Ordering Trees via Immanants - An Update

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Summary

Let n be a positive integer and λ be a partition of n. For an $n \times n$ complex matrix $A = (a_{ij})$, the immanant of A with respect to λ, denoted by $d_\lambda(A)$, is defined by

$$d_\lambda(A) := \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n denotes the permutation group of $\{1, 2, ..., n\}$ and χ_λ denotes the irreducible character of S_n associated with the partition λ. Familiar examples of immanants include the determinant and permanent, which correspond to the alternating character $\chi_{(1^n)}$ and the trivial character $\chi_{(n)}$ respectively. The normalized immanant $\overline{d}_\lambda(A)$ is defined by $\overline{d}_\lambda(A) := d_\lambda(A)/\chi_\lambda(\mathrm{id})$, where id denotes the identity permutation in S_n.

Let G be a graph on n vertices with vertex set $\{v_1, v_2, ..., v_n\}$. Its Laplacian matrix $L(G) = (l_{ij})$ is defined by

$$l_{ij} = \begin{cases} \deg_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j, \text{ and } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

We define an equivalence relation $\sim$ on the set of trees with n vertices, which we denote by $\mathcal{T}_n$, as follows:

$$T_1 \sim T_2 \;\Leftrightarrow\; \overline{d}_\lambda(L(T_1)) = \overline{d}_\lambda(L(T_2)) \;\;\forall\;\text{ immanants } d_\lambda.$$

A partial order $\leq$ on the equivalence classes $\mathcal{T}_n/\sim$ is defined by

$$T_1 \leq T_2 \;\Leftrightarrow\; \overline{d}_\lambda(L(T_1)) \leq \overline{d}_\lambda(L(T_2)) \;\;\forall\;\text{ immanants } d_\lambda.$$

(Note that there exist nonisomorphic trees on n vertices whose Laplacian matrix share the same immanant values for all immanants - see for example [9] - and so the partial order $\leq$ cannot be defined on $\mathcal{T}_n$ itself.) In this project when we speak of a tree $T \in \mathcal{T}_n$, we identify the tree T as a representative of the equivalence class of $\mathcal{T}_n/\sim$ to which it belongs. If for two trees T and T', we have $T \leq T'$, we say that tree T' *dominates* T.

It was shown in [9] that for a fixed positive integer n, we have the ordering

$$S(n) \leq T \leq P(n),$$

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where $S(n)$ and $P(n)$ denote the star and path on n vertices respectively. In this project, we sharpen this result by proving that for all immanants d_λ other than those associated with the partitions $\lambda = (1^n), (2, 1^{n-2})$, we have the strict inequality

$$d_\lambda(L(S(n))) < d_\lambda(L(T)) < d_\lambda(L(P(n))). \tag{1}$$

The key steps in proving this relation are as follows:

For a tree T, we define $a_T(j)$ to be the number of j-vertex orientations that can be assigned to the tree T. It is shown that we can express $d_\lambda(L(T))$ is terms of the $a_T(j)$'s as follows:

$$d_\lambda(L(T)) = \sum_{i=1}^{[n/2]} a_T(i) \sum_{j=0}^{i} \chi_\lambda(j) \binom{i}{j}. \tag{2}$$

The next important result is that the binomial sums of characters

$$\sum_{j=0}^{i} \chi_\lambda(j) \binom{i}{j} \geq 0 \tag{3}$$

whenever λ is a partition of n and $n \geq 2i$. An important corollary of this result is:

Let χ_λ be an irreducible character of S_m, where $m \geq 4$. Then

$$\chi_\lambda(0) + 2\chi_\lambda(1) + \chi_\lambda(2) \geq 0, \tag{4}$$

equality holding if and only if $\lambda = (1^m)$ or $(2, 1^{m-2})$.

Therefore, if for two trees T and T' on n vertices, we have $a_T(j) \leq a_{T'}(j)$ for all $1 \leq j \leq [n/2]$, then $T \leq T'$.

To prove equation (1), it suffices from (2), (3) and (4) to prove that $a_{S(n)}(j) \leq a_T(j) \leq a_{P(n)}(j)$ for all j, and $a_{S(n)}(2) < a_T(2) < a_{P(n)}(2)$. The result for stars $S(n)$ is easy since $S(n)$ has no j-vertex orientations for all $j \geq 2$, and a tree T has no 2-vertex orientations if and only if $T = S(n)$. The result for paths $P(n)$ follows by using these results:

1. If, in a tree T, there exist paths $\{w, v_1, v_2, ..., v_r\}$ and $\{w, v_{r+1}, v_{r+2}, ..., v_n\}$ for some $1 \leq r < n$ such that $\deg(v_i) \leq 2$ for every i and $\deg(v_r) = \deg(v_n) = 1$, let us call the tree T' obtained from T by replacing the edge $\{w, v_{r+1}\}$ with the edge $\{v_r, v_{r+1}\}$ a *majorant* of T. Then $a_T(j) \leq a_{T'}(j)$ for all j and $a_T(2) < a_{T'}(2)$.
2. Furthermore, every tree T on n vertices ($n \geq 4$) that is not isomorphic to the path has a majorant.

Let p and q be positive integers with $p \leq q$. Take the star $S(p+1)$ and append $q-1$ leaves to one of the end-vertices. Call this a *double star* and denote it by $D(p, q)$. We have the following result:

For positive integers p and q ($p \leq q$) satisfying $p + q = n$, the double stars $D(p, q)$ are linearly ordered with respect to the ordering $\leq$. Furthermore,

$$D(1,n-1) \leq D(2,n-2) \leq D(3,n-3) \leq ... \leq D([\tfrac{n}{2}],[\tfrac{n+1}{2}]).$$

Let p, q and r be positive integers with $p \leq q$. Take the path $P(r+2)$ and append $p-1$ leaves to one of the end-vertices, and $q-1$ leaves to the other. Call this an *r-broom* and denote it by $B(r;p,q)$. We have the following result:

For positive integers p and q $(p \leq q)$ satisfying $p+q+1=n$, the 1-brooms $B(1;p,q)$ are linearly ordered with respect to the ordering $\leq$. Furthermore,

$$B(1;1,n-2) \leq B(1;2,n-3) \leq B(1;3,n-4) \leq ... \leq B(1;[\tfrac{n-1}{2}],[\tfrac{n}{2}]).$$

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