

NUS UROPS Abstract Paper
The Study Of Transcendental Numbers and Exponential
Functions

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Abstract

Although the ancient Greeks knew the existence of irrational numbers, the theory of transcendental numbers is only about 150 years old. It was born in 1844 when Liouville established, for the first time, the existence of transcendental numbers. Since then, the advances of the theory, especially around Liouville's and hermite's theorems on linear forms in logarithms, have proved useful in many areas of number theory. This Report will focus on the algebraic and transcendence properties of some values, going into detail how some algebraic numbers are approximated by some other real numbers, for example rational, amongst others. To top it off, we shall look at 2 of the major forms of the transcendental numbers, the usual exponential, e ; and the pi, π . We will sketch the present state of knowledge on this topic and describe some of the tools that are involved in the proofs, addressing out open questions and potential avenues for further progress.

We looked at the different basic directions of research in the theory of transcendental numbers. We gave the following definitions(or theorems),

A rational number is a number that can be expressed in the form of a/b , where a and b are integers with $b > 0$.

A real number is a *rational* number if and only if it can be expressed as a repeating decimal. Of course, some definitions of the *transcendental* numbers are given, Also, by definition, a number which is not the Root of *any* polynomial equation with integer Coefficients, meaning that it's not an Algebraic number of any degree, is said to be *transcendental*.

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This definition guarantees that every transcendental number must also be irrational, since a rational number, by definition, is an algebraic number of degree 1.

Now, we shall give a brief outline on what we had worked on for the whole of this research. We will first discuss the many ways of approximation of different types of numbers; these include

- Approximation of real numbers by algebraic numbers
- Simultaneous approximation
- Approximation of algebraic numbers by rational numbers (using Liouville's theorem)
- Approximation of algebraic numbers by algebraic numbers

In particular, the last section on "Approximation of algebraic numbers by algebraic numbers" will be dedicated to the refinements and generalizations of Liouville's theorem, and 3 of these refinements will be discussed, they are

- Thue's theorem
- Roth's theorem
- Thue-Siegel-Roth's theorem

The next part of the research was conducted on the basic exponential function, e .

Fields of interest that we went into are:

- The different properties of the e -function using number theory.
- Discussion of *Lindemann's theorem*.
- Gelfond-Schneider's theorem (which is, in short, to proving that α^β is also transcendental).

Finally, we looked at the methods used in actually computing transcendental numbers on modern computer systems. The computations used will be applied to the calculation of the Exponential-functions, e^x , which are transcendental.

Transcendental functions can be computed in software by a variety of algorithms. The algorithms that are most suitable for implementation on modern computer architectures usually comprise of three steps: *reduction, approximation, and reconstruction.*