Comparing the closed almost disjointness and dominating numbers

by

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Abstract. We prove that if there is a dominating family of size $\aleph_1$, then there are $\aleph_1$ many compact subsets of $\omega^\omega$ whose union is a maximal almost disjoint family of functions that is also maximal with respect to infinite partial functions.

1. Introduction. Recall that two infinite subsets $a$ and $b$ of $\omega$ are almost disjoint or a.d. if $a \cap b$ is finite. A family $A$ of infinite subsets of $\omega$ is said to be almost disjoint or a.d. in $[\omega]^\omega$ if its members are pairwise almost disjoint. A Maximal Almost Disjoint family, or MAD family in $[\omega]^\omega$ is an infinite a.d. family in $[\omega]^\omega$ that is not properly contained in a larger a.d. family.

Two functions $f$ and $g$ in $\omega^\omega$ are said to be almost disjoint or a.d. if they agree in only finitely many places. We say that a family $A \subset \omega^\omega$ is a.d. in $\omega^\omega$ if its members are pairwise a.d., and we say that an a.d. family $A \subset \omega^\omega$ is MAD in $\omega^\omega$ if $\forall f \in \omega^\omega \exists h \in A \ |f \cap h| = \aleph_0$. Identifying functions with their graphs, every a.d. family in $\omega^\omega$ is also an a.d. family in $[\omega \times \omega]^\omega$; however, it is never MAD in $[\omega \times \omega]^\omega$ because any function is a.d. from the vertical columns of $\omega \times \omega$. MAD families in $\omega^\omega$ that become MAD in $[\omega \times \omega]^\omega$ when the vertical columns of $\omega \times \omega$ are thrown in were considered by Van Douwen.

We say that $p \subset \omega \times \omega$ is an infinite partial function if it is a function from some infinite set $A \subset \omega$ to $\omega$. An a.d. family $A \subset \omega^\omega$ is said to be Van Douwen if for any infinite partial function $p$ there is $h \in A$ such that $|h \cap p| = \aleph_0$. $A$ is Van Douwen iff $A \cup \{c_n : n \in \omega\}$ is a MAD family in $[\omega \times \omega]^\omega$, where $c_n$ is the $n$th vertical column of $\omega \times \omega$. The first author showed in [3] that Van Douwen MAD families always exist.

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Recall that $b$ is the least size of an unbounded family in $\omega^\omega$, $d$ is the least size of a dominating family in $\omega^\omega$, and $a$ is the least size of a MAD family in $[\omega]^{\omega}$. It is well known that $b \leq a$. Whether $a$ could consistently be larger than $d$ was an open question for a long time, until Shelah achieved a breakthrough in [4] by producing a model where $d = \aleph_2$ and $a = \aleph_3$. However, it is not known whether $a$ can be larger than $d$ when $d = \aleph_1$; this is one of the few major remaining open problems in the theory of cardinal invariants posed during the earliest days of the subject (see [5] and [2]). In this note we take a small step towards resolving this question by showing in this paper provides a partial positive answer to their question.

**Theorem 1.** Assume $d = \aleph_1$. Then there exist $\aleph_1$ compact subsets of $\omega^\omega$ whose union is a Van Douwen MAD family.

The cardinal invariant $a_{\text{closed}}$ was recently introduced and studied by Brendle and Khomskii [1] in connection with the possible descriptive complexities of MAD families in certain forcing extensions of $L$.

**Definition 2.** $a_{\text{closed}}$ is the least $\kappa$ such that there are $\kappa$ closed subsets of $[\omega]^{\omega}$ whose union is a MAD family in $[\omega]^{\omega}$.

Obviously, $a_{\text{closed}} \leq a$. Brendle and Khomskii showed in [1] that $a_{\text{closed}}$ behaves differently from $a$ by producing a model where $a_{\text{closed}} = \aleph_1 < \aleph_2 = b$. They asked whether $s = \aleph_1$ implies that $a_{\text{closed}} = \aleph_1$. As $s \leq d$, our result in this paper provides a partial positive answer to their question.

2. The construction. Assume $d = \aleph_1$ in this section. We will build $\aleph_1$ compact subsets of $\omega^\omega$ whose union is a Van Douwen MAD family. To this end, we will construct a sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ of finitely branching subtrees of $\omega^{<\omega}$ such that $\bigcup_{\alpha < \omega_1} [T_\alpha]$ has the required properties. Henceforth, $T \subset \omega^{<\omega}$ will mean $T$ is a subtree of $\omega^{<\omega}$.

**Definition 3.** Let $T \subset \omega^{<\omega}$. Let $A \in [\omega]^{\omega}$ and $p : A \to \omega$. For any ordinal $\xi$ and $\sigma \in T$ define $\text{rk}_{T,p}(\sigma) \geq \xi$ to mean

$$\forall \zeta < \xi \exists \tau \in T \exists l \in A \ [\tau \supset \sigma \land |\sigma| \leq l < |\tau| \land \tau(l) = p(l) \land \text{rk}_{T,p}(\tau) \geq \zeta].$$

Note that if $\eta \leq \xi$ and $\text{rk}_{T,p}(\sigma) \geq \xi$, then $\text{rk}_{T,p}(\sigma) \geq \eta$, and that for a limit ordinal $\xi$, if $\forall \zeta < \xi \ [\text{rk}_{T,p}(\sigma) \geq \zeta]$, then $\text{rk}_{T,p}(\sigma) \geq \xi$. Also, for any $\sigma, \tau \in T$, if $\sigma \subset \tau$ and $\text{rk}_{T,p}(\tau) \geq \xi$, then $\text{rk}_{T,p}(\sigma) \geq \xi$. Moreover, if $\text{rk}_{T,p}(\sigma) \not\geq \xi$ and if $\tau \in T$ and $l \in A$ are such that $\tau \supset \sigma$, $|\sigma| \leq l < |\tau|$, and $p(l) = \tau(l)$, then there is $\zeta < \xi$ such that $\text{rk}_{T,p}(\tau) \not\geq \zeta$. Therefore, if there is $f \in [T]$ with $|f \cap p| = \aleph_0$, and if $\sigma \subset f$ and there is some ordinal $\xi$ such that $\text{rk}_{T,p}(\sigma) \not\geq \xi$, then there is some $\sigma \subset \tau \subset f$ and some ordinal $\zeta < \xi$ such that $\text{rk}_{T,p}(\tau) \not\geq \zeta$, thus allowing us to construct an infinite, strictly
descending sequence of ordinals. So if \( f \in [T] \) with \(|f \cap p| = \aleph_0\), then for any \( \sigma \in f \) and any ordinal \( \xi \), \( \text{rk}_{T,p}(\sigma) \geq \xi \). On the other hand, suppose that \( \sigma \in T \) with \( \text{rk}_{T,p}(\sigma) \geq \omega_1 \). Then there is \( \tau \in T \) with \( \tau \supseteq \sigma \) and \( l \in A \) such that \(|\sigma| \leq l < |\tau|\), \( p(l) = \tau(l) \), and \( \text{rk}_{T,p}(\tau) \geq \omega_1 \), allowing us to construct \( f \in [T] \) with \( \sigma \subset f \) such that \(|f \cap p| = \aleph_0\).

**Definition 4.** Suppose \( T \subset \omega^{<\omega} \), \( A \in [\omega]^{\omega} \), and \( p : A \to \omega \). Assume that \( p \) is a.d. from each \( f \in [T] \). Then define \( H_{T,p} : T \to \omega_1 \) by

\[
H_{T,p}(\sigma) = \min\{\xi : \text{rk}_{T,p}(\sigma) \not\geq \xi + 1\}.
\]

Note the following features of this definition:

1. \( \forall \sigma, \tau \in T \ [\sigma \subset \tau \Rightarrow H_{T,p}(\sigma) \geq H_{T,p}(\tau)] \).
2. For all \( \sigma, \tau \in T \) with \( \sigma \subset \tau \), if there exists \( l \in A \) such that \(|\sigma| \leq l < |\tau|\) and \( p(l) = \tau(l) \), then \( H_{T,p}(\tau) < H_{T,p}(\sigma) \).

On the other hand, notice that if there is a function \( H : T \to \omega_1 \) such that \((*)_1\) and \((*)_2\) hold with \( H_{T,p} \) replaced with \( H \), then \( p \) must be a.d. from \([T]\).

**Definition 5.** \( I \) is said to be an interval partition if \( I = \langle i_n : n \in \omega \rangle \), where \( i_0 = 0 \), and \( \forall n \in \omega \ [i_n < i_{n+1}] \). For \( n \in \omega \), \( I_n \) denotes the interval \([i_n, i_{n+1})\).

Given two interval partitions \( I \) and \( J \), we say that \( I \) dominates \( J \) and write \( J \leq^* I \) if \( \forall \infty n \in \omega \ \exists k \in \omega \ [J_k \subset I_n] \).

It is well known that \( \mathfrak{d} \) is also the size of the smallest family of interval partitions dominating any interval partition. So fix a sequence \( \langle I^\alpha : \alpha < \omega_1 \rangle \) of interval partitions such that:

1. \( \forall \alpha \leq \beta < \omega_1 \ [I^\alpha \leq^* I^\beta] \).
2. For any interval partition \( J \), there exists \( \alpha < \omega_1 \) such that \( J \leq^* I^\alpha \).

Fix an \( \omega_1 \)-scale \( \langle f_\alpha : \alpha < \omega_1 \rangle \) such that \( \forall \alpha < \omega_1 \ \forall n \in \omega \ [f_\alpha(n) < f_\alpha(n+1)] \). For each \( \alpha \geq 1 \), define \( e_\alpha \) and \( g_\alpha \) by induction on \( \alpha \) as follows. If \( \alpha \) is a successor, then \( e_\alpha : \omega \to \alpha \) is any onto function, and \( g_\alpha = f_\alpha \). If \( \alpha \) is a limit, then let \( \{e_n : n \in \omega\} \) enumerate \( \{e_\xi : \xi < \alpha\} \). Now, define \( e_\alpha : \omega \to \alpha \) and \( g_\alpha \in \omega^\omega \) such that

1. \( \forall n \in \omega \ [g_\alpha(n) \leq g_\alpha(n+1)] \).
2. \( \forall n \in \omega \ \forall i \leq n \ \forall j \leq f_\alpha(n) \ \exists k < g_\alpha(n) \ [e_\alpha(k) = e_i(j)] \).

Observe that such an \( e_\alpha \) must be a surjection. For each \( n \in \omega \), put

\[
w_\alpha(n) = \{e_\alpha(i) : i \leq g_\alpha(n)\}.
\]

Now fix \( \alpha < \omega_1 \) and assume that \( T_\epsilon \subset \omega^{<\omega} \) has been defined for each \( \epsilon < \alpha \) such that each \( T_\epsilon \) is finitely branching and \( \bigcup_{\epsilon < \alpha} T_\epsilon \) is an a.d. family in \( \omega^\omega \). Let \( \langle \epsilon_n : n \in \omega \rangle \) enumerate \( \alpha \), possibly with repetitions. For a tree
$T \subset \omega^{<\omega}$ and $l \in \omega$, write

$$T|l = \{ \sigma \in T : |\sigma| \leq l \} \quad \text{and} \quad T(l) = \{ \sigma \in T : |\sigma| = l \}.$$  

We will define a sequence of natural numbers $0 = l_0 < l_1 < \cdots$ and determine $T_\alpha|l_n$ by induction on $n$. First, $T_\alpha|l_0 = \{0\}$. Assume that $l_n$ and $T_\alpha|l_n$ are given. Suppose also that we are given a sequence of natural numbers $\langle k_i : i < n \rangle$ such that

1. $\forall i < j < n \ [k_i < k_{i+1}]$.
2. $I^\alpha_{k_i} \subset [0, l_n)$. 

Let $\sigma^*$ denote the member of $T_\alpha(l_n)$ that is rightmost with respect to the lexicographical ordering on $\omega^{l_n}$. Suppose we are also given $L_n : T_\alpha(l_n) \setminus \{\sigma^*\} \to W_n$, an injection. Here $W_n$ is the set of all pairs $\langle p_0, \bar{h} \rangle$ such that:

3. There are $s \in [\omega]^{<\omega}$, and numbers $i_0 < j_0 \leq n$ such that
   a. $s \subset \bigcup_{i \in [i_0,j_0)} I^\alpha_{k_i}$,
   b. for each $i \in [i_0,j_0)$, $|s \cap I^\alpha_{k_i}| = 1$,
   c. $p_0 : s \to \omega$ such that $\forall m \in s \ [p_0(m) \leq f_\alpha(m)]$.

4. There is $j_1 < n$ such that $\bar{h} = \langle h_{\epsilon_i} : i \leq j_1 \rangle$ (if $\alpha = 0$, this means that $\bar{h} = 0$). For each $i \leq j_1$, $h_{\epsilon_i} : T_{\epsilon_i} \upharpoonright \text{max}(s) + 1 \to w_\alpha(\text{max}(s) + 1)$ such that $(*)_1$ and $(*)_2$ hold with $T$ replaced with $T_{\epsilon_i} \upharpoonright \text{max}(s) + 1$, $H_{T,p}$ replaced with $h_{\epsilon_i}$, $A$ with $s$, and $p$ with $p_0$.

Assume that for each $i < n$, we are also given $\sigma_i \in T_\alpha(l_i)$, which we will call the active node at stage $i$. Note that $T_\alpha(l_0) = \{0\}$, and so $\sigma_0 = 0$. For each $\sigma \in T_\alpha(l_n)$, let $\Delta(\sigma) = \text{max}(\{0 \cup \{ i < n : \sigma_i = \sigma|l_i \})$. For, $\sigma, \tau \in T_\alpha(l_n)$, say $\sigma < \tau$ if either $\Delta(\sigma) < \Delta(\tau)$, or $\Delta(\sigma) = \Delta(\tau)$ and $\sigma$ is to the left of $\tau$ in the lexicographic ordering on $\omega^{l_n}$. Let $\sigma_n$ be the $\prec$-minimal member of $T_\alpha(l_n)$. Then $\sigma_n$ will be active at stage $n$. The meaning of this is that none of the other nodes in $T_\alpha(l_n)$ will be allowed to branch at stage $n$. Choose $k_n$ greater than all $k_i$ for $i < n$ such that $I^\alpha_{k_n} \subset [l_n, \infty)$. Let $V_n$ be the set of all pairs $\langle p_1, \bar{h} \rangle$ such that:

5. There exist $s$ and a natural number $i_1 \leq n$ such that
   a. $s \subset \bigcup_{i \in [i_1,n+1)} I^\alpha_{k_i}$,
   b. for each $i \in [i_1,n+1)$, $|s \cap I^\alpha_{k_i}| = 1$,
   c. $p_1 : s \to \omega$ such that $\forall m \in s \ [p_1(m) \leq f_\alpha(m)]$.

6. There is $j_2 \leq n$ such that $\bar{h} = \langle h_{\epsilon_i} : i \leq j_2 \rangle$. For each $i \leq j_2$, $h_{\epsilon_i} : T_{\epsilon_i} \upharpoonright \text{max}(s) + 1 \to w_\alpha(\text{max}(s) + 1)$, such that $(*)_1$ and $(*)_2$ are satisfied with $T$ replaced with $T_{\epsilon_i} \upharpoonright \text{max}(s) + 1$, $H_{T,p}$ replaced with $h_{\epsilon_i}$, $A$ with $s$, and $p$ with $p_1$.

Note that $V_n$ is always finite. Now, the construction splits into two cases.
CASE I: $\sigma_n \neq \sigma^*$. Put $\langle p_0, \bar{h} \rangle = L_n(\sigma_n)$. Let $i_0 < n$ be as in (7) above, and let $j_1 < n$ be as in (8). Let

$$U_n = \{ \langle p_1, \bar{h} \rangle \in V_n : p_0 \subset p_1 \land i_0 = i_1 \land j_1 < j_2 \land \forall i \leq j_1 [h_{e_i} \subseteq \text{dom}(h_{e_i}) = h_{e_i}] \}.$$ 

Here $i_1$ is as in (9), and $j_2$ is as in (10) with respect to $\langle p_1, \bar{h} \rangle$. Now choose $l_{n+1} > l_n$ large enough so that $I_{kn}^\alpha \subseteq [l_n, l_{n+1})$ and so that it is possible to pick $\{\tau_x : x \in U_n \} \subset \omega^{|l_{n+1}|}$ and $\{\tau_\sigma : \sigma \in T_\alpha(l_n) \} \subset \omega^{|l_{n+1}|}$ such that the following conditions are satisfied:

1. For each $x \in U_n$, $\tau_x \supset \sigma_n$, and for each $\sigma \in T_\alpha(l_n)$, $\tau_x \supset \sigma$.
2. For each $x, y \in U_n$, if $x \neq y$, then there exists $m \in [l_n, l_{n+1})$ such that $\tau_x(m) \neq \tau_y(m)$. For each $x \in U_n$, there exists $m \in [l_n, l_{n+1})$ such that $\tau_x(m) \neq \tau_{\sigma_n}(m)$. For $x = \langle p_1, \bar{h} \rangle \in U_n$, if $\{i^* \} = \text{dom}(p_1) \cap I_{kn}^\alpha$, then $p_1(i^*) = \tau_x(i^*)$.
3. For each $x \in U_n$ and $\sigma \in T_\alpha(l_n)$, if $\sigma \neq \eta$, then $\forall m \in [l_n, l_{n+1}) [\tau_\sigma(m) \neq \tau_\eta(m)]$.
4. For each $i \leq n$, $\tau \in T_\epsilon(l_{n+1})$, $\sigma \in T_\alpha(l_n)$ and $m \in [l_n, l_{n+1})$, $\tau(m) \neq \tau_\sigma(m)$. For each $x \in U_n$, $i \leq j_2$, $\tau \in T_\epsilon(l_{n+1})$ and $m \in [l_n, l_{n+1})$, if $\tau_x(m) = \tau(m)$, then $m \in \text{dom}(p_1)$ and $p_1(m) = \tau_x(m)$.

Define $L_{n+1}$ as follows. For any $x \in U_n$, $L_{n+1}(\tau_x) = x$. For any $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$, $L_{n+1}(\sigma_n) = L_n(\sigma)$. This finishes Case I.

CASE II: $\sigma_n = \sigma^*$. For each $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$, let $\langle p_0(\sigma), \bar{h}(\sigma) \rangle = L_n(\sigma)$. Let $i_0(\sigma) < n$ witness (7) for $L_n(\sigma)$ and let $j_1(\sigma) < n$ witness (8) for $L_n(\sigma)$. Let $U_n$ be the set of all $\langle p_1, \bar{h} \rangle \in V_n$ such that there is no $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$ so that

$$p_0(\sigma) \subset p_1 \land i_0(\sigma) = i_1 \land j_1(\sigma) < j_2 \land \forall i \leq j_1(\sigma) [h_{e_i} \subseteq \text{dom}(h_{e_i}) = h_{e_i}].$$ 

Here $i_1 \leq n$ and $j_2 \leq n$ witness (9) and (10) respectively with respect to $\langle p_1, \bar{h} \rangle$. Choose $l_{n+1} > l_n$ large enough so that $I_{kn}^\alpha \subseteq [l_n, l_{n+1})$ and so that it is possible to choose $\{\tau^*\}$, $\{\tau_x : x \in U_n\}$, and $\{\tau_\sigma : \sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}\}$, subsets of $\omega^{|l_{n+1}|}$, satisfying the following conditions:

15. $\tau^* \supset \sigma_n$. For each $x \in U_n$, $\tau_x \supset \sigma_n$. For each $\sigma \in T_\alpha(l_{n+1}) \setminus \{\sigma_n\}$, $\tau_\sigma \supset \sigma$.
16. $\tau^*$ is the right most branch of $T_\alpha(l_{n+1})$. For each $x \in U_n$, there exists $m \in [l_n, l_{n+1})$ such that $\tau^*(m) \neq \tau_x(m)$. For each $x, y \in U_n$, if $x \neq y$, then there is $m \in [l_n, l_{n+1})$ so that $\tau_x(m) \neq \tau_y(m)$. For each $x = \langle p_1, \bar{h} \rangle \in U_n$, if $\{i^* \} = I_{kn}^\alpha \cap \text{dom}(p_1)$, then $p_1(i^*) = \tau_x(i^*)$.
17. For each $x \in U_n$ and $m \in [l_n, l_{n+1})$, $\tau_x(m) \neq \tau^*(m)$. For each $\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}$ and for each $m \in [l_n, l_{n+1})$, $\tau^*(m) \neq \tau_\sigma(m)$, and for each $x \in U_n$, $\tau_\sigma(m) \neq \tau_x(m)$. For each $\sigma, \eta \in T_\alpha(l_n) \setminus \{\sigma_n\}$, if $\sigma \neq \eta$, then for all $m \in [l_n, l_{n+1})$, $\tau_\sigma(m) \neq \tau_\eta(m)$. 


(18) For each \(i \leq n, \tau \in T_{e_i}(l_{n+1}), m \in [l_n, l_{n+1}), \) and \(\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}, \tau^*(m) \neq \tau(m) \) and \(\tau_\sigma(m) \neq \tau(m)\). For each \(x = \langle p_1, \vec{h} \rangle \in U_n, i \leq j_2, \tau \in \tau_{e_i}(l_{n+1})\) and \(m \in [l_n, l_{n+1})\), if \(\tau_x(m) = \tau(m)\), then \(m \in \text{dom}(p_1)\) and \(p_1(m) = \tau_x(m)\).

For each \(\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}\), define \(L_{n+1}(\tau_\sigma) = L_n(\sigma)\). For each \(x \in U_n\), set \(L_{n+1}(\tau_x) = x\). This completes the construction. We now check that it is as required.

**Lemma 6.** For each \(f \in [T_\alpha]\), there are infinitely many \(n \in \omega\) such that \(\sigma_n = f|l_n\).

**Proof.** For each \(n \in \omega\) put \(\Theta(n) = \min \{\Delta(\sigma) : \sigma \in T_\alpha(l_n)\}\). It is clear from the construction that \(\Theta(n + 1) \geq \Theta(n)\). If the lemma fails, then there are \(m\) and \(\tau \in T_\alpha(l_{m+1})\) with the property that for infinitely many \(n > m+1\), there is a \(\sigma \in T_\alpha(l_n)\) such that \(\Theta(n) = \Delta(\sigma) = m\) and \(\sigma|l_{m+1} = \tau\). Let \(\tau\) be the leftmost node in \(T_\alpha(l_{m+1})\) with this property. Choose \(n_1 > n_0 > m + 1\) and \(\sigma \in T_\alpha(l_{n_1})\) such that \(\Theta(n_1) = \Theta(n_0) = \Delta(\sigma) = m\), \(\sigma|l_{m+1} = \tau\), and there is no \(\eta \in T_\alpha(l_{n_0})\) such that \(\Delta(\eta) = m\) and \(\eta|l_{m+1}\) is to the left of \(\tau\). Note that \(\Delta(\sigma|l_{n_0}) = m\). So \(\sigma_{n_0}\) is to the left of \(\sigma|l_{n_0}\), and \(\sigma_{n_0}|l_{m+1}\) is not to the left of \(\tau\), whence \(\sigma_{n_0}|l_{m+1} = \tau\). But then there is some \(n \in [m+1, n_0)\) where \(\sigma|l_n\) was active, a contradiction. ■

Note that Lemma 6 implies that for any \(\sigma \in T_\alpha\), there is a unique minimal extension of \(\sigma\) which is active. Lemma 6 also implies that there are infinitely many \(n\) where Case II occurs.

**Lemma 7.** \(T_\alpha\) is finitely branching and \(\bigcup_{\epsilon \leq \alpha} [T_\epsilon]\) is a.d. in \(\omega^\omega\).

**Proof.** It is clear from the construction that \(T_\alpha\) is finitely branching. Fix \(f, g \in [T_\alpha]\) with \(f \neq g\). Let \(n = \max\{i \in \omega : f|l_i = g|l_i\}\). It is clear from the construction that \(\forall m \geq l_{n+1}[f(m) \neq g(m)]\).

Next, fix \(\epsilon < \alpha\). Suppose \(\epsilon = \epsilon_i\). Let \(h \in [T_{e_i}]\) and \(f \in [T_\alpha]\), and suppose for a contradiction that \(|h \cap f| = \aleph_0\). So there are infinitely many \(n \in \omega\) such that \(f|l_n\) and \(f|l_{n+1} = \tau_x\) for some \(x \in U_n\). This is because if \(n \geq i\) and \(f|l_n = \sigma_n\) and \(f|l_{n+1} = \tau_x\) for some \(x \in U_n\), then when Case I occurs, (14) says that \(f|l_{n+1} = \tau_x\) for any \(\sigma \in T_\alpha(l_n)\), while when Case II occurs, by (18), \(f|l_{n+1} \neq \tau^*\) and also \(f|l_{n+1} \neq \tau_\sigma\) for any \(\sigma \in T_\alpha(l_n) \setminus \{\sigma_n\}\). So \(f|l_{n+1} = \tau_x\) for some \(x \in U_n\), and \(f|l_n = \sigma_n\). Now, put \(x = \langle p_{1,n}, h_n \rangle\). Note that in this case \(L_{n+1}(f|l_{n+1}) = x\). For such \(n\), let \(j_2(n)\) be as in (10) with respect to \(x_n\). So for infinitely many such \(n\), \(j_2(n) \geq i\). But then for infinitely many such \(n\), \(h_{e_i,n}(h_\max(\text{dom}(p_{1,n})) + 1) < h_{e_i,n}(h|l_n)\), producing an infinite strictly descending sequence of ordinals. ■
LEMMA 8. For each $A \in [\omega]^\omega$ and $p : A \rightarrow \omega$, there are $\alpha < \omega_1$ and $f \in [T_\alpha]$ such that $|p \cap f| = \aleph_0$.

Proof. Suppose for a contradiction that there are $A \in [\omega]^\omega$ and $p : A \rightarrow \omega$ such that $p$ is a.d. from $[T_\alpha]$, for each $\alpha < \omega_1$. Let $M < H(\theta)$ be a countable elementary submodel containing everything relevant. Let $\alpha = M \cap \omega_1$. For each $\epsilon < \alpha$, let $H_\epsilon$ denote $H_{\epsilon,p}$, and note that $H_\epsilon$ and $\text{ran}(H_\epsilon)$ are members of $M$. Let $\xi_\epsilon = \sup(\text{ran}(H_\epsilon)) + 1 < \alpha$. Find $g \in M \cap \omega^\omega$ such that for $n \in \omega$, $H_\epsilon^\omega T_\epsilon \upharpoonright n \subset \{e_{\xi_\epsilon}(j) : j \leq g(n)\}$. Since $\forall \omega n \in \omega \ [g(n) \leq f_\alpha(n)]$, it follows from (4) that for all but finitely many $n \in \omega$, and all $\sigma \in T_\epsilon \upharpoonright n$, $H_\epsilon(\sigma) \in w_\alpha(n)$. Now, find an infinite $q \subseteq p$ such that $\forall m \in \text{dom}(q) \ [q(m) \leq f_\alpha(m)]$ and $\forall \omega n \in \omega \ [\text{dom}(q) \cap I^\alpha_n = 1]$. Note that for any $\epsilon < \alpha$, $(\ast_1)$ and $(\ast_2)$ are satisfied with $T$ replaced with $T_\epsilon$, $H_{\epsilon,p}$ replaced with $H_\epsilon$, $A$ with $\text{dom}(q)$, and $p$ with $q$. But now, it follows from the construction that there is $f \in [T_\alpha]$ such that for infinitely many $n \in \omega$, there is $m \in [l_n, l_{n+1}] \cap \text{dom}(q)$ such that $q(m) = f(m)$.

We describe how to find such an $f \in [T_\alpha]$. We have $\forall \omega n \in \omega \ [\text{dom}(q) \cap I^\alpha_n = 1]$. For each $n \in \omega$ such that $|\text{dom}(q) \cap I^\alpha_n| = 1$, let $m_n$ be the unique member of $\text{dom}(q) \cap I^\alpha_n$. We observed above that for any $\epsilon < \alpha$, for all but finitely many $n \in \omega$, and each $\sigma \in T_\epsilon \upharpoonright n$, $H_\epsilon(\sigma) \in w_\alpha(n)$. It follows that for any $i \in \omega$, there is $u_i \geq i$ such that for each $j < i$ and each $n \geq u_i$, $m_n$ is defined and $\forall \sigma \in T_\epsilon \upharpoonright m_n + 1 \ [H_\epsilon(\sigma) \in w_\alpha(m_n + 1)]$. Choose $n^* \geq u_0$ so that Case II occurs at stage $n^*$. Let $\eta_0 = \sigma_{n^*}$. Define $s_0 = \{m_{n^*}\}$ and $q_0 = q|s_0$. Put $h_0 = \langle h_0 \rangle$, where $h_0 = H_{c_0}(\text{max}(s_0) + 1)$. Note that $h_0$ is a map from $T_{c_0} \upharpoonright \text{max}(s_0) + 1$ to $w_\alpha(\text{max}(s_0) + 1)$, and so $x_0 = \langle q_0, h_0 \rangle \in V_{n^*}$. Since $m_{n^*} \notin I^\alpha_{k_i}$ for any $i < n^*$, it follows that $x_0 \in U_{n^*}$. Let $\eta_1 = \tau_{x_0} \supseteq \eta_0$. Notice that $\eta_1(m_{n^*}) = q(m_{n^*})$. Notice also that $\eta_1$ is not the rightmost branch of $T_\alpha(l_{(n^* + 1)})$, and so if $\sigma$ is any extension of $\eta_1$ that happens to be active at a certain stage, then Case I necessarily occurs at that stage. Finally, note that $L_{n^* + 1}(\eta_1) = x_0$. Now, for each $n > n^*$, let $s_n = \{m_j : n^* \leq j \leq n\}$, and put $q_n = q|s_n$. For any $i > 0$ and $n > n^*$, if $n \geq u_i$, then for each $j \leq i$, define $h^n_j = H_{c_j}(\text{max}(s_n) + 1)$. Put $\bar{h}^n_i = \langle h^n_j : j \leq i \rangle$ and $x^n_i = \langle q_n, \bar{h}^n_i \rangle$. Note that for any $i > 0$ and $n > n^*$, if $n \geq u_i$, then $x^n_i \in V_n$. Moreover, if at stage $n$, Case I occurs and $L_n(\sigma_n) = x^p_{i-1}$ for some $v \in \omega$, then $x^n_i \in V_n$; here $x^n_0 = x_0$ for all $v \in \omega$. Now, it is easy to see that there is a branch $g \in [T_\alpha]$ such that $\eta_1 \subset g$ and $\forall n \geq n^* + 1 \ [L_n(g)|l_n = x_0]$. This is because for any $n > n^* + 1$, given $g|l_n$ such that $\eta_1 \subset g|l_n$ and $L_n(g)|l_n = x_0$, if $\sigma$ is the unique minimal extension of $g|l_n$ that is active, then $\tau_\sigma \supset g|l_n$ and $L_{u+1}(\tau_\sigma) = x_0$, where $u$ is the stage at which $\sigma$ is active. Applying Lemma 6 to $g$, find $n^{**}$ such that $n^{**} > n^*, n^{**} \geq u_1$, and $\sigma_{n^{**}} = g|l_{n^{**}}$. It follows that $x^{n^{**}}_1 \in U_{n^{**}}$. Let $\eta_2 = \tau_{x^{n^{**}}_1} \supseteq \eta_1$. Note that $\eta_2(m_{n^{**}}) = q(m_{n^{**}})$ and
that $L^*_n+n^*_1(\eta_2) = x^*_1$. Continuing in this fashion, we get
\[ f = \bigcup_{n \in \omega} \eta_n \in [T_\alpha] \quad \text{with} \quad |f \cap q| = \omega. \]

3. Remarks and questions. The construction in this paper is very specific to $\omega_1$; indeed, it is possible to show that $d$ is not always an upper bound for $a_{\text{closed}}$. A modification of the methods of Section 4 of [4] shows that if $\kappa$ is a measurable cardinal and if
\[ \lambda = \text{cf}(\lambda) = \lambda^\kappa > \mu = \text{cf}(\mu) > \kappa, \]
then there is a c.c.c. poset $P$ such that $|P| = \lambda$, and $P$ forces that $b = d = \mu$ and $a = a_{\text{closed}} = c = \lambda$.

As mentioned in Section [1], we see the result in this paper as providing a weak positive answer to the following basic question, which has remained open for long.

**Question 9.** If $d = \aleph_1$, then is $a = \aleph_1$?

There are also several open questions about upper and lower bounds for $a_{\text{closed}}$.

**Question 10** (Brendle and Khomskii [1]). If $s = \aleph_1$, then is $a_{\text{closed}} = \aleph_1$?

**Question 11.** Is $h \leq a_{\text{closed}}$?

Regarding Question [10], it is proved in Brendle and Khomskii [1] that if $V$ is any ground model satisfying CH, then any finite support iteration of Suslin c.c.c. posets in $V$ forces that $a_{\text{closed}} = \aleph_1$. It is well known that $V$ remains a splitting family after such a finite support iteration of Suslin c.c.c. posets. Showing a positive answer to Question [10] would be an improvement of the result in this paper.

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