{\sigma \in \mathscr{S}} \log\left(\frac{e^{\nu} + e^{l(\sigma)}}{e^{-\nu} + e^{l(\sigma)}}\right) = 0 \mod 2\pi i, \tag{5.3}$$

where the sum converges absolutely.

Theorem 5.4 *Suppose M has discrete length spectrum on its torus fiber. Then*

$$\sum_{\sigma \in \mathscr{C}_{\mathrm{L}}} \log\left(\frac{e^{\nu} + e^{l(\sigma)}}{e^{-\nu} + e^{l(\sigma)}}\right) = \pm\lambda \mod 2\pi i, \tag{5.4}$$

where the sign depends only on our conventions of orientation and where λ is the complex length of the chosen longitude.

As a corollary we have

Corollary 5.5 *Let M be a once-punctured torus bundle over the circle with pseudo-Anosov monodromy and let $\overline{M}(p/q)$ be the closed 3-manifold obtained from M by performing Dehn surgery on ∂M with surgery slope $p/q \in \mathbf{Q}$. Then, except for finitely many surgery slopes, $\overline{M}(p/q)$ has a hyperbolic structure and, moreover, the following identity holds:*

$$\sum_{\sigma \in \mathscr{C}_{\mathrm{L}}} \log\left(\frac{e^{\nu} + e^{l(\sigma)}}{e^{-\nu} + e^{l(\sigma)}}\right) = \pm\lambda \mod 2\pi i, \tag{5.5}$$

where the meanings of λ, ν and $\mathscr{C}_{\mathrm{L}}$ etc are defined as before for the (incomplete) hyperbolic structure on M induced from that of $\overline{M}(p/q)$ and the sign depends only on our conventions of orientation.

5.4 Periodic generalized Markoff maps

In this section we consider generalized Markoff maps which are invariant under a hyperbolic element of the mapping class group $\mathcal{MCG}$ of $\mathbb{T}$.

Let $\mathcal{X}_\tau$ be the space of τ-representations $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ modulo conjugation by elements in $\mathrm{SL}(2, \mathbf{C})$. Consider the action of the mapping class group of $\mathbb{T}$, $\mathcal{MCG} \cong \mathrm{SL}(2, \mathbf{Z})$, on $\mathcal{X}_\tau$ via its induced action on Γ. Any $H \in \mathcal{MCG}$ induces an automorphism H_* of Γ, and H acts on $\mathcal{X}_\tau$ by

$$H(\rho)(g) = \rho(H_*(g)),$$

for any $\rho \in \mathcal{X}_\tau$ and $g \in \Gamma$. Bowditch [7] studied representations $\rho \in \mathcal{X}_{\mathrm{tp}} = \mathcal{X}_{-2}$ stabilized by a cyclic subgroup $\langle H \rangle < \mathcal{MCG}$ generated by a hyperbolic element and proved a variation of the McShane's identity for such representations. This result can be generalized for τ-representations as follows. The case $\tau = -2$ with $\mathfrak{h}$ replaced by h is Bowditch's variation (Theorem A in [7]).

Theorem 5.6 *Suppose a τ-representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$, where $\tau \neq \pm 2$, is stabilized by a hyperbolic element $H \in \mathrm{SL}(2, \mathbf{Z}) \cong \mathcal{MCG}$ and ρ satisfies the BQ-conditions on $\hat{\Omega}/\langle H_* \rangle$, that is,*

(i) $\mathrm{tr}\rho(g) \notin [-2, 2]$ *for all classes* $[g] \in \hat{\Omega}/\langle H_* \rangle$, *and*

(ii) $|\mathrm{tr}\rho(g)| \leq 2$ *for only finitely many classes* $[g] \in \hat{\Omega}/\langle H_* \rangle$.

Then we have

$$\sum_{[g] \in \hat{\Omega}/\langle H_* \rangle} \mathfrak{h}\big(\mathrm{tr}\rho(g)\big) = 0, \tag{5.6}$$

where the function $\mathfrak{h} = \mathfrak{h}_\tau$ is defined as in §2.3 and where the sum converges absolutely.

We prove Theorem 5.6 by reformulating it in terms of generalized Markoff maps as follows.

Theorem 5.7 *Suppose that $\phi \in \mathbf{\Phi}_\mu$ $(\mu \neq 0, 4)$ is invariant under the action of a hyperbolic element $H \in \mathrm{SL}(2, \mathbf{Z}) \cong \mathcal{MCG}$ and ϕ satisfies the BQ-conditions on $\Omega/\langle H \rangle$, that is,* (i) $\phi^{-1}([-2, 2]) = \emptyset$, *and* (ii) $|\phi(X)| \leq 2$ *for only finitely many classes $[X] \in \Omega/\langle H \rangle$. Then*

$$\sum_{[X] \in \Omega/\langle H \rangle} \mathfrak{h}\big(\phi(X)\big) = 0 \mod 2\pi i, \tag{5.7}$$

where the sum converges absolutely and the function $\mathfrak{h} = \mathfrak{h}_\tau$ is defined as in §2.3 with $\tau = \mu - 2$ and $\nu = \cosh^{-1}(-\tau/2) = \cosh^{-1}(1 - \mu/2)$.

Proof. The proof is essentially the same as that Bowditch gave in [7] for the case where $\mu = 0$, except that now we use the generalized function $\mathfrak{h}$ and the generalized ϕ-weight $\psi(\vec{e})$. To give a sketch of the proof, note that $H \in \mathrm{SL}(2, \mathbf{Z})$ leaves invariant an infinite arc $\beta \subset \Sigma$. Let $e_j, j \in \mathbf{Z}$ be successive edges in β such that H acts on $\{e_j \mid j \in \mathbf{Z}\}$ as a shift by m on the indices, that is, $He_j = e_{j+m}$ for some integer $m > 0$. Let $T = \bigcup_{j=1}^{m-1} e_j$ (T is a vertex on β if $m = 1$) and let $C(T)$ be the circular set of directed edges around T. In particular, $\vec{e}_0, -\vec{e}_m \in C(T)$, where $\vec{e}_j$ is directed from e_{j-1} to e_{j+1}. Now let $C^*(T) = C(T) \backslash \{\vec{e}_0, -\vec{e}_m\}$. Then by the properties if ψ, one has $\sum_{\vec{e} \in C^*(T)} \psi(\vec{e}) = 0 \mod 2\pi i$ since $\psi(\vec{e}_0) = \psi(\vec{e}_m)$. Applying Proposition 4.11 to each $\vec{e} \in C^*(T)$ and then summing the resulting identities (4.12) for all $\vec{e} \in C^*(T)$, one obtains the desired identity (5.7) since $2\,\hat{\mathfrak{h}} = \mathfrak{h} \mod 2\pi i$. □

Remark 5.8 We make the remark that in the space of generalized Markoff maps which are invariant under a fixed hyperbolic element $H \in \mathrm{SL}(2, \mathbf{Z})$ the subset consisting of those generalized Markoff maps which satisfy the BQ-conditions on $\Omega/\langle H \rangle$ is open. This can be similarly proved as Theorem 4.7 by Bowditch's argument in [8]. This tells us that M would continue to have discrete length spectrum on its torus fiber if it is deformed slightly from the complete case to incomplete case.

We may also reformulate Theorem 5.4 in terms of generalized Markoff maps.

Let $H \in \mathrm{SL}(2, \mathbf{Z}) \cong \mathcal{MCG}$ be the monodromy of the once-punctured torus bundle M over the circle. Let $\hat{\rho} : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$ be the representation for the hyperbolic structure on M and we also denote by $\hat{\rho}$ its restriction to $\Gamma = \pi_1(\mathbb{T}) \lhd \pi_1(M)$, where we regard $\mathbb{T}$ as the fiber of M. Let $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ be a lifted representation of $\hat{\rho} : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ to $\mathrm{SL}(2, \mathbf{C})$.

Suppose this $\rho : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ is a τ-representation and let $\phi \in \mathbf{\Phi}_\mu$ be the corresponding μ-Markoff map, where $\mu = \tau + 2$. Then ϕ is invariant under H as defined in the statement of Theorem 5.7. With the notation above and in §4.3, Theorem 5.4 can be reformulated as follows.

Theorem 5.9 *If M has discrete length spectrum on its torus fiber then*

$$\sum_{[X] \in \Omega_{\mathrm{L}}/\langle H \rangle} \mathfrak{h}\big(\phi(X)\big) = \lambda \mod 2\pi i, \tag{5.8}$$

where $\mathfrak{h} = \mathfrak{h}_\tau$ is defined as in §2.3 and the sum converges absolutely.

5.5 Proof of Theorem 5.4

In this section we give the proof of Theorem 5.4.

Proof of Theorem 5.4. In order to prove Theorem 5.4, we need to compute the sum on the left hand of (5.8). We shall assume that the monodromy is positive. Let us fix an orientation on the meridian of ∂M consistent with the orientation of the fibre. We shall use the upper half-space model of $\mathbb{H}^3$, so that its ideal boundary is identified with the extended complex plane, $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, which has $\mathrm{PSL}(2, \mathbf{C})$ acting in the usual way. We can normalize our representation $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$ so that 0 and ∞ are the fixed points of the images of the meridian and longitude. Then the similarity structure on ∂M is given by $\mathbf{C}\backslash\{0\}$ modulo the subgroup of $\rho(\pi_1(M))$ generated by the images of the meridian and the longitude.

Suppose a, b are free generators of the fibre group Γ such that the ordered pair of simple closed curves on $\mathbb{T}$ that they represent have algebraic intersection number $+1$. The commutator, $[a, b] = aba^{-1}b^{-1}$ is peripheral in $\mathbb{T}$, and so represents a meridian of ∂M. Then $\rho([a, b])$ describes the translation $\zeta \mapsto e^{2\nu}\zeta$.

Now, to each region $X \in \Omega$, we shall associate an incomplete line $\tilde{\Delta}(X)$ in the developing image of M in $\mathbb{H}^3$ so that $\tilde{\Delta}(X)$ is normal to the z-axis and has one end, $q(X)$, lying in the z-axis. Let $p(X) \in \mathbf{C}\backslash\{0\}$ be the ideal point which is the orthogonal projection of $q(X)$ on $\mathbf{C}\backslash\{0\}$ along the direction of $\tilde{\Delta}(X)$.

We construct $\tilde{\Delta}(X)$ as follows. Recall that X corresponds to some simple closed curve $\gamma(X)$ on $\mathbb{T}$. Let $\delta(X)$ be an arc on $\mathbb{T}$ with both endpoints at the puncture such that $\gamma(X) \cap \delta(X) = \emptyset$. The homotopy class of $\delta(X)$ relative to its endpoints is well-defined. We are identifying $\mathbb{T}$ with a fibre of M, so $\mathbb{T}$ is naturally homotopy equivalent to the infinite cyclic cover of M. Under this equivalence, $\delta(X)$ has a unique realization as a geodesic $\Delta(X)$ in M whose developing images in $\mathbb{H}^3$ are normal to the missing lines. Choose a lift, $\tilde{\Delta}(X)$, of $\Delta(X)$ to $\mathbb{H}^3$ which is normal to the z-axis. Any other choice of lift would give us an image of $\tilde{\Delta}(X)$, hence an image of $p(X)$, under the cyclic action generated by $\zeta \mapsto e^{\nu}\zeta$. Hence $\tilde{\Delta}(X)$ and $p(X)$ are well-defined up to the action of $\zeta \mapsto e^{\nu}\zeta$.

Consider the bi-infinite sequence, $(X_j)_{j\in\mathbf{Z}}$, of all regions of Ω_{L}. We may choose $\tilde{\Delta}(X_j)$ so that the transformation $\zeta \mapsto e^{\lambda}\zeta$ acts on them as a shift. This is guaranteed by the geometric triangulation of ∂M described earlier.

It is not hard to see that in the geometric triangulation of $\mathbf{C}\backslash\{0\}$, the vertex corresponding to X_j is given by one of the images of $p(X_j)$ under the action generated by $\zeta \mapsto e^{\nu}\zeta$, so we may as well assume that it actually equals $p_j = p(X_j)$. The choice of p_j naturally determines that of p_{j-1} and p_{j+1}, and so, inductively, p_j for all $j \in \mathbf{Z}$. Now the sequence (p_j) is periodic under the transformation corresponding to the chosen longitude of ∂M. This transformation is given by $\zeta \mapsto e^{\lambda}\zeta$, where $\lambda = \lambda(\partial M)$ is the length of the longitude. This corresponds to the action of $\langle H\rangle$ on Ω which has the effect of shifting the sequence (X_j). Let m be the number of steps through which this sequence is shifted. Thus,

$$\lambda = \log p_m - \log p_0 = \sum_{j=1}^{m}\big(\log p_j - \log p_{j-1}\big) \mod 2\pi i.$$

We thus want to compute the numbers $\log p_j - \log p_{j-1}$ modulo $2\pi i$. Let $\vec{e}_j$ be the directed edge given by $X_j \cap X_{j-1}$, whose head lies in β, the axis of H in the binary tree Σ, and let C_{L} be the set $\{\vec{e}_1, \cdots, \vec{e}_m\}$ of directed edges.

Fix some $j \in \{1, \cdots, m\}$. Let $X = X_{j-1}$, $Y = X_j$, and let Z be the region at the head of $\vec{e}_j$. As described earlier, we can find free generators a, b for Γ which correspond, respectively, to the regions X and Y. Moreover, we can suppose that a and b are as described earlier and Z is represented by ab. By our discussion in §2.7, in particular, by Lemma 2.43, we have

$$\log p_j - \log p_{j-1} = \Psi(x, y, z) \mod 2\pi i,$$

where $x = \operatorname{tr}\rho(a) = \phi(X)$, $y = \operatorname{tr}\rho(b) = \phi(Y)$ and $z = \operatorname{tr}\rho(ab) = \phi(Z)$. Thus

$$\log p_j - \log p_{j-1} = \psi(\vec{e}_j) \mod 2\pi i.$$

It follows that we have, modulo $2\pi i$,

$$\begin{aligned}\lambda(\partial M) &= \log p_m - \log p_0 = \sum_{j=1}^{m} (\log p_j - \log p_{j-1}) \\ &= \sum_{\vec{e}\in C_{\mathrm{L}}} \psi(\vec{e}) = \sum_{[X]\in\Omega_{\mathrm{L}}/\langle H\rangle} \mathfrak{h}(\phi(X)) \\ &= \sum_{\sigma\in\mathscr{S}_{\mathrm{L}}} \log\left(\frac{e^{\nu}+e^{l(\sigma)}}{e^{-\nu}+e^{l(\sigma)}}\right).\end{aligned}$$

This proves Theorem 5.4 under certain choices of the orientations involved. □

Proof of Corollary 5.5. By Thurston's Hyperbolic Dehn Surgery Theorem (see [50]), $\overline{M}(p/q)$ has a complete hyperbolic structure for all but a finite number of surgery slopes $p/q \in \mathbf{Q}$. By further excluding a finite number of slopes, we may assume that the induced incomplete hyperbolic structure on M is obtained by a slight deformation from the unique complete hyperbolic structure on M. By Remark 5.8, M then has discrete length spectrum on its torus fiber $\mathbb{T}$ and the conclusion follows from Theorem 5.4. □

Chapter 6

Classical Schottky Groups and McShane's Identity

In this chapter we extend the generalization of McShane's identity, (3.19), for hyperbolic surfaces with geodesic boundary to an identity for marked classical Schottky groups by analytic continuation. This is possible because the classical Schottky space, appropriately parametrized, is a connected open subset of the parametrization space and both sides of McShane's identity are analytic functions on the the classical Schottky space.

In §6.1 we give some basic facts about classical Schottky groups and the parametrizations of the marked classical Schottky space. In §6.2 we state and prove the main result, Theorem 6.4, of this chapter. Finally, in §6.3, we analyze an example to show that our generalization of McShane's identity to classical Schottky groups does give new identities on the corresponding hyperbolic surfaces of fuchsian Schottky groups.

6.1 Marked classical Schottky groups

In this section we give the basic facts about marked classical Schottky groups. See [27], [28] for a complete study of various Schottky spaces.

We define a **Schottky group** Γ as a finitely generated free subgroup of $\mathrm{PSL}(2, \mathbf{C})$, where every non-trivial element of Γ is loxodromic (including hyperbolic), and where Γ acts discontinuously somewhere on the extended complex plane, $\mathbf{C}_\infty$.

Then for every set of free generators, $\{a_1, \cdots, a_p\}$, of a Schottky group Γ, there is a region $D \subset \mathbf{C}_\infty$, where D is bounded by $2p$ disjoint simple closed curves, $C_1, C_1', \cdots, C_p, C_p'$ so that, for $i = 1, \cdots, p$, $a_i(C_i) = C_i'$, and $a_i(D) \cap D = \emptyset$. It is a well-known fact that these curves are not uniquely determined, even up to the free homotopy in $\Omega(\Gamma)$, the set of discontinuity of Γ.

A Schottky group Γ together with an ordered set $\{a_1, \cdots, a_p\}$ of free generators is called (by us) a **marked Schottky group** and is denoted by $\Gamma = \langle a_1, \cdots, a_p \rangle$.

A marked Schottky group $\Gamma = \langle a_1, \cdots, a_p \rangle$ is **classical** if one can choose the set of disjoint simple closed curves $C_1, C_1', \cdots, C_p, C_p'$ so that they are all euclidean circles. A Schottky group Γ is called *classical* if there is a set of generators $\{a_1, \cdots, a_p\}$ such that the marked Schottky group $\Gamma = \langle a_1, \cdots, a_p \rangle$ is classical.

Fix a marked Schottky group $\Gamma_0 = \langle a_1, \cdots, a_p \rangle$. The **Schottky space**, $\mathcal{S}_{\mathrm{alg}}$, is defined as the space of discrete faithful representations of Γ_0 in $\mathrm{PSL}(2, \mathbf{C})$, modulo conjugation, where the image group is again a Schottky group.

The **classical Schottky space**, $\mathcal{S}^{\mathrm{c}}_{\mathrm{alg}}$, is the space of representations $\rho \in \mathcal{S}_{\mathrm{alg}}$ modulo conjugation such that $\rho(\Gamma)$ is classical on some set of generators.

Remark 6.1 It was shown by Marden [26] that there exist non-classical Schottky groups for every $p \geq 2$. An explicit example of a non-classical Schottky group was constructed by Yamamoto [52]. On the other hand, Button [12] has proved that all fuchsian Schottky groups are classical (but in general not on every set of generators).

Definition 6.2 The **marked classical Schottky space**, $\mathcal{S}_{\rm alg}^{\rm mc}$, is the space of representations $\rho \in \mathcal{S}_{\rm alg}$ modulo conjugation such that $\rho(\Gamma) = \langle \rho(a_1), \cdots, \rho(a_p)\rangle$ is classical on the set of free generators $\{\rho(a_1), \cdots, \rho(a_p)\}$.

Next we give a natural parametrization of the marked classical Schottky space $\mathcal{S}_{\rm alg}^{\rm mc}$ by the fixed ideal points and the traces of the associated set of ordered free generators.

For this purpose we need to choose a lift of a representation in $\mathrm{PSL}(2, \mathbf{C})$ of Γ to a representation in $\mathrm{SL}(2, \mathbf{C})$. Given a marked Schottky group $\Gamma = \langle a_1, \cdots, a_p\rangle \subset \mathrm{SL}(2, \mathbf{C})$ where $p \geq 2$, we may normalize it by conjugation so that

$$\mathrm{Fix}^- a_1 = 0, \ \ \mathrm{Fix}^+ a_1 = \infty \ \ \text{and} \ \ \mathrm{Fix}^- a_2 = 1.$$

Then we parametrize $\Gamma = \langle a_1, \cdots, a_p\rangle \subset \mathrm{SL}(2, \mathbf{C})$ with this marking by

$$(\,\mathrm{Fix}^+ a_2, \mathrm{Fix}^- a_3, \mathrm{Fix}^+ a_3, \cdots, \mathrm{Fix}^- a_p, \mathrm{Fix}^+ a_p; \mathrm{tr}\, a_1, \cdots, \mathrm{tr}\, a_p) \in \mathbf{C}_\infty^{2p-3} \times \mathbf{C}^p.$$

With this normalized parametrization we have

Lemma 6.3 (Maskit [28]) *The marked classical Schottky space $\mathcal{S}_{\rm alg}^{\rm mc}$ is a path connected open subset of $\mathbf{C}_\infty^{2p-3} \times \mathbf{C}^p$.*

Proof. Here we would like to sketch the idea of the proof; see Maskit [28] for a detailed proof. We use the conformal ball model of hyperbolic 3-space $\mathbb{H}^3$. The ideal sphere is then the unit sphere S_∞. Consider a marked classical Schottky group $\Gamma = \langle a_1, \cdots, a_p\rangle \subset \mathrm{Isom}^+(\mathbb{H}^3)$. Then there are open disks D_j, D'_j, $j = 1, \cdots, p$, whose boundary circles are denoted C_j, C'_j respectively, such that $a_j(C_j) = C'_j$ and $a_j(D_j) \cap D'_j = \emptyset$. Now we may deform $\Gamma = \langle a_1, \cdots, a_p\rangle$ in $\mathcal{S}_{\rm alg}^{\rm mc}$, keeping the idea fixed points of all a_j unchanged, so that each $\mathrm{tr}\, a_j$ has sufficiently large norm. Hence we may assume that under the deformation the circles C_j, C'_j shrink towards the respective ideal fixed points of a_j and so that their sizes become sufficiently small. After that we may deform the marked classical Schottky group in $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$ continuously to an arbitrary standard one with this property. Since any marked classical Schottky group can be so deformed, the space $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$ is path connected. That $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$ is open is easy and well-known. □

Finally let us say a few words about the fundamental domain in $\mathbb{H}^3$ of a classical Schottky group. Let $\Gamma = \langle a_1, \cdots, a_p \rangle$ be a classical Schottky group with the set of disjoint discs D_j, D'_j as before. Suppose the circles C_j, C'_j bound respectively geodesic planes E_j, E'_j in $\mathbb{H}^3$. For $j = 1, \cdots, p$, let H_j be the open half space of $\mathbb{H}^3$ bounded by D_j and E_j; similarly for H'_j. Then $\mathcal{D} := \mathbb{H}^3 - \bigcup_{j=1}^{p} H_j - \bigcup_{j=1}^{p} H'_j$ is a fundamental domain in $\mathbb{H}^3$ of the classical Schottky group Γ. Note that $D = \overline{\mathcal{D}} \cap \mathbf{C}_\infty$ is the fundamental domain in the extended complex plane $\mathbf{C}_\infty$ of Γ as described at the beginning of this section. It is well-known that the quotient hyperbolic 3-manifold $\mathcal{H} = \mathbb{H}^3/\Gamma = \mathcal{D}/\Gamma$ is a handlebody of genus p. Note that $\mathbf{C}_\infty/\Gamma = D/\Gamma$ is a conformal surface of genus p which is the conformal boundary of the handlebody $\mathcal{H}$.

6.2 McShane's identities for Schottky groups

In this section we state and prove our main theorem, Theorem 6.4 below, of this chapter which gives an extension of the generalized McShane's identity (3.17) or (3.19) to an identity for marked classical Schottky groups.

Let $\Gamma_0 = \langle a_1, \cdots, a_p \rangle$ be a marked classical Schottky group. Consider a continuous path in the marked classical Schottky space $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$, that is, a continuous family of discrete faithful representations $\rho_t : \Gamma_0 \to \mathrm{PSL}(2, \mathbf{C})$, $t \in [0, 1]$ so that the images are marked classical Schottky groups

$$\Gamma(t) := \rho_t(\Gamma_0) = \langle a_1(t), \cdots, a_p(t) \rangle,\ t \in [0, 1],$$

where $a_j(t) = \rho_t(a_j)$, $j = 1, \cdots, p$. We are interested in the cases where $\Gamma(0) = \langle a_1(0), \cdots, a_p(0) \rangle \subset \mathrm{PSL}(2, \mathbf{R})$ is a fuchsian Schottky group. Let M be the corresponding compact hyperbolic surface which is the convex core of $\mathbb{H}^2/\Gamma(0)$. Let the geodesic boundary components of M be $\Delta_0, \Delta_1, \cdots, \Delta_q$ where $q \geq 0$. Recall that in Chapter 3 we have obtained a generalized McShane's identity (3.19) (see also [34]) for such a surface with a distinguished geodesic boundary Δ_0. We restate it as follows:

$$\sum_{\alpha,\,\beta} G\left(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + \sum_{j=1}^{q} \sum_{\beta} S\left(\frac{l_0}{2}, \frac{l_j}{2}, \frac{|\beta|}{2}\right) = \frac{l_0}{2}, \tag{6.1}$$

where $l_j \geq 0$ (we assume $l_0 > 0$) is the hyperbolic length of the geodesic boundary component Δ_j, $j = 0, 1, \cdots, q$, and in the first sum on the left-hand side of (6.1) the summation is taken over all unordered pair of simple closed geodesics α, β on M (note that one of α and β might be a boundary geodesic Δ_j) such that α and β bound with Δ_0 an embedded pair of pants on M and in the sub-sum of the second sum on the left-hand side of (6.1) the summation is taken over all interior simple closed geodesics β such that β bounds with Δ_0 and Δ_j an embedded pair of pants on M. Here we have assumed that M is not a thrice-punctured sphere, $S_{0,3}$, for in that case we will obtain only the trivial identity (6.12).

For $g \in \Gamma(0)$, we denote $g(t) = \rho_t\, \rho_0^{-1}(g) \in \Gamma(t)$. (We may assume $\rho(0) = \mathrm{id}$ for simplicity; then $g(t) = \rho_t(g)$.)

The other half translation length $\lceil \frac{l(\gamma)}{2} \rfloor$. For $\gamma \in \mathrm{SL}(2, \mathbf{C})$, we have defined the half translation length of γ by

$$\frac{l(\gamma)}{2} = \cosh^{-1}\left(\frac{\operatorname{tr}\gamma}{2}\right) \in \mathbf{C}/2\pi i\mathbf{Z}. \tag{6.2}$$

Now we define the *other* half translation length of $\gamma \in \mathrm{SL}(2, \mathbf{C})$ by

$$\left\lceil \frac{l(\gamma)}{2} \right\rfloor := \frac{l(-\gamma)}{2} = \cosh^{-1}\left(-\frac{\operatorname{tr}\gamma}{2}\right) = \frac{l(\gamma)}{2} + \pi i. \tag{6.3}$$

With the above notation we have the following extension of the generalized McShane's identity (6.1) to an identity for marked classical Schottky groups.

Theorem 6.4 *For a continuous family of marked classical Schottky groups* $\Gamma(t) = \langle a_1(t), \cdots, a_p(t)\rangle \subset \mathrm{SL}(2, \mathbf{C})$, $t \in [0,1]$, *such that* $\Gamma(0) = \langle a_1(0), \cdots, a_p(0)\rangle \subset \mathrm{SL}(2, \mathbf{R})$ *is a fuchsian Schottky group, let* M *be the corresponding compact hyperbolic surface which is the convex core of* $\mathbb{H}^2/\Gamma(0)$ *and let the geodesic boundary components of* M *be* $\Delta_0, \Delta_1, \cdots, \Delta_q$ *where* $q \geq 0$. *Then we have for* $t \in [0,1]$

$$\begin{aligned} &\sum_{\alpha,\beta} G\Big(\Big\lceil \frac{l(\Delta_0(t))}{2}\Big\rfloor, \Big\lceil \frac{l(\alpha(t))}{2}\Big\rfloor, \Big\lceil \frac{l(\beta(t))}{2}\Big\rfloor\Big) \\ &\quad + \sum_{j=1}^{q}\sum_{\beta} S\Big(\Big\lceil \frac{l(\Delta_0(t))}{2}\Big\rfloor, \Big\lceil \frac{l(\Delta_j(t))}{2}\Big\rfloor, \Big\lceil \frac{l(\beta(t))}{2}\Big\rfloor\Big) \\ &\quad = \Big\lceil \frac{l(\Delta_0(t))}{2}\Big\rfloor \qquad \mathrm{mod}\ \pi i, \end{aligned} \tag{6.4}$$

where in the first sum on the left-hand side of (6.4) *the summation is taken over all unordered pairs of simple closed geodesics* α, β *on* M *such that* α *and* β *bound with* Δ_0 *an embedded pair of pants on* M *(note that one of* α, β *might be a boundary geodesic* Δ_j*), and in the sub-sum of the second sum on the left-hand side of* (6.4) *the summation is taken over all simple closed geodesics* β *such that* β *bounds with* Δ_0 *and* Δ_j *an embedded pair of pants on* M. *Moreover, each series on the left-hand side of* (6.4) *converges absolutely.*

Proof. We shall prove that the series on the left-hand side of (6.4) converge uniformly on compact subsets of the marked classical Schottky space $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$. Then each side of (6.4) is a holomorphic function modulo πi on the space $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$. By Theorem 3.17 the identity (6.4) holds on the totally real subspace in a neighborhood of $\Gamma(0) = \langle a_1(0), \cdots, a_p(0)\rangle$ in $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$; hence we know that it also holds modulo πi for each $t \in [0,1]$ by analytic continuation.

Given a compact subset $\mathcal{K}$ of $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$, we have a constant $\kappa > 0$ such that each marked classical Schottky group $\Gamma = \langle a_1, \cdots, a_p\rangle \in \mathcal{K}$ has a fundamental domain $\mathcal{D}$ in $\mathbb{H}^3$ as described at the end of §6.1 such that the minimum hyperbolic distance between any pair of its bounding geodesic planes $E_1, E_1', \cdots, E_p, E_p'$ is $\geq \kappa$. Then we have the following length estimate lemma.

Lemma 6.5 *If $g \in \Gamma$ is a cyclically reduced word in letters $a_1, a_1^{-1}, \cdots, a_p, a_p^{-1}$ with word length $\|g\|$, then the closed geodesic γ which g represents in the quotient hyperbolic 3-manifold $\mathcal{D} = \mathbb{H}^3/\Gamma$ has hyperbolic length $\geq \kappa\|g\|$.*

Proof. Choose in $\mathbb{H}^3$ an arbitrary lift, $\tilde{\gamma}$, of the closed geodesic γ. Note that $\mathbb{H}^3$ is paved up by the images of a fundamental domain $\mathcal{D}$ under the action of elements of Γ, that is, $\mathbb{H}^3 = \bigcup_{g' \in \Gamma} g'(\mathcal{D})$. It can be shown that the line $\tilde{\gamma}$ in $\mathbb{H}^3$ passes through $\|g\|$ successive images of $\mathcal{D}$ "periodically" dictated by the word g. Thus the hyperbolic length of γ, which equals the length of the part of $\tilde{\gamma}$ lying in the union of these $\|g\|$ successive images of $\mathcal{D}$, is at least $\kappa\,\|g\|$. □

Now we prove the uniform convergence of the first series in (6.4) for $t \in [0, 1]$. By the above lemma there is a constant $\kappa > 0$ such that for every $t \in [0, 1]$ and every $g(t) \in \Gamma(t)$, we have $L(g(t)) \geq \kappa\|g(t)\|$, where $L(g(t))$ is the hyperbolic length of the closed geodesic that $g(t)$ represents in the quotient hyperbolic 3-manifold $\mathbb{H}^3/\Gamma(t)$, and where $\|g(t)\|$ is the cyclically reduced word length of $g(t)$ in the letters $a_1(t)^{\pm 1}, \cdots, a_p(t)^{\pm 1}$.

Note that the fuchsian classical Schottky group $\Gamma(0) \subset \mathrm{PSL}(2, \mathbf{R})$ has a fundamental domain $\mathcal{D}(0)$ in $\mathbb{H}^3$ whose intersection with $\mathbb{H}^2 \subset \mathbb{H}^3$ is a fundamental domain of $\Gamma(0)$ in $\mathbb{H}^2$. Let $\mathcal{G}$ be the set of the unordered pairs $\{\alpha, \beta\}$ of simple closed geodesics α, β on the hyperbolic surface M such that α, β cobound with Δ_0 an embedded pair of pants. Then the pairs $\{\alpha, \beta\}$ in $\mathcal{G}$ can be counted by considering $\|\alpha\| + \|\beta\|$, the sum of their cyclically reduced word lengths. By an application of the argument in [5] (see §3.6 for an outline) for the behaviors of simple closed geodesics on M using the fundamental domain $\mathcal{D}(0) \cap \mathbb{H}^2$ mentioned above we know there is a polynomial P such that the numbers of pairs $\{\alpha, \beta\}$ in $\mathcal{G}$ such that $\|\alpha\| + \|\beta\| = n$ is no greater that $P(n)$. Note that the Birman–Series' argument in [5] works as well here for pairs of disjoint simple closed geodesics on the surface M since the simple diagrams that the pairs determine on the fundamental domain $\mathcal{D}(0) \cap \mathbb{H}^2$ contain complete information to reconstruct them.

Note that for $t \in [0,1]$ and a simple closed geodesic α on M, the real part $\Re\, l(\alpha(t))$ of the complex translation length $l(\alpha(t))$ is equal to the hyperbolic length $L(\alpha(t))$ of the closed geodesic that $\alpha(t)$ represents in the quotient hyperbolic manifold $\mathbb{H}^3/\Gamma(t)$. Note also that for each simple closed geodesic γ on M, there exists a constants $c_1(\gamma), c_2(\gamma) > 0$ such that $c_1(\gamma) \leq \Re\, l(\gamma(t)) \leq c_2(\gamma)$ for all $t \in [0,1]$.

Now for each pair $\{\alpha, \beta\}$ in $\mathcal{G}$, we have $L(\alpha(t)) + L(\beta(t)) \geq \kappa(\|\alpha\| + \|\beta\|)$ for all $t \in [0,1]$, and hence $\Re\, l(\alpha(t)) + \Re\, l(\beta(t)) = L(\alpha(t)) + L(\beta(t)) \to +\infty$ uniformly as $n = \|\alpha\| + \|\beta\| \to \infty$.

By the definition of G, we have

$$
\begin{aligned}
& G\big(\big\lceil \tfrac{l(\Delta_0(t))}{2} \big\rfloor, \big\lceil \tfrac{l(\alpha(t))}{2} \big\rfloor, \big\lceil \tfrac{l(\beta(t))}{2} \big\rfloor\big) \\
&\quad = \log\left(\frac{\exp\big\lceil \frac{l(\Delta_0(t))}{2} \big\rfloor + \exp\big(\big\lceil \frac{l(\alpha(t))}{2} \big\rfloor + \big\lceil \frac{l(\alpha(t))}{2} \big\rfloor\big)}{\exp\big(-\big\lceil \frac{l(\Delta_0(t))}{2} \big\rfloor\big) + \exp\big(\big\lceil \frac{l(\alpha(t))}{2} \big\rfloor + \big\lceil \frac{l(\beta(t))}{2} \big\rfloor\big)}\right) \\
&\quad = \log\left(1 + \frac{2\sinh\big\lceil \frac{l(\Delta_0(t))}{2} \big\rfloor}{\exp\big(-\big\lceil \frac{l(\Delta_0(t))}{2} \big\rfloor\big) + \exp\big(\big\lceil \frac{l(\alpha(t))}{2} \big\rfloor + \big\lceil \frac{l(\beta(t))}{2} \big\rfloor\big)}\right). \qquad (6.5)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& \big|\exp\big(-\big\lceil \tfrac{l(\Delta_0(t))}{2} \big\rfloor\big) + \exp\big(\big\lceil \tfrac{l(\alpha(t))}{2} \big\rfloor + \big\lceil \tfrac{l(\alpha(t))}{2} \big\rfloor\big)\big| \\
&\quad \geq \big|\exp\big(\big\lceil \tfrac{l(\alpha(t))}{2} \big\rfloor + \big\lceil \tfrac{l(\alpha(t))}{2} \big\rfloor\big)\big| - \big|\exp\big(-\big\lceil \tfrac{l(\Delta_0(t))}{2} \big\rfloor\big)\big| \\
&\quad = \exp\big(\tfrac{\Re\, l(\alpha(t))}{2} + \tfrac{\Re\, l(\beta(t))}{2}\big) - \exp\big(-\tfrac{\Re\, l(\Delta_0(t))}{2}\big) \qquad (6.6) \\
&\quad \geq 1 - \exp\big(-\tfrac{\Re\, l(\Delta_0(t))}{2}\big) \\
&\quad \geq 1 - \exp\big(-c_1(\Delta_0)\big). \qquad (6.7)
\end{aligned}
$$

Since $|\log(1+u)| \leq 2|u|$ for all $u \in \mathbf{C}$ such that $|u| \leq 1/2$, it follows from (6.5) and (6.6) that there is a constant $C > 0$, depending only on the family $\{\Gamma(t)\}_{t \in [0,1]}$, such that for all but a finite number of pairs $\{\alpha, \beta\}$ in $\mathcal{G}$ we have

$$
\begin{aligned}
\big|G\big(\big\lceil \tfrac{l(\Delta_0(t))}{2} \big\rfloor, \big\lceil \tfrac{l(\alpha(t))}{2} \big\rfloor, \big\lceil \tfrac{l(\beta(t))}{2} \big\rfloor\big)\big| &\leq C \cdot \exp\big(-\tfrac{L(\alpha(t)) + L(\beta(t))}{2}\big) \\
&\leq C \cdot \exp\big(-\tfrac{\kappa(\|\alpha\| + \|\beta\|)}{2}\big). \qquad (6.8)
\end{aligned}
$$

The claim below tells us that the left-hand side of (6.8) is always finite. Hence (6.8) actually holds for all pairs $\{\alpha, \beta\}$ in $\mathcal{G}$. It then follows from Lemma 3.47 that the fist series in (6.4) converges absolutely and uniformly for $t \in [0, 1]$.

Claim. For each pair $\{\alpha, \beta\}$ in $\mathcal{G}$ and for all $t \in [0, 1]$, we have

$$\exp\left(\pm \left\lceil \tfrac{l(\Delta_0(t))}{2} \right\rfloor\right) + \exp\left(\left\lceil \tfrac{l(\alpha(t))}{2} \right\rfloor + \left\lceil \tfrac{l(\beta(t))}{2} \right\rfloor\right) \neq 0. \tag{6.9}$$

When $\pm$ is $-$ in (6.9), the inequality follows from (6.7). When $\pm$ is $+$ in (6.9), it follows from the following equivalent inequality:

$$\left\lceil \tfrac{l(\Delta_0(t))}{2} \right\rfloor + \pi i \neq \left\lceil \tfrac{l(\alpha(t))}{2} \right\rfloor + \left\lceil \tfrac{l(\beta(t))}{2} \right\rfloor \mod 2\pi i. \tag{6.10}$$

To prove (6.10), suppose $\left\lceil \tfrac{l(\Delta_0(t))}{2} \right\rfloor + \pi i = \left\lceil \tfrac{l(\alpha(t))}{2} \right\rfloor + \left\lceil \tfrac{l(\beta(t))}{2} \right\rfloor \mod 2\pi i$ holds for some $t = t_0 \in [0, 1]$. We may assume (by replacing α and β by their inverses and/or conjugates in $\Gamma(0)$, if necessary) that $\Delta_0 = \alpha\beta$ and hence $\Delta_0(t_0) = \alpha(t_0)\beta(t_0)$. Now it is easy to know (say, by the cosine rule (2.8) of Fenchel) that $\alpha(t_0)$ and $\beta(t_0)$ have the same axis in $\mathbb{H}^3$, hence either $\Gamma(t_0)$ is not a discrete subgroup of $\mathrm{SL}(2, \mathbf{C})$ or the representation $\rho(t_0) : \Gamma \to \mathrm{SL}(2, \mathbf{C})$ is not faithful. In either case we have a contradiction. This proves (6.10) and hence the above claim.

The absolute and uniform convergence for the other series in (6.4) can be proved similarly . □

Corollary 6.6 *If $q = 0$ in Theorem 6.4 (namely, the surface M has only one boundary component), then we have*

$$\sum_{\alpha, \beta} G\left(\left\lceil \frac{l(\Delta_0(t))}{2} \right\rfloor, \left\lceil \frac{l(\alpha(t))}{2} \right\rfloor, \left\lceil \frac{l(\beta(t))}{2} \right\rfloor\right) = \left\lceil \frac{l(\Delta_0(t))}{2} \right\rfloor \mod 2\pi i, \tag{6.11}$$

(note that here the identity holds modulo $2\pi i$ instead of πi) where the sum is taken over all unordered pairs of simple closed geodesics α, β on M such that α and β bound with Δ_0 an embedded pair of pants on M and the series converges absolutely. □

6.3 An example: thrice-punctured sphere

In this section we consider an example of fuchsian Schottky group $\Gamma_1 = \langle a, b\rangle$ whose surface M_1 is a hyperbolic thrice-punctured sphere with geodesic boundary. Let the three geodesic boundary components be denoted $\Delta_0, \Delta_1, \Delta_2$ with hyperbolic lengths $l_0 = L(\Delta_0) > 0, l_1 = L(\Delta_1) > 0, l_2 = L(\Delta_2) > 0$ respectively. For this hyperbolic surface we have the following trivial identity:

$$G\left(\frac{l_0}{2}, \frac{l_1}{2}, \frac{l_2}{2}\right) + S\left(\frac{l_0}{2}, \frac{l_1}{2}, \frac{l_2}{2}\right) + S\left(\frac{l_0}{2}, \frac{l_2}{2}, \frac{l_1}{2}\right) = \frac{l_0}{2}. \tag{6.12}$$

There is, however, a non-trivial identity for M_1 derived from the marked fuchsian Schottky group $\Gamma_1 = \langle a, b\rangle$ where the marking $\langle a, b\rangle$ is given by a usual marking of the one-hole torus group. Consider the fuchsian classical Schottky group $\Gamma_0 = \langle a_0, b_0\rangle \subset \mathrm{PSL}(2, \mathbf{R})$ whose corresponding hyperbolic surface is a one-hole torus, T, with geodesic boundary, where a_0, b_0 correspond to two simple closed geodesics on T intersecting once. Let the geodesic boundary component of T be denoted Δ, with hyperbolic length $l > 0$. Then for such T we have the generalized McShane's identity (3.2):

$$\sum_{\alpha} G\left(\frac{l}{2}, \frac{l(\alpha)}{2}, \frac{l(\alpha)}{2}\right) = \frac{l}{2},$$

or equivalently

$$\sum_{\alpha} 2\tanh^{-1}\left(\frac{\sinh(l/2)}{\cosh(l/2) + \exp l(\alpha)}\right) = \frac{l}{2}, \tag{6.13}$$

where the sum is taken over all interior simple closed geodesics α on T.

Since the marked classical Schottky space $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$ is connected, we can choose a continuous path in $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$ from $\Gamma_0 = \langle a_0, b_0\rangle$ to $\Gamma_1 = \langle a, b\rangle$ preserving the markings. Then Corollary 6.6 tells us that the identity (6.13) also holds modulo $2\pi i$ for the hyperbolic thrice-punctured sphere $S_{0,3}$ with geodesic boundary if we sum over the closed geodesics on $S_{0,3}$ whose free homotopy classes correspond to those of

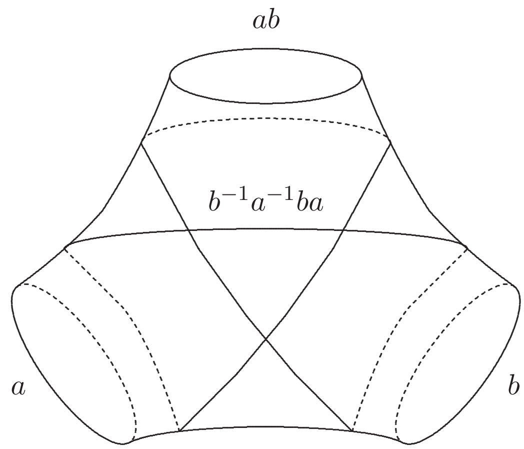

Figure 6.1: A commutator curve on $S_{0,3}$

interior simple closed geodesics on the hyperbolic one-hole torus T. Note that the gaps are measured along a closed geodesic γ on $S_{0,3}$ which represents the free homotopy class of, say, the commutator $b^{-1}a^{-1}ba$. See Figure 6.1 for an illustration of one such closed curve γ on $S_{0,3}$. Note also that on $S_{0,3}$ there are only three simple closed geodesics which are the geodesic boundary components.

To express this identity explicitly, let us recall the sequence of pairs of words, $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$, on letters $a^{\pm}, b^{\pm}$ constructed in §4.4. In particular, we have

$$(L_{\frac{0}{1}}, R_{\frac{0}{1}}) = (b^{-1}ab,\, a), \quad (L_{\frac{1}{2}}, R_{\frac{1}{2}}) = (ab,\, ba),$$
$$(L_{\frac{1}{1}}, R_{\frac{1}{1}}) = (b,\, a^{-1}ba), \quad (L_{\frac{2}{1}}, R_{\frac{2}{1}}) = (a^{-1},\, a^{-1}c),$$

where $c := b^{-1}a^{-1}ba$. Note that the conjugacy classes of all the words $R_{\frac{p}{q}}$, $\frac{p}{q} \in [0,2)$, in the free group $\langle a, b\rangle$ are exactly all the classes of non-trivial, non-peripheral, unoriented simple closed curves on the one-hole torus T.

We may choose a lift of $\Gamma_1 = \langle a, b\rangle$ into $\mathrm{SL}(2,\mathbf{C})$ so that

$$\operatorname{tr} a < -2, \quad \operatorname{tr} b < -2 \quad \text{and} \quad \operatorname{tr} ab < -2. \tag{6.14}$$

Then $(x, y, z) = (\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} ab)$ gives a μ-Markoff triple with $\mu > 20$. Recall that $\nu = \cosh^{-1}(1 - \mu/2) = \cosh^{-1}(\mu/2 - 1) + \pi i$.

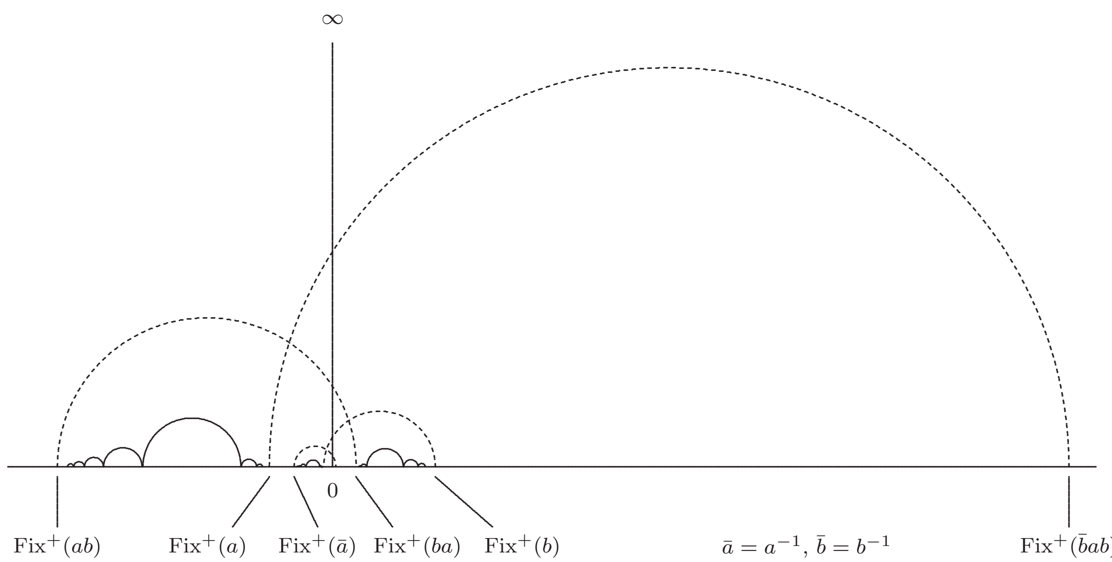

Figure 6.2: Gaps for $S_{0,3}$

Then for each free homotopy class of curves on $S_{0,3}$ represented by the word $R_{\frac{p}{q}}$, the hyperbolic length of the unique closed geodesic on $S_{0,3}$ in this class is given by

$$l\big(R_{\frac{p}{q}}\big) = 2\cosh^{-1}\big(\tfrac{1}{2}\operatorname{tr} R_{\frac{p}{q}}\big) = \cosh^{-1}\big(\tfrac{1}{2}\operatorname{tr}^2 R_{\frac{p}{q}} - 1\big). \tag{6.15}$$

The above described non-trivial identity for $S_{0,3}$ can thus be expressed as

$$\begin{aligned}\sum_{\frac{p}{q}\in[0,2)} \mathfrak{h}\big(\operatorname{tr} R_{\frac{p}{q}}\big) &= \sum_{\frac{p}{q}\in[0,2)} 2\tanh^{-1}\left(\frac{\sinh\nu}{\cosh\nu + \exp l\big(R_{\frac{p}{q}}\big)}\right) \\ &= \sum_{\frac{p}{q}\in[0,2)} \log\left(\frac{\exp\nu + \exp l\big(R_{\frac{p}{q}}\big)}{\exp(-\nu) + \exp l\big(R_{\frac{p}{q}}\big)}\right) \\ &= \nu \mod 2\pi i. \end{aligned} \tag{6.16}$$

Remark 6.7 This is actually the identity (4.8) we have derived for the μ-Markoff map corresponding to the μ-Markoff triple $(x, y, z) = (\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} ab)$.

Next let us describe the true picture of how the gaps distribute. Note that on the right hand side of (6.16) we have $\nu = \cosh^{-1}(\mu/2 - 1) + \pi i$. In the sum on the left-hand side of (6.16) all terms except three are negative real. The three terms correspond to the gaps associated to the boundary *simple* closed geodesics in the classes a, b and ab; each such gap value has imaginary part π. (See Lemma 6.9 at the end of this section for a proof of these assertions in terms of generalized Markoff maps.) This explains why the equality (6.16) holds modulo $2\pi i$.

Now Figure 6.2 shows how the gaps locate and why they sum to ν modulo $2\pi i$, where we have normalized the $(\mu - 2)$-representation corresponding to the μ-Markoff triple $(\operatorname{tr} a,\ \operatorname{tr} b,\ \operatorname{tr} ab)$ by conjugation so that the commutator $b^{-1}a^{-1}ba$ has the oriented line $[0, \infty]$ as its axis. Here the upper-half plane model of $\mathbb{H}^2$ can be thought of as the intersection of the upper-half space model of $\mathbb{H}^3$ with the vertical plane passing through the real line $\mathbf{R}$ in the complex plane $\mathbf{C}$. For $\frac{p}{q} \in [0, 2)$, the gap value $\mathfrak{h}\big(\operatorname{tr} R_{\frac{p}{q}}\big)$ is the complex length from $\operatorname{Fix}^+\big(R_{\frac{p}{q}}\big)$ to $\operatorname{Fix}^+\big(L_{\frac{p}{q}}\big)$ measured along the oriented line $[0, \infty]$ in $\mathbb{H}^3$. For each pair of words $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$, $\frac{p}{q} \in [0, 2]$, we draw the geodesic in the upper-half plane model of $\mathbb{H}^2$ which has $\operatorname{Fix}^+\big(R_{\frac{p}{q}}\big)$ and $\operatorname{Fix}^+\big(L_{\frac{p}{q}}\big)$ as ideal endpoints to represent the corresponding gap. The four dotted ones correspond to $\frac{p}{q} = \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}$; each of them crosses the y-axis. All the other gaps for $\frac{p}{q} \in [0, 2)$ lie and fill in the three intervals $[\operatorname{Fix}^+(ab), \operatorname{Fix}^+(a)]$, $[\operatorname{Fix}^+(b), \operatorname{Fix}^+(ba)]$ and $[\operatorname{Fix}^+(a^{-1}), \operatorname{Fix}^+(a^{-1}ba)]$ as the figure shows. It is shown in Lemma 6.8 below that the complex length from $\operatorname{Fix}^+(a^{-1}) = \operatorname{Fix}^+(L_{\frac{2}{1}})$ to $\operatorname{Fix}^+(b^{-1}ab) = \operatorname{Fix}^+(L_{\frac{0}{1}})$ measured along the oriented line $[0, \infty]$ is equal to ν. Hence all the gap values sum to ν modulo $2\pi i$.

Lemma 6.8 *Given* $A, B \in \mathrm{SL}(2, \mathbf{C})$ *such that* $\operatorname{tr}(B^{-1}A^{-1}BA) \neq \pm 2$, *let* $\nu = \cosh^{-1}(-\frac{1}{2}\operatorname{tr}(B^{-1}A^{-1}BA))$. *Then* ν *equals the complex length from* $\operatorname{Fix}^+(A^{-1})$ *to* $\operatorname{Fix}^+(B^{-1}AB)$ *along the oriented axis* $\mathbf{a}(B^{-1}A^{-1}BA)$.

Proof. There exist $Q, R, P \in \mathrm{SL}(2, \mathbf{C})$ such that $Q^2 = R^2 = P^2 = -I$ and $A = -RQ$, $B = -PR$. Hence $BA = -PQ$, $B^{-1}AB = -RPRQPR$ and $B^{-1}A^{-1}BA = -(RPQ)^2 = -K^2$, where $K := RPQ$. It follows that the complex translation length $l(K) = \cosh^{-1}\big(\frac{1}{2}\operatorname{tr}(K^2)\big) = \nu$.

Since $KA^{-1}K^{-1} = (RPQ)(-QR)(-QPR) = -RPRQPR = B^{-1}AB$, the conjugation by K maps the axis $\mathbf{a}(A^{-1})$ to the axis $\mathbf{a}(B^{-1}AB)$ and hence the attractive fixed point $\mathrm{Fix}^+(A^{-1})$ to the attractive fixed point $\mathrm{Fix}^+(B^{-1}AB)$. By Lemma 2.17 and Definition 2.29, $l(K)$ equals the complex length from $\mathrm{Fix}^+(A^{-1})$ to $\mathrm{Fix}^+(B^{-1}AB)$ along $\mathbf{a}(B^{-1}A^{-1}BA) = \mathbf{a}(K)$. This proves Lemma 6.8. □

Finally, we prove the earlier assertions that all gap values except three are real and negative, while the three exceptional ones all have imaginary part πi.

Lemma 6.9 *Let ϕ be the μ-Markoff map generated by a real μ-Markoff triple (x_0, y_0, z_0) at a vertex v_0 such that $x_0, y_0, z_0 < -2$. Let $X_0, Y_0, Z_0 \in \Omega$ be the regions meeting v_0 and let $\mathfrak{h} = \mathfrak{h}_\tau$ be the gap function defined by (2.31). Then we have $\mathfrak{h}(\phi(X)) < 0$ for all $X \in \Omega \backslash \{X_0, Y_0, Z_0\}$ and each of three values $\mathfrak{h}(x_0)$, $\mathfrak{h}(y_0), \mathfrak{h}(z_0)$ has imaginary part πi.*

Proof. Since $x_0, y_0, z_0 < -2$, we have

$$\mu = x_0^2 + y_0^2 + z_0^2 - x_0 y_0 z_0 = x_0^2 + y_0^2 + z_0^2 + |x_0||y_0||z_0| > 20.$$

Consequently, $x_0^2 < \mu$, $y_0^2 < \mu$ and $z_0^2 < \mu$.

Claim. For all $X \in \Omega$, we have $|\phi(X)| > 2$ and $\phi(X)^2 > \mu$.

To prove the claim, let us define the distance $d(X)$ of $X \in \Omega$ from the fixed vertex v_0 to be the number of edges in a shortest path in the binary tree Σ connecting X to v_0. Thus $d(X) = 0$ if and only if $X \in \{X_0, Y_0, Z_0\}$.

We first prove the claim for $X \in \Omega$ with $d(X) = 1$. In this case, without loss of generality, we may assume that X meet both Y_0 and Z_0. Thus by the edge condition on the edge $e = Y_0 \cap Z_0$, we have $x = y_0 z_0 - x_0$ and $|x| = |y_0||z_0| + |x_0|$. It follows that $|x| > |y_0|$, $|x| > |z_0|$ and

$$x^2 = x_0^2 + y_0^2 z_0^2 + 2\,|x_0||y_0||z_0| > x_0^2 + y_0^2 + z_0^2 + |x_0||y_0||z_0| = \mu.$$

Now for each $X \in \Omega$ with $d(X) > 1$, there exist $Y, Z \in \Omega$ such that X, Y and Z meet at a vertex and $d(Y), d(Z) < d(X)$. We may assume that $d(Y) > d(Z)$.

Then $d(Y) = d(X) - 1$. To prove the claim by induction on $d(X)$, we only need to show that $|\phi(Y)| < |\phi(X)|$.

Let $W \in \Omega$ be the other region that also meets both Y and Z. Then by the edge condition on the edge $e = Y \cap Z$, we have $x = yz - w$. The induction hypotheses tell us that $|y| > |z| > 2$ and $|y| > |w|$. Hence $|x| \geq |y||z| - |w| > 2|y| - |y| = |y|$. This proves the claim.

By the claim, for each $X \in \Omega$, $l(x) = \cosh^{-1}(\frac{1}{2}x^2 - 1) > 0$ since $x^2 > 4$.

Now we can prove that $\mathfrak{h}(x) < 0$ for all $X \in \Omega$ with $d(X) > 0$.

To this end, let us write $\nu = \tilde{\nu} + \pi i$. Then $\tilde{\nu} := \cosh^{-1}(\frac{1}{2}\mu - 1) > 0$ and

$$\mathfrak{h}(x) = \log\left(\frac{-e^{\tilde{\nu}} + e^{l(x)}}{-e^{-\tilde{\nu}} + e^{l(x)}}\right). \tag{6.17}$$

Since $e^{l(x)} > 1 > e^{-\tilde{\nu}}$ and $-e^{\tilde{\nu}} + e^{l(x)} < -e^{-\tilde{\nu}} + e^{l(x)}$, we have

$$\begin{aligned}
\mathfrak{h}(x) < 0 \quad &\Longleftrightarrow \quad -e^{\tilde{\nu}} + e^{l(x)} > 0 \\
&\Longleftrightarrow \quad l(x) > \tilde{\nu} \\
&\Longleftrightarrow \quad \cosh^{-1}(\tfrac{1}{2}x^2 - 1) > \cosh^{-1}(\tfrac{1}{2}\mu - 1) \\
&\Longleftrightarrow \quad x^2 > \mu.
\end{aligned}$$

Finally, we need to show that $\mathfrak{h}(x_0)$ has imaginary part πi. By the above argument, this is equivalent to $-e^{\tilde{\nu}} + e^{l(x_0)} < 0$, hence to $x_0^2 < \mu$, which is shown to be true at the beginning of the proof.

Similarly, $\mathfrak{h}(y_0)$ and $\mathfrak{h}(z_0)$ all have imaginary part πi. This finishes the proof of Lemma 6.9. $\square$

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