INVESTIGATING SYLOW 2-SUBGROUPS OF SYMMETRIC AND ALTERNATING GROUPS

Kok Yik Siong

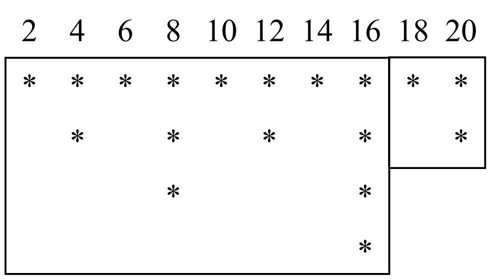
Abstract. In this project, we search for an efficient way of constructing the Sylow 2-subgroups of symmetric and alternating groups. From this, we observe the various invariants of these subgroups in search of a sufficient condition for Sylow 2-subgroups.

A central problem of group theory is the classification of all possible groups. This has been achieved recently, over a period of more than a hundred years. One of the new questions that group theorists have set themselves for this new century is to understand the Sylow 2-subgroups of finite simple groups.

Though only the Sylow 2-subgroups of symmetric and alternating groups are examined, the techniques and arguments will be similar to that of the classical groups. In essence, the process of constructing Sylow 2-subgroups will be the same, starting with a small group as the basic building block, and defining extensions upon these and copies of the same block.

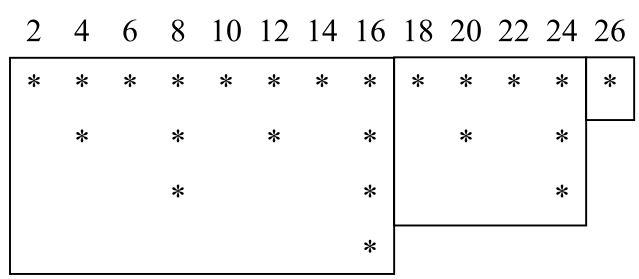
To determine the possible order of the Sylow 2-subgroups of S_n is equivalent to finding the highest power of 2 that divides the order of S_n, which is n!. Thus a subgroup P of S_n can be a Sylow-2 subgroup of S_n only if its order is given by 2^m, where m is given by the expression in Theorem 2.9, for $p = 2$.

$\underline{n = 20}$



$m = 18 = 15 + 3$

$\underline{n = 26}$



$m = 23 = 15 + 7 + 1$

Representing the additional powers of 2 that each even number contributes to m by '*', we notice that in each rectangle, the number of '*' is one less than a power of two. By observing an obvious pattern, it can be shown that there cannot be two rectangles of the same size.

Let $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$, where $t_n = 2^n-1$. Since this is a superincreasing sequence, then if $\sum_{i=1}^{\infty} x_i t_i = m$ has a solution for some m, then we have the following two results.

Theorem 1

$m = \left\lfloor \frac{n}{2^1} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \ldots$ for some n iff $\sum_{i=1}^{\infty} x_i t_i = m$ has a solution.

Corollary 1

If a subgroup P of $S_n(A_n)$ is one of its Sylow 2-subgroups, then $o(P) = 2^m(2^{m-1})$, where $\sum_{i=1}^{\infty} x_i t_i = m$ has a solution.

Using the following theorem for permutation groups, we can easily determine if two permutations are conjugate.

Theorem 2

Let $\alpha = (a_1 a_2 a_3 \ldots a_n)$ and β be two permutations from S_n, with $a_1, a_2, a_3, \ldots, a_n$ being the characters that S_n is acting upon. Then $\beta\alpha\beta^{-1} = (\beta(a_1)\beta(a_2)\beta(a_3)\ldots\beta(a_n))$.

Lemma 1

Let H and K be two subgroups of S_n of order c, with $HK = H \times K$. Define $x \in S_n$ s.t. $xHx^{-1} = K$, $x \notin H, K$ and $o(x) = 2$. Consider $h \in H, k \in K$ where $xhx = k$ and $h, k \neq e$. Then

(i) $o(xh) = 2q$

(ii) $xh \neq hx$

(iii) $xh, hx \notin H$ where $o(h) = q = o(k)$ and q is even.

Lemma 2

Let H and K be two subgroups of S_n of order c, with $HK = H \times K$. Suppose $\exists\, x \in S_n$ s.t. $xHx^{-1} = K$ and $o(x) = 2$. Let $B = HKL$ where $L = \langle x \rangle$. Then $HK \lhd B$ and $|B| = |H||K||L| = 2c^2$.

The purpose of the previous two lemmas is to demonstrate some of the properties of the new group when two copies of a group are further extended by group of order 2. These will allow us to define P_n and Q_n by a recursive process.

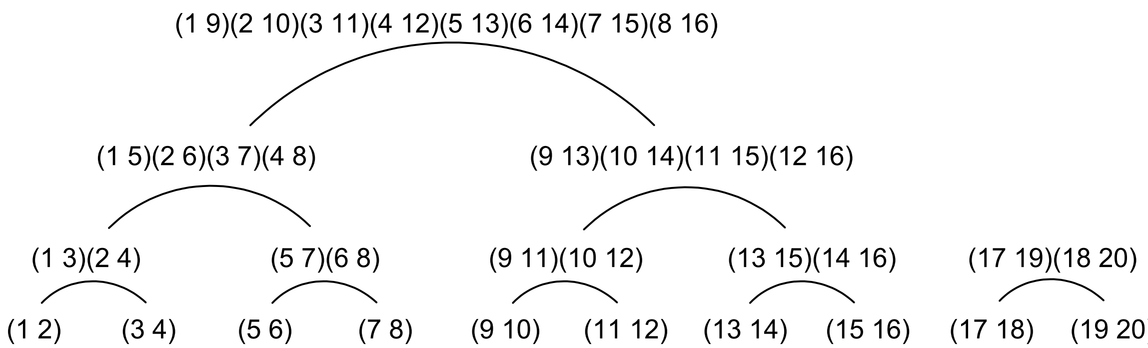


Fig 1 Tree diagram for Sylow 2-subgroup of S_{20} and S_{21}

This diagram illustrates how the group is built up, upon the groups generated by these permutations. By the use of the above lemmas, we can show the following result, which proves that this construction does indeed generate P_n.

Theorem 3

Given $n \in N$, the m permutations determined by the layering process generate a group of order and this would be a Sylow 2-subgroup of S_n.

Construction of Sylow 2-subgroup of A_n

Having seen the construction process for S_n, we hope to adopt a similar approach to do likewise for A_n. Looking at the layers of the Sylow 2-subgroup for S_n from the second upwards, we have seen that they can be rewritten as the Sylow 2-subgroup for the symmetric group of $\left\lfloor \frac{n}{2} \right\rfloor$ characters, namely 1’, 2’, 3’, …, n_1’. We can say that this group (which we denote by H)

acts (by conjugation) on the set {1', 2', 3', …, n_1'}, with the action being closed within the set. We define the first layer as all distinct products arising from an even number of transpositions from {1', 2', 3', …, n_1'}.

Having constructed P_n and Q_n in terms of layers, it is now very convenient to discuss the maximum order of the elements in P_n and Q_n. We have defined r earlier to be the natural number where $2^r \le n < 2^{r+1}$. r is just the number of layers that P_n (or Q_n) has. When we consider the 2^r characters in the same partition, we have at the 1st layer 2^{r-1} transpositions. Inductively, we have $2^{r-r} = 2^0 = 1$ permutation at the r^{th} layer. Hence there are r layers in a partition of size 2^r.

It can be established that the maximum order of elements in P_n is 2^r, where $2^r \le n < 2^{r+1}$. Since for two elements a, b to commute means to have $ba = ab$, multiplying by b^{-1} from the right, we can rewrite this relation as $bab^{-1} = a$. Thus given an element a, looking out for elements that commute is equivalent to looking for elements that fixes a under conjugation.

In the case of P_n, every element can be expressed as a product of the m permutations that were determined by the layering process. Thus any element of S_n commutes with all the elements of P_n if and only if it commutes with these m permutations.

In the first layer, suppose we have a transposition, say (a_1 a_2). Then either (a_1 a_2) is a disjoint cycle of β or both a_1 and a_2 do not appear in the cycle representation of β. For each partition of size 2^i that n is partitioned into, we have 2^{i-1} transpositions at the lowest layer. In the layers above, for each pair of transpositions, we have a combination of permutations that sends one to another. If β commutes with all the elements of P_n, then β must be fixed by all the permutations in the layers of P_n above the first. Thus if (a_1 a_2) is a disjoint cycle of β, then β must contain all the other transpositions in the same partition as (a_1 a_2). Suppose the transpositions in the first layer of P_n are partitioned as $\{\alpha_{11},...,\alpha_{1r}\}$, $\{\alpha_{21},...,\alpha_{2s}\}$,…, $\{\alpha_{k1},...,\alpha_{kt}\}$, then $Z(P_n) = <\alpha_{11}...\alpha_{1r}, \alpha_{21}...\alpha_{2s},...,\alpha_{k1}...\alpha_{kt}>$. Using a similar approach, we have $Z(Q_n) = Z(P_n)$ for $n \equiv 0$ or 1 (mod 4) and $Z(Q_n) = Z(P_{n-1})$ for $n \equiv 2$ or 3 (mod 4). It turns out that for centralisers, it becomes easier, in that $Z(P_n) = C(P_n)$ *and* $C(Q_n) = Z(Q_n)$