Transcendental Numbers

The study of π

Lau Wee Lip Jonathan, Dr. Lim Chong Hai
Department of Mathematics, National University of Singapore
Semester II, 2000/2001

June 22, 2001

Abstract

Introduction

The purpose of this project is to study two classes that all complex numbers are classified into, algebraic numbers and transcendental numbers. A number α is said to be *algebraic* if it is a root of the polynomial

$$f(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad f(x) \not\equiv 0$$

with rational coefficients. To be more precise, a complex number α is said to be *algebraic of degree* n if it is the root of a polynomial of degree n over $\mathbb{Q}$. A complex number α (which maybe a real number) that does not satisfy the above prerequisite is said to be *transcendental.* In other words, a transcendental number is not a root of any polynomial with algebraic cofficients which is not identicially zero. One particular point to note is that the class of transcendental real numbers are irrational. This is because given any $\alpha \in \mathbb{Q}$, where $\alpha = p/q$, $(p, q) \in \mathbb{Z} \times \mathbb{N}$, it satisfies the polynomial

$$f(x) = x - \frac{p}{q}, \qquad f(x) \in \mathbb{Q}[x].$$

This gives us the first important criterion to determine if a number is algebraic or transcendental. However, this does not mean that all irrational numbers are transcendental. This is a sufficient but not necessary condition. For example, the number $\sqrt{2}$ is an algebraic number of degree 2, satisfying the function, $f(x) = x - 2$. The main crux of this report is studying methods of approximation of real numbers by real numbers finally leading to the approximation of algebraic numbers by algebraic numbers. This gives us the criterion in which a number τ must satisfy before deeming it as algebraic or transcendental. In fact Liouville (1809 - 1882) gave a criterion that any algebraic number of degree n must satisfy. It can be shown that the criterion limits the extent to which a real algebraic number of degree n can be approximated by rational numbers. The project goes on further to prove the the transcendence of π. We shall employ the method attributed to Hermite which was further developed by Lindemann in 1882 to prove the transcendence of π. In the process, Lindemann also provided basis for the proof of the impossibility of squaring the circle. This will also be touch upon in the project.

Approximation of algebraic numbers

We approach the problem by approximation methods to identify whether a number is rational or irrational. We define *a real number α to have rational approximations of order $\varphi(q)$ if there exist a constant $c > 0$ depending only on α and the function $\varphi(q)$ such that the inequality*

$$0 < |\alpha - \frac{p}{q}| < c\varphi(q)$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. We can show that all irrational numbers have rational approximations of order $1/q^2$. We introduce the concept of *irreducible polynomials* and use it to provide the first rigorous proof, by Liouville in 1844, for the existance of transcendental numbers. He proved: *If α is a root of an irreducible polynomial with rational coefficients of degree n, $n \geq 1$, then there exist a constant $c(\alpha) > 0$ such that the following inequality holds for any rational integers p and q, with $q > 0$ and $p/q \neq \alpha$:*

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^n}.$$

This criterion is of such nature that we can easily construct real numbers that violates it for every $n > 1$. Any such number will have to be transcendental. It also tells us that an algebraic number cannot be approximated "too closely" by rational numbers. By approximating algebraic numbers using algebraic numbers, we can obtain a stronger form of Liouville Theorem, and do multiple approximations for algebraic numbers.

Transcendence of π and Applications

The theorem developed by Liouville is not sufficient to prove the transcendence of π. In order to do so, we need to first prove the irrationality of π and use contradiction to show that π is transcendental. We make use of *Euler's Identity*:

$$e^{i\pi} + 1 = 0$$

to prove this case. This result will be applied to prove the impossibility of *squaring the circle*, or *quadrature of the circle* as it is sometimes called. This is one of the three classical problems in Greek mathematics which were extremely influential in the development of geometry, the other 2 being doubling the cube and trisecting an angle. To approach the problem, we introduce the concept of *constructible numbers* and prove that they form a subfield in $\mathbb{R}$. By showing that all constructibles are algebraic numbers, we prove the result.

References

[1] Andrei Borisovich SHIDLOVSKII, *Transcendental Numbers*, Walter de Gruyter, Berlin

[2] Robert G. BARTLE & Donald R. SHERBERT, *Introduction to Real Analysis 3rd Edition*, John Wiley & Sons, Inc.

[3] I. NIVEN, *"A Simple Proof That π is Irrational"*, Bulletin of American Mathematical Society, vol. 53 (1947), p.509

[4] I.N. HERSTIEN, *Abstract Algebra 3rd Edition*, John Wiley & Sons, Inc.

[5] David M BURTON, *Elementary Number Theory, 3rd Edition*, Wm. C.Brown Publishers

[6] *Squaring a circle*, webpage, http://www.cut-the-knot.com/impossible/sq_circle.html

[7] *Squaring the circle*, webpage,
http://www-groups.dcs.st-andrews.ac.uk/~history/HistTopics/Squaring_the_circle.html

[8] *A history of* π, webpage,
http://www-groups.dcs.st-andrews.ac.uk/~history/HistTopics/Pi_through_the_ages.html

[9] *A chronology of* π, webpage,
http://www-groups.dcs.st-andrews.ac.uk/~history/HistTopics/Pi_chronology.html