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CALENDARS, INTERPOLATION, GNOMONS AND ARMILLARY SPHERES IN THE WORK OF GUO SHOUJING (1231–1314)

NG SAY TIONG

HELMER ASLAKSEN

Department of Mathematics

National University of Singapore

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Introduction

In today's modern society, when we talk about calendars, almost everyone will associate it with the Gregorian calendar, which is the calendar most commonly used today. The Gregorian calendar is a solar calendar, which follows the movement of the sun. It uses days to approximate the length of the tropical year. A tropical year can be thought of as the time from one spring equinox to the next. A year in the Gregorian calendar will approximate the time the earth takes to complete a revolution around the sun. A year in a Gregorian calendar consists of 365 days (in most cases) divided into 12 months, with each month having the same number of days for every year. However, since the length of a tropical year in days is not a whole number, to prevent the calendar from going "out of tune" with the movement of the sun, the calendar itself has a special feature. Certain years of the calendar have an extra day which is inserted to the second month, and are called *leap years*. The determination of a leap year under the Gregorian calendar follows a simple algorithm: if the year can be divided by 4 and not by 100, or it can be divided by 400, then it will be a leap year. For example, year 1900 is not a leap year, while year 2000 is. In this way the years of the Gregorian calendar can almost keep in tune with the movement of the sun. However, because of its simplicity in its structure, it is a fixed calendar and is hardly interesting.

The same thing cannot be said about the Chinese calendar, the calendar that is commonly used in Chinese society together with the Gregorian calendar. The Chinese has been using their calendars since the time of the Warring State (春秋戰國時代, 770–221 B.C), and unlike the Gregorian calendar, the Chinese calendar itself has a different structure, since it is a lunisolar calendar. In a lunisolar calendar, the month, which we called lunar month, follows the actual movement of the moon. The first day of a new month will be the day when a new moon occurs. Furthermore, the lunisolar calendar use months to approximate the tropical year, and in a lunisolar calendar, a year normally consists of 12 lunar months. However, the movement of the moon is very complex, and therefore the days in every lunar month has to be calculated. So, unlike the case of the Gregorian calendar, the number of days in a certain month of the Chinese calendar is not fixed for every year. Moreover, because the length of 12 lunar months is about 11 days shorter than the length of a year, the Chinese calendar has a special feature called *leap month*, an extra month that is introduced to certain years of the calendar. This is very much like the extra day inserted in the Gregorian calendar. The differences are that a whole month instead of only a day is inserted so that the calendar can have the years keeping in tune with the movement of the sun, and the extra month can be inserted anywhere throughout the year, whereas the extra day in the Gregorian calendar can only be inserted in the second month. Note that there is another kind of calendar, called lunar calendar, which also has the month following the movement of the moon, but does not use months to approximate the tropical year. (To know more about the modern Chinese calendar, refer to [1])

Because the Chinese calendar has always been a lunisolar calendar, a lot of developments can be seen in the calendar throughout the history of China. As new theories developed and more accurate methods of calculations were derived, Chinese astronomers would edit and compile a new calendar in replacement of the previous one. Also, because Chinese Emperors of ancient times needed to know the various astronomical occurrences that were going to happen, so that they could show his people that they were truly "the sons of the heavens" who knew every details of the heavens, the predictions of the various astronomical occurrences, like eclipses, became an important job for the astronomers. They had to come out with methods of calculations to calculate the time of occurrence of these phenomenon, and all these were reflected in the calendars that they had edited and compiled. Of all these calendars, perhaps the most outstanding one was the Shou Shi calendar (授時歷, 1281 A.D), edited by Wang Xun (王恂, 1235–1281 A.D), Guo Shoujing (郭守敬, 1231–1316 A.D), and their co-workers in the Yuan (元朝, 1260–1368 A.D) Dynasty. The calendar contained many astronomical values that were very accurate at that time, and when they were compared with the modern values, the errors in them are very small, which shows the degree of accuracy that had been achieved in the Shoushi calendar. This is due to the fact that new methods of calculations, recorded in the calendar itself, had been derived and were used in the various calculations. Even Laplace, the famous mathematician, was astounded by the accuracy of the results and even used one of them as the proof to his theory that the angle between the ecliptic and the celestial equator was becoming smaller. Because of the success that the Shoushi calendar had enjoyed, we find it worthwhile to go about understanding this work by Guo and his co-workers better.

The Developments of the Chinese Calendar Throughout its History

Before we go into the Shoushi calendar, let us look at the developments of the Chinese calendar throughout its history, and see why the Shoushi calendar was one of the best calendar in the list of the calendars that had been edited and compiled. To understand these developments better, we will have to know some terminologies used by the Chinese astronomers in the Chinese calendar. Firstly, we have the 24 jie qi (節气). It is a terminology used in the calendar to track the seasonal changes throughout a year. With the 24 jie qi the farmers were able to know when they should start planting the seeds of their crops, when they should collect their grown crops, when they should start preparing themselves for winter etc. Chinese astronomers had defined the 24 jie qi in two different ways. Initially, they thought that the motion of the sun with respect to the earth was constant, and it moves in an average speed and completes one revolution in a year. Therefore they would divide the number of days in a year by 24, and this average value will be the number of days between successive qi. The cycle will start from the winter solstice, and after the number of days corresponding to the average value obtained, there would be a change in qi. This was the so-called ping qi (平气). However, after they discovered that the motion is actually inconsistent, they still started the cycle at the winter solstice, but defined a change in qi when the sun with respect to the earth had actually moved 15 degrees by observation. This was called ding qi (定气). Then we have the ping shuo (平朔) and ding shuo (定朔). Ping shuo means that the motion of the moon was taken to be constant by the Chinese astronomers when they were calculating the numbers of days in a month, and thus we have an "average moon". Ding shuo, on the other hand, means that the Chinese astronomers computed the number of days in a month based on the actual inconsistent motion of the moon, and the first day of a month must be the day that a new moon occurred on. Also, Chinese astronomers define precession of the equinoxes as sui cha (歲差).

Initially, Chinese astronomers thought that the motion of the moon was constant, and in the calendars they had compiled, they would first define the average time from one new moon to the next and used the value to compute the number of days in each month. The value of this average time differed from calendars to calendars since the editors of each of these calendars computed the value differently. However, all the different values that were computed did not differ from the actual value 29.5306 days by much. Since it is not a whole number and is about 29.5 days, the Chinese calendars normally have a small month consisting of 29 days and then followed by a big month of 30 days. Occasionally, a small month was followed by two successively big months, since the average value was slightly more than 29.5 days, and when the excess time added up to 1 day, one more big month would be needed. This was called the lian da (聯大). Therefore, using these so-called ping shuo (average moon) theory, the calendars could only have 2 big months together, and was not possible for having 2 short months to be together.

In the later period of the Eastern Han Dynasty (后漢朝, 25–220 A.D), the Chinese astronomer by the name of Jia Kui (賈逵, ?–92 A.D) was the first to discover that the motion of the moon is inconsistent. This discovery was recorded in his Sifen calendar (后漢四分歷 , 85 A.D), but the various calculations he made for his calendar did not base on this discovery, but rather on the basis that the motion of the moon is constant. The first time that the theory of the inconsistent motion of the moon was brought into consideration in the calculations occurred in the Qian Xiang calendar (乾象歷 , 223 A.D). It was the calendar that was compiled by Kan Ze (闞澤) in the period of the Three Kingdoms (三國時代, 220–280 A.D). Although the calendar still used ping shuo to compute the number of days in a month, in the calculations for the eclipses, the inconsistent motion of the moon was taken into account. This greatly increased the accuracy of the prediction of the eclipses, but because the motion of the sun was still taken to be moving at a constant speed, there was rooms left for improvement.

Calendars compiled after the Qian Xiang calendar all used the theory of the inconsistent motion of the moon in the computations. However, the calendars still used ping shuo to compute the number of days in a month. The notion of ding shuo was finally proposed by the Chinese astronomers He Chengtian (何承天), when he compiled the Yuan Jia calendar (元嘉歷, 445 A.D). However, using ding shuo would mean strings of 3 big months and strings of 2 small months occurring in the calendar, which never happened in calendars before it. Therefore, the authorities felt uncomfortable about it and did not want it to be introduced into the calendar. Thus due to the pressure from the authorities, He Chengtian was forced to use ping shuo to compute the number of days in a month. The first calendar that used ding shuo for computation was the Wu Yin calendar (戊寅歷, 619 A.D), compiled by Fu Renjun (傅仁均). However, the occurrence of a string of 4 big months caused much dispute because it differed too much from the old practice, and the basis of the computations of the months was forced to revert back to ping shuo. It was only after the Lin De calendar (麟德歷, 665 A.D) compiled in the Tang Dynasty (唐朝, 618–907 A.D) that ding shuo was commonly accepted in the computation of the number of days in a month.

Although the inconsistent motion of the moon was discovered early in the Eastern Han Dynasty, the inconsistent motion of the sun with respect to the earth was only discovered during the North and South Dynasty (南北朝, 386–589 A.D), by the Chinese astronomer called Zhang Zixin (張子信, 6th century A.D). The astronomer Liu Zhuo (劉焯, 544–610 A.D) of the Sui Dynasty (隋朝, 589–618 A.D) agreed with this theory, and came out with the concept of ding qi in his Huang Ji calendar (皇极歷, 600 A.D). However, he misunderstood the actual motion of the sun. He thought that there would be a sudden change in the speed of the sun at certain times in a year, where the speed changed from the fastest to the slowest or vice-versa. This is incorrect since the change in the speed of the sun is gradual; its speed would increase from the slowest to the fastest, then when it is at the fastest speed, it will begin to slow down until it is at the slowest speed again, and the process continues. This actual movement of the sun was fully understood by the famous astronomer Yi Xing (一行, 683–727 A.D), and he used the theory in his computations, for example the calculations for the time of eclipses, in his Da Yan calendar (大衍歷, 729 A.D), getting accurate results in the process. Chinese astronomers later used this theory in their computations for the calendars. However, because the notion of the 24 jie qi only concerns with the change in seasons, and the change is normally not noticeable, the calendars continue to use ping qi when denoting the seasonal change in terms of qi. The notion of ding qi only took centre stage in the calendars compiled in the Qing Dynasty (清朝, 1616–1912 A.D).

Precession of the equinoxes was discovered by Chinese astronomers, which they defined as sui cha, during the North and South Dynasty. It was discovered by Zu Chongzhi (祖衝之, 429–500 A.D), and he defined that the axis of the earth moved a degree after 45 years and 11 months. This is way off the modern value, but the discovery of the precision was in itself a great achievement, and this theory was adapted later during the making of the calendars after the Da Ming calendar (大明歷, 510 A.D) that was compiled by Zu Chongzhi himself. Later in the Shoushi calendar, Guo Shoujing defined the axis moved a degree after 66 years and 8 months. All the above developments concerning the Chinese calendars can be summarized by the table as follows:

Theory/concept	When first introduced	When commonly accepted
Inconsistent motion of the moon	First discovered by Jia Kui (?–92 A.D) in the Eastern Han period (25–200 A.D); first discussed in the Sifen calendar (85 A.D) of the Eastern Han Dynasty (25–200 A.D)	After the Qian Xiang calendar (223 A.D) of the Three Kingdoms (220–280 A.D).
Inconsistent motion of the sun	First discovered by Zhang Zixin (6th century A.D) during the North and South Dynasty (386–589 A.D); first mentioned in the Huang Ji calendar (600 A.D) of the Sui Dynasty (589–618 A.D)	After the Da Yan calendar (729 A.D) of the Tang Dynasty (618–907 A.D)
Ping shuo	From the first Chinese calendars in the Shang Dynasty (商朝, 1523–1027 B.C)	
Ding shuo	First proposed in the Yuan Jia calendar (445 A.D) of the North and South Dynasty (386–589 A.D); first used in the Wu Yin calendar (619 A.D) of the Tang Dynasty (618–907 A.D)	After the Lin De calendar (665 A.D) of the Tang Dynasty (618–907 A.D)
Ping qi	Zhuan Xu calendar (顓頊歷, 174 B.C) of the Warring States (770–221 B.C)	
Ding qi	Huang Ji calendar (600 A.D) of the Sui Dynasty (589–618 A.D)	After the Shi Xian Calendar (時憲歷) of the Qing Dynasty (1645–1911 A.D)
Precession (sui cha)	First discovered by Zu Chongzi (429–500 A.D) of the North and South Dynasty (386–589 A.D); first mentioned in the Da Ming calendar (510 A.D) of the North and South Dynasty (386–589 A.D)	

One of the reasons why the Shou Shi calendar was so successful was that it was able to build on a solid foundation that the calendars before it had laid out and furthermore came out with new innovations. We will now discuss some of the various innovations, and the various discoveries made by Guo Shoujing that were used in his Shou Shi calendar.

The Method of Interpolation

One of the most important mathematical developments made by Guo Shoujing, which was recorded in the Shou Shi calendar, was on the method of interpolation. The method of interpolation was originally discovered by Liu Zhuo during the Sui Dynasty. However, compared to the method of interpolation used by Liu Zhuo, the method of interpolation recorded in the Shou Shi calendar had shown vast improvement and produced much more accurate results.

What is the method of interpolation? Suppose that we have a function $f(x) = x$. We know that the values of $f(1), f(2), f(3), \ldots$ are $1, 2, 3, \ldots,$ so what is the value of $f(1+0.5)$? The answer is simple: $f(1+0.5) = 1+0.5 = 1.5$. However, this is a simple linear function, therefore we can calculate the intermediate value $f(1+s)$ for $0 < s < 1$ easily. What if we have a non-linear function, like $f(x) = x^2$? The values of $f(1), f(2), f(3), \ldots$ are $1, 4, 9, \ldots$, but an intermediate value like $f(1+0.4)$ is much more difficult to calculate from the above known data, and requires to establish certain formulas, using the method of interpolation. Liu Zhuo is the first person to grasp this method, and it was used in his Huang Ji calendar for calculation.

Modern Methods of Interpolation: Newton's Divided Difference and Forward Difference Methods

Before we go into the interpolation method that was used by the Chinese, it is worthwhile to go through the modern methods of interpolation, and observe the similarity between them. The modern methods of interpolation commonly used today to find a formula to fit the data gathered are the Newton's divided difference and forward difference methods. As the name suggested, these methods were discovered and formulated by Isaac Newton, the famous physicist and mathematician. We will now give a brief introduction on the method of divided difference.

Given a function f, there exists an unique polynomial P_n of degree n that agrees with the function f at the distinct points $x_0, x_1, \ldots, x_n$. The polynomial P_n has the representation

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

for appropriate constants $a_0, a_1, \ldots, a_n$.

To determine the constants, the divided-difference notation will be used. The zeroth divided difference of the function f, with respect to x_i, is denoted as $f[x_i]$ and is simply the evaluation of f at x_i, i.e.,

$$f[x_i] = f(x_i).$$

The remaining divided differences are defined inductively; the first divided difference of f with respect to x_i and x_{i+1} is denoted as $f[x_i, x_{i+1}]$ and is defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

When we have determined the $(k-1)$-th divided differences $f[x_i, x_{i+1}, \ldots, x_{i+k-1}]$ and $f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}]$, the k-th divided difference relative to $x_i, x_{i+1}, \ldots, x_{i+k}$ is given by

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{x+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

The inductive nature of the divided difference is best described by the following table:

x	$f(x)$	First Divided Differences	Second Divided Differences	Third Divided Differences
x_0	$f[x_0]$			
		$f[x_0,x_1]=\frac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2]=\frac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3]=\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2,x_3]=\frac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]=\frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2,x_3,x_4]=\frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3,x_4]=\frac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5]=\frac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3,x_4,x_5]=\frac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4,x_5]=\frac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

With the divided-difference notation, it is easy to determine the constants $a_0, a_1, \dots, a_n$. The constant a_k is equal to $f[x_0, x_1, \dots, x_k]$, i.e.,

$$a_k = f[x_0, x_1, \dots, x_k]$$

for each $k = 0, 1, \dots, n$. Therefore P_n can be rewritten as

$$\begin{aligned} P_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + \\ &\quad f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}) \\ &= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \dots, x_k](x - x_0) \cdots (x - x_{k-1}). \end{aligned}$$

(The procedure of evaluating the constants $a_0, a_1, \dots, a_n$ can be found in the book [2])

When $x_0, x_1, \dots, x_n$ are arranged consecutively with equal spacing, then the above formula can be expressed in a simplified form. Introducing the notation $h = x_{i+1} - x_i$ for each $i = 0, 1, \ldots, n-1$ and $x = x_0 + sh$, the difference $x - x_i$ can be written as

$$x - x_i = (s - i)h.$$

So the above formula becomes

$$\begin{aligned} P_n(x) = P_n(x_0 + sh) &= f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2] \\ &\quad + \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0, x_1, \ldots, x_n] \\ &= \sum_{k=0}^{n} s(s-1)\cdots(s-k+1)h^k f[x_0, x_1, \ldots, x_k] \\ &= \sum_{k=0}^{n} \binom{s}{k} k! h^k f[x_0, x_1, \ldots, x_k], \end{aligned}$$

where $\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$ is the binomial-coefficient notation.

Introducing the forward difference notation $\triangle$, defined as $\triangle p_n = p_{n+1} - p_n$ for $n \geq 0$ and higher powers $\triangle^k p_n$ defined recursively by

$$\triangle^k p_n = \triangle^{k-1}(\triangle p_n)$$

for $k \geq 2$, we see that

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \triangle f(x_0), \\ f[x_0, x_1, x_2] &= \frac{1}{2h} \left(\frac{\triangle f(x_1) - \triangle f(x_0)}{h} \right) = \frac{1}{2h^2} \triangle^2 f(x_0), \end{aligned}$$

and in general

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k! h^k} \triangle^k f(x_0).$$

Therefore the original equation can be further expressed as

$$P_n(x) = \sum_{k=0}^{n} \binom{s}{k} \triangle^k f(x_0),$$

and this is known as the Newton forward-difference formula. Note that both Newton's divided-difference formula and Newton's forward difference formula can be used for functions of any degree.

With the knowledge of modern method of interpolation, we are now ready to take a look at the method of interpolation that was derived and used by the Chinese. It is interesting to see that with the vast interest in mathematics (much of it related to astronomy and the planning of calendars) throughout the history of China, it was not until the Sui Dynasty that the method of interpolation was discovered and used. Before the time of the Eastern Han Dynasty, people believed that the speed of the movement of the sun, the moon and the five planets (Mercury, Venus, Mars, Jupiter, Saturn) was constant. However, in the later period of the Eastern Han, Jia Kui discovered that the movement of the moon is sometimes fast, sometimes slow. In the time of the North and South Dynasty, Zhang Zixin discovered that the movement of the sun is also sometimes fast, sometimes slow. All this evidence began to convince astronomers that the former belief of constant movement of the heavenly bodies was wrong, and they began to adapt the new theory of the inconsistent movement of the heavenly bodies. However, with this new theory, the calculations of the movement of the heavenly bodies necessary for the making of a calendar became much more complicated, and this brought about the need for the method of interpolation.

Generally speaking, what the method of interpolation does is to provide a formula that "fits" the data gathered. The earliest method of interpolation was recorded in Liu Zhuo's Huang Ji calendar as follows:

Suppose that the time interval between observations is l and the results of the observations are $f(l), f(2l), f(3l), \ldots, f(nl), \ldots$ Then to predict the result of an observation at a certain time, say $nl + s$, where $0 < s < l$, the formula Liu Zhuo used was

$$f(nl+s) = f(nl) + \frac{s}{2l}(\triangle_1 + \triangle_2) + \frac{s}{l}(\triangle_1 - \triangle_2) - \frac{s^2}{2l^2}(\triangle_1 - \triangle_2),$$

where $\triangle_1 = f(nl+l) - f(nl)$, $\triangle_2 = f(nl+2l) - f(nl+l)$. Let us simply the formula:

$$
\begin{aligned}
&f(nl+s)\\
&= f(nl) + \frac{s}{2l}(f(nl+l) - f(nl) + f(nl+2l) - f(nl+l)) + \frac{s}{l}(f(nl+l)-\\
&\quad f(nl) - f(nl+2l) + f(nl+l)) - \frac{s^2}{2l^2}(f(nl+l) - f(nl) - f(nl+2l) + f(nl+l))\\
&= f(nl) + \frac{s}{l}(-\frac{f(nl)}{2} + \frac{f(nl+2l)}{2} + 2f(nl+l) - f(nl) - f(nl+2l))-\\
&\quad \frac{s^2}{2l^2}(f(nl+l) - f(nl) - (f(nl+2l) - f(nl+l)))\\
&= f(nl) + \frac{s}{l}(f(nl+l) - f(nl)) + \frac{s}{l}(f(nl+l) - \frac{f(nl)}{2} - \frac{f(nl+2l)}{2})-\\
&\quad \frac{s^2}{2l^2}(f(nl+l) - f(nl) - (f(nl+2l) - f(nl+l)))\\
&= f(nl) + \frac{s}{l}(f(nl+l) - f(nl)) + \left(\frac{s^2}{2l^2} - \frac{s}{2l}\right)(f(nl+2l) - 2f(nl+l) + f(nl))\\
&= f(nl) + \frac{s}{l}(f(nl+l) - f(nl)) + \left(\frac{s^2}{2l^2} - \frac{sl}{2l^2}\right)(f(nl+2l) - 2f(nl+l) + f(nl))\\
&= f(nl) + \frac{s}{l}(f(nl+l) - f(nl)) + \left(\frac{s}{2l^2}\right)(s-l)(f(nl+2l) - 2f(nl+l) + f(nl))\\
&= f(nl) + \frac{s}{l}(f(nl+l) - f(nl)) + \left(\frac{(s)(s-l)}{2l^2}\right)(f(nl+2l) - 2f(nl+l) + f(nl)).
\end{aligned}
$$

Compare this to the formula of interpolation for the Newton divided difference of degree 2

$$P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1),$$

where $x = nl + s$ (the time that we want the result of), $x_0 = nl$ and $x_i = nl + il$. We have

$$P_2(nl+s) = f(nl) + \frac{f(nl+l) - f(nl)}{l}(s) + \frac{f(nl+2l) - 2f(nl+l) + f(nl)}{2l^2}(s)(s-l),$$

which it is the same as the formula produced by Liu Zhuo.

Note that in the reference book [4], in section 4.1, there are errors in the explanations for the "equal time interval method of interpolation". The book states that, if the results of the observation at times w, $2w$, …, nw, …, with the constant time interval w, are $f(w)$, $f(2w)$, …, $f(nw)$, …, then the result of the observation at time $w + s$, $0 < s < w$, can be found by the formula

$$f(w+s) = f(w) + s\triangle + \frac{s(s-1)}{2!}\triangle^2 + \frac{s(s-1)(s-2)}{3!}\triangle^3 + \cdots.$$

In defining the notations $\triangle$, $\triangle^2$ and $\triangle^3$, the book first defines six other new notations, namely

$$\triangle_1' = f(2w) - f(w),$$

$$\triangle_2' = f(3w) - f(2w),$$
$$\triangle_3' = f(4w) - f(3w),$$
$$\triangle_1^2 = \triangle_2' - \triangle_1' , \qquad \triangle_2^2 = \triangle_3' - \triangle_2' , \qquad \triangle_1^3 = \triangle_2^2 - \triangle_1^2 .$$

The book then defines $\triangle$, $\triangle^2$ and $\triangle^3$ as

$$\triangle = \triangle_1' ,$$
$$\triangle^2 = \triangle_1^2 ,$$
$$\triangle^3 = \triangle_1^3 .$$

Clearly, the book is stating the formula for Newton's forward difference method of interpolation (degree 2). However, not only that these definitions of the notations $\triangle$, $\triangle^2$ and $\triangle^3$ (which is obviously the forward difference notations) do not clearly show the recursive nature of them, there is a serious mistake made in the definition of s. In the formula for Newton's forward difference method of interpolation, as introduced before, the s here takes on a different meaning. It is related to the time w' that we are interested in finding the result of observation in the following way:

$$w' = w + sw,$$

and not just simply $w' = w + s$. If we are going to define the time that we are interested in finding the result of observation as $w + s$, $0 < s < w$, then the formula for Newton's divided difference method of interpolation (degree 2) shall be used, i.e.,

$$\begin{aligned} f(w+s) &= f(w) + \frac{f(2w) - f(w)}{w}(s) + \frac{f(3w) - 2f(2w) + f(w)}{2w^2}(s)(s-w) \\ &= f(w) + \frac{\triangle_1'}{w}(s) + \frac{\triangle_2' - \triangle_1'}{2w}(s)(s-w), \end{aligned}$$

where $\triangle_1' = f(2w) - f(w)$ and $\triangle_2' = f(3w) - f(2w)$.

Also, the book states that, again with the results of the observation at times w, $2w$, $\ldots$, nw, $\ldots$, with the constant time interval w, being $f(w)$, $f(2w)$, $\ldots$, $f(nw)$, $\ldots$, the formula for the method of interpolation that Liu Zhuo had derived in finding the result of observation at time $w+s$, $0 < s < w$, is

$$f(w+s) = f(w) + \frac{s}{2}(\triangle_1' + \triangle_2') + (s)(\triangle_1' - \triangle_2') - \frac{s^2}{2}(\triangle_1' - \triangle_2').$$

However, this was not the formula that Liu Zhuo had derived. Instead, his formula, using the same notations as above, should have been

$$f(w+s) = f(w) + \frac{s}{2w}(\triangle_1' + \triangle_2') + \frac{s}{w}(\triangle_1' - \triangle_2') - \frac{s^2}{2w^2}(\triangle_1' - \triangle_2').$$

These mistakes can be easily overlooked because the book is able to show that the "method of equal interval interpolation" that it has stated is the same as the formula it states as the formula of interpolation derived by Liu Zhuo. (which is surprising considering the fact that both of them are wrong in the first place and the book can still show a clear relation between them). Therefore, since it is usually the main English text that people studying the history of Chinese Mathematics will refer to, we see it worthwhile pointing out these mistakes.

Liu Zhuo applied his method of interpolation, which is called *the equal interval second difference method of interpolation* using modern terms, to his calculations of the positions of the sun and the moon. These calculations were mainly used to predict eclipses. Another breakthrough was achieved in the method of interpolation during the mid Tang Dynasty, when Yi Xing independently came up with *the unequal interval second difference interpolation* formula for calculations. The advantage of his method over that of Liu Zhuo's was that it can be applied to data taken at unequal intervals of time. This method of interpolation was recorded in the Da Yan calendar, which was compiled and edited by Yi Xing himself. The formula that was used for the calculations, using modern algebraic notation, is as follows:

Assume that L_1, L_2 are two unequal time intervals and at times $w, w+L_1, w+L_1+L_2$, the results of observations are $g(w), g(w+L_1), g(w+L_1+L_2)$, then the result at time $w+s$ using the unequal interval formula of Yi Xing, is equivalent to

$$g(w+s) = g(w) + s\left(\frac{\triangle_1 + \triangle_2}{L_1 + L_2}\right) + s\left(\frac{\triangle_1}{L_1} - \frac{\triangle_2}{L_2}\right) - \frac{s^2}{L_1 + L_2}\left(\frac{\triangle_1}{L_1} - \frac{\triangle_2}{L_2}\right),$$

where $\triangle_1 = g(w+L_1) - g(w)$, $\triangle_2 = g(w+L_1+L_2) - g(w+L_1)$.

Again if we use the formula of Newton's divided difference (degree 2), we have

$$
\begin{aligned}
&P_2(w+s)\\
&= g(w) + g[w+L_1, w](s) + g[w+L_1+L_2, w+L_1, w](s)(s-L_1)\\
&= g(w) + (s)\left(\frac{g(w+L_1)-g(w)}{L_1}\right) + (s)(s-L_1)\left(\frac{\frac{g(w+L_1+L_2)-g(w+L_1)}{L_2} - \frac{g(w+L_1)-g(w)}{L_1}}{L_1+L_2}\right)\\
&= g(w) + (s)\left(\frac{g(w+L_1)-g(w)}{L_1} - L_1\left(\frac{\frac{g(w+L_1+L_2)-g(w+L_1)}{L_2} - \frac{g(w+L_1)-g(w)}{L_1}}{L_1+L_2}\right)\right)\\
&\quad + \frac{s^2}{L_1+L_2}\left(\frac{g(w+L_1+L_2)-g(w+L_1)}{L_2} - \frac{g(w+L_1)-g(w)}{L_1}\right)\\
&= g(w) + (s)\left(\frac{g(w+L_1)-g(w)}{L_1} + \frac{g(w+L_1)-g(w)}{L_1+L_2}\right)\\
&\quad - \frac{sL_1}{L_1+L_2}\left(\frac{g(w+L_1+L_2)-g(w+L_1)}{L_2}\right)\\
&\quad + \frac{s^2}{L_1+L_2}\left(\frac{g(w+L_1+L_2)-g(w+L_1)}{L_2} - \frac{g(w+L_1)-g(w)}{L_1}\right).
\end{aligned}
$$

Letting $\triangle_1 = g(w+L_1) - g(w)$, $\triangle_2 = g(w+L_1+L_2) - g(w+L_1)$, we have

$$
\begin{aligned}
&P_2(w+s)\\
&= g(w) + s\left(\frac{\triangle_1}{L_1} + \frac{\triangle_1}{L_1+L_2}\right) - s\left(1 - \frac{L_2}{L_1+L_2}\right)\left(\frac{\triangle_2}{L_2}\right) + \frac{s^2}{L_1+L_2}\left(\frac{\triangle_2}{L_2} - \frac{\triangle_1}{L_1}\right)\\
&= g(w) + s\left(\frac{\triangle_1}{L_1} - \frac{\triangle_2}{L_2} + \frac{\triangle_2}{L_1+L_2} + \frac{\triangle_1}{L_1+L_2}\right) - \frac{s^2}{L_1+L_2}\left(\frac{\triangle_1}{L_1} - \frac{\triangle_2}{L_2}\right)\\
&= g(w) + s\left(\frac{\triangle_1}{L_1} - \frac{\triangle_2}{L_2}\right) + s\left(\frac{\triangle_1+\triangle_2}{L_1+L_2}\right) - \frac{s^2}{L_1+L_2}\left(\frac{\triangle_1}{L_1} - \frac{\triangle_2}{L_2}\right),
\end{aligned}
$$

which is the same as the formula given by Yi Xing.

This method of unequal interval second difference interpolation, together with the method of equal interval second difference interpolation, was of vast importance and had great influences on the calendars that were compiled and edited after the Huang Ji calendar and the Da Yan calendar. For example, the method of equal interval second difference interpolation was used in the Lin De calendar by Li Chunfeng (李淳風), and the method of unequal interval second difference interpolation was used in Xu Ang's (徐昂) Xuan Ming calendar (宣明歷, 882 A.D). The formula that was recorded in the Xuan Ming calendar, using the same notations as above in the method of unequal interval second difference interpolation, is as follows:

$$g(w+s) = g(w)+s\left(\frac{\triangle_1}{L_1}\right)+\frac{sL_1}{L_1+L_2}\left(\frac{\triangle_1}{L_1}-\frac{\triangle_2}{L_2}\right)-\frac{s^2}{L_1+L_2}\left(\frac{\triangle_1}{L_1}-\frac{\triangle_2}{L_2}\right),$$

which was consistent with the formula given by Yi Xing.

The two methods of second order difference interpolation given by Liu Zhuo and Yi Xing assumed that the motion of the heavenly bodies satisfied a quadratic function, since the formulas were of degree 2. However this does not coincide with the facts, since the motion of the heavenly bodies is not a quadratic polynomial; it is described by a more complicated function. Therefore, although the two formulas gave satisfying results, they are not accurate enough. This problem was ingeniously improved on by Wang Xun, Guo Shoujing and their fellow co-workers. They produced the famous Shou Shi calendar, and adapted the principle of third order interpolation to calculate the tables for the positions of the heavenly bodies. This was one of the five major innovations in the Shou Shi calendar.

The method of third order difference interpolation was recorded in the Shou Shi calendar in a different way as from that in the Huang Ji and the Da Yan calendar. From the winter solstice to the spring equinox, it is found to be 88.91 days. This period of 88.91 days are then divided into six equal periods each of 14.82 days. The interval between each period is defined as l. At the points $l, 2l, \ldots, 6l$, the path of the sun is observed and then from each observation $l, 2l, \ldots, 6l$ degrees are subtracted. This is because using the old definition of degrees by Chinese mathematicians, the sun moves, on average, one degree each day and thus in nl days it moves nl degrees. This will give the so-called *accumulated difference*, that is:

$$\text{Accumulated difference of } nl \text{ days} = \text{actual degrees moved in } nl \text{ days} - nl.$$

The table of accumulated differences that was recorded in the calendar is as follows:

Period $= n$	Accumulated Days $= nl$	Accumulated Difference $= f(nl)$
1	14.82	7058.0250
2	29.64	12976.3920
3	44.46	17693.7462
4	59.28	21148.7328
5	74.10	23279.9970
6	88.92	24026.1840

Note that in the Shou Shi calendar, one degree is taken as 10000 divisions.

Therefore in the preceding table, the value 7058.0250 is actually 0.70580250 degree.

Then Guo Shoujing and his co-workers divided the accumulated differences by the number of days (nl) and called the result the *average daily difference*. Then they composed a new table of average daily differences $F(nl)$, and used the differences of the average daily differences to get the values of the *first differences* $\triangle F(nl)$. Next, they successively subtracted the first differences $\triangle F(nl)$ to get the *second differences* $\triangle^2 F(nl)$. The values of the *third differences* $\triangle^3 F(nl)$ were founded by successively subtracting the second differences $\triangle^2 F(nl)$. Therefore a table as followed was obtained:

Period (n)	Average Daily Difference $= F(nl)$	First Difference $= \triangle F(nl)$	Second Difference $= \triangle^2 F(nl)$	Third Difference $= \triangle^3 F(nl)$
0				
1	476.25			
		-38.45		
2	437.80		-1.38	
		-39.83		0
3	397.97		-1.38	
		-41.21		0
4	356.76		-1.38	
		-42.59		0
5	314.17		-1.38	
		-43.97		
6	270.20			

The third differences are all of value 0, which implies that we will be finding a formula of degree 2. In order to derive the formula, the values of $F(0)$, $\triangle F(0)$, $\triangle^2 F(0)$ are required. ¿From the preceding table of the average daily differences, the value of $\triangle^2 F(0)$ is known, which is -1.38. Then using this value, the value of $\triangle F(0)$ can be obtained, since $\triangle F(0) = \triangle F(l) - \triangle^2 F(0) = -38.45 - (-1.38) = -37.07$. The value of $F(0)$ is obtained in a similar way, i.e. $F(0) = F(l) - \triangle F(0) = 476.25 - (-37.07) = 513.32$. Then using the method of second order difference interpolation, we have

$$\frac{f(nl)}{nl} = F(nl) = F(0) + n\triangle F(0) + \frac{n(n-1)}{2!}\triangle^2 F(0) = 513.32 - 37.07n - 1.38\frac{n(n-1)}{2}.$$

Note that this is similar to the Newton's forward difference method.

Letting $m = nl$ (the number of days after winter solstice),

$$\frac{f(m)}{m} = F(m) = 513.32 - 37.07\frac{m}{l} - 1.38\frac{m}{2l}(\frac{m}{l} - 1),$$

and since $l = 14.82$, the equation become

$$f(m) = 513.32m - 2.46m^2 - 0.0031m^3,$$

where $f(m)$ is the accumulated difference m days after the winter solstice.

Putting $m = 1, 2, 3, 4, \ldots$ in turn and substituting into $f(m)$ shall at once give the successive accumulated daily differences. Let $a = 513.32, b = -2.46, c = -0.0031$, we see that if we substitute $1, 2, 3, 4, \ldots$ in turn it is at once clear that the a's in $f(1), f(2), f(3), f(4), \ldots$ increase as multiples of $1, 2, 3, 4, \ldots$; the b's as $1^2, 2^2, 3^2, 4^2, \ldots$ and the c's as $1^3, 2^3, 3^3, 4^3, \ldots$. So in the Shou Shi calendar the value $a = 513.32$ was called the *linear difference* (定差); $b = -2.46$ was called the *square difference* (平差); $c = -0.0031$ was called the *cubic difference* (立差). Later on the mathematicians of the Qing period called this method of accumulated differences the *technique of linear, square, and cubic differences* (定平立三差術).

However, in carrying out the calculations to find the accumulated differences for each day, Guo Shoujing and his co-workers did not make use of the formula by substituting the number of days but still used the method of tabular calculations. Using

$$\begin{aligned} F(0) &= 0, \\ F(1) &= a + b + c, \\ F(2) &= 2a + 4b + 8c, \\ F(3) &= 3a + 9b + 27c, \\ &\vdots \end{aligned}$$

the following table was computed

No of days after the winter solstice $= m$	Accumulated Difference $= f(m)$	First Difference $= \triangle f(m)$	Second Difference $= \triangle^2 f(m)$	Third Difference $= \triangle^3 f(m)$
0				
1	$a+b+c$			
		$a+3b+7c$		
2	$2a+4b+8c$		$2b+12c$	
		$a+5b+19c$		$6c$
3	$3a+9b+27c$		$2b+18c$	
		$a+7b+37c$		$6c$
4	$4a+16b+64c$		$2b+24c$	
		$a+9b+61c$		
5	$5a+25b+125c$			

¿From the table, it was easy to obtain

$$\begin{aligned}
\triangle^3 f(0) &= 6c = -0.00186 \\
\triangle^2 f(0) &= \triangle^2 f(1) - \triangle^3 f(0) = 2b+12c-6c = 2b+6c = -4.9386 \\
\triangle f(0) &= \triangle f(1) - \triangle^2 f(0) = a+3b+7c-(2b+6c) = a+b+c = 510.8569
\end{aligned}$$

in the Shou Shi calendar, the value 510.8569 was called jia fen (加分), -4.9386 was called ping li he cha (平立合差), and -0.0186 was called jia fen li cha (加分立差)

After obtaining $f(0)$, $\triangle f(0)$, $\triangle^2 f(0)$, $\triangle^3 f(0)$, it is easy to list out the accumulated differences day by day. That is to say, knowing

No of days after the winter solstice (m)	Accumulated Difference $(f(m))$	First Difference $(\triangle f(m))$	Second Difference $(\triangle^2 f(m))$	Third Difference $(\triangle^3 f(m))$
0	0			
		510.8569		
1	⋮		-4.9386	
		⋮		-0.0186
⋮			⋮	
				⋮

the table can be completed by successively adding from right to left, and knowing that the third difference always stay at the constant value of -0.0186.

For example, to find $\triangle^2 f(1)$, add -0.0186 to -4.9386 to get -4.9572, and to find $\triangle^2 f(2)$, add -0.0186 to -4.9572 to get -4.9758; to find $\triangle f(1)$, add -4.9386 to 510.8569 to get 505.9183. Therefore, the following table was obtained by Guo Shoujing and his co-workers:

No of days after the winter solstice	Accumulated Difference	First Difference	Second Difference	Third Difference
0	0			
		510.8569		
1	510.8569		-4.9386	
		505.9183		-0.0186
2	1016.7752		-4.9572	
		500.9611		-0.0186
3	1517.7363		-4.9758	
		495.9853		-0.0186
4	2013.7216		-4.9944	
		490.9909		⋮
5	2504.7125		⋮	
		⋮		
⋮	⋮			

This is the final table of accumulated daily differences as recorded in the Ming Shi (明史). Note that the original copy of the Shou Shi calendar did not survive till today; a lot of information on the Shou Shi calendar was based on the Yuan Shi (元史) and the Ming Shi. In particular, the Ming Shi recorded the Da Tong calendar, which although sounded like a different calendar by the name, it used the same mathematical methods and the various astronomical values as the Shou Shi calendar. Therefore the method of interpolation and the tables that were recorded in the Ming Shi were in fact the same as those in the Shou Shi calendar.

Again, note that in the reference book [4], there is again errors in table 5.7, which is largely due to the fact that the book confuses with the sign used. As stated from the book, this table shows how Guo and his co-workers went about finding the accumulated differences for each day after the winter solstice by the method of tabular calculations. It uses the same notations a, b and c to denote the values of 513.32, -2.46 and -0.0031 respectively. However, the table has the values as follows:

No of days after the winter solstice $= m$	Accumulated Difference $= f(m)$	First Difference $= \triangle f(m)$	Second Difference $= \triangle^2 f(m)$	Third Difference $= \triangle^3 f(m)$
0				
1	$a - b - c$			
		$a - 3b - 7c$		
2	$2a - 4b - 8c$		$-2b - 12c$	
		$a - 5b - 19c$		$-6c$
3	$3a - 9b - 27c$		$-2b - 18c$	
		$a - 7b - 37c$		$-6c$
4	$4a - 16b - 64c$		$-2b - 24c$	
		$a - 9b - 61c$		
5	$5a - 25b - 125c$			

It is clear that the book has changed the signs of the values of a, b and c to be all positive during the construction of this table, i.e $a = 513.32$, $b = 2.46$ and $c = 0.0031$. Due to this inconsistency in the definitions of the values of a, b and c, the book again brings about confusion and we again point this mistake out so that people referring to this book can avoid this confusion. Also, note that there is an error in the calculation for the value of $\triangle f(0)$ in this reference book; it should be 510.8569, not 510.8560

Let us use the values of $f(0) = 0$, $\triangle f(0) = 510.8569$, $\triangle^2 f(0) = -4.9386$ and $\triangle^3 f(0) = -0.0186$ and apply the formula for third order difference interpolation, we have

$$f(m) = 510.8569m - 4.9386\frac{m(m-1)}{2!} - 0.0186\frac{m(m-1)(m-2)}{3!}$$

as the accumulated differences m days after the winter solstice. After simplication, we have

$$f(m) = 513.32m - 2.46m^2 - 0.0031m^3,$$

which is the same equation as obtained before.

We can see that the authors of the Shou Shi calendar, on the basis of second order difference interpolation (using average daily differences), brought about a formula which was equivalent to the method of third order difference interpolation. This in turn gave rise to greater accuracy to the calculations of the positions of the heavenly bodies, and with such accuracy made Shou Shi calendar one of the best calendars in the world at that time.

The Astronomical Tools Used by Guo Shoujing

In the previous section, we discussed the method of third order difference interpolation that Guo Shoujing and his co-workers had derived. However, this is not the only method that they derived; there were some other mathematical methods that were considered revolutionary at that point of time and were also recorded in the Shou Shi calendar. One of the methods was related to what we now call spherical trigonometry, but not as complete as the spherical trigonometry that we now have. (For more detailed description on this method, refer to [4]). With these new mathematical methods, Guo and his co-workers were able to make more accurate astronomical calculations, which helped them to make better astronomical predictions, especially in the case of eclipses. However, the success of the Shou Shi calendar was not restricted to these newly developed and discovered mathematical methods; it also included the precision of the astronomical tools that Guo and his co-workers built to collect astronomical data. As Guo Shoujing put it "Accurate observational results were fundamental to the planning of a good calendar, and the accuracy of these results depended greatly on the precision of the astronomical tools used"(厲之本在于測驗, 而測驗之器莫先儀表). Therefore, a lot of effort was put into the building of these astronomical tools, and the ways that Guo and his co-workers went about modifying some of the ancient astronomical tools to improve accuracy of the observations, and to simplify them so that they became much more easier to handle, were considered to be truly ingenious.

The use of an astronomical tool called the gnomon started very early in the history of Chinese astronomy. Historical records revealed that Chinese astronomers had been using this tool since the Shang Dynasty. The gnomon was very simple in its construction; it was actually a rod of a certain height placed vertically on flat ground. Then by measuring the noon shadows of the rod that was cast on the ground day by day, ancient Chinese astronomers could determine the time when the solstices occurred, by observing the changes in the daily measurements of the noon shadows of the rod (gnomon). For example, at the winter solstice, the length of the noon shadow of the gnomon would be at its longest. However, the gnomon itself had some major shortcomings. With a short gnomon, although it gave a much more distinct noon shadow, the length of its noon shadow would be short, and minor change in the change of the length of the noon shadow would be hard to detect. And since at the times of the solstices, the changes in the length of the noon shadow of the gnomon would be very small, a short gnomon would not be of much use in determining the times of the solstices.

With this in mind, it is not hard to see why Guo and his co-workers built a gnomon of 40 feet in height. According to historical records (the Yuan Shi), the vertical rod itself was actually 50 feet long, with 14 feet buried in ground for stability. Therefore there was 36 feet of the rod that was above ground. The other 4 feet was accounted for by two bronze dragons, which held between them a horizontal crossbar about 6 feet long aligned in the east-west direction. This rod was placed in the centre of an impressive building that can still be found at Dengfeng in the Henan Province. It was called the Zhuogong's Tower for the measurement of the Sun's Shadow (周公測景台), which is a brick tower about 9 meters high, the top being a square platform, called the Star Observation Platform (觀星台), that is about 8 meters by 8 meters, and the faces sloping so that the base of the tower is substantially larger than the top. The edges of the top platform are oriented in the north-south and the east-west directions. From the middle of the north edge, a vertical channel is cut in the north face of the tower, and from the bottom of this channel, which is vertically below the edge of the platform, runs a shadow-measuring scale over 120 feet long, which carries graduations that was used to measure the length of the noon shadows of the gnomon. This scale was accurately leveled by means of two water-filled grooves running at the sides of the scale throughout its whole length. This scale was called the Sky Measuring Scale (量天尺). When the gnomon was placed inside the building, which was vertically above the end of the Sky Measuring Scale, the top of the gnomon, which consisted of the 6 feet crossbar, would be around wrist level for someone working on the Star Observation Platform. Also, a plumb-line from the crossbar down the channel would enable the astronomers to find the zero point of the scale quite accurately.

With such a long gnomon, the noon shadows would be long, and the small change in the length of the noon shadows of the gnomon around the times of the solstices could be observed. However, the shadow would be light and ill-defined. Furthermore, look at Fig 1 (adapted from [9]). When the sun casts a shadow of the gnomon PQ, the stretch QR will be in deep shadow, and the stretch RS will be lit by part of the sun's disk, and the shadow grows progressively lighter from R to S, where it disappears altogether. The actual length of the shadow is TQ, where T is the point about halfway between R and S, which corresponds to the centre of the sun. However, it is impossible for the eye to judge the point T; in fact, the eye considers the shadow as ending at (or very close to) R, where the shadows begins to fade, and this leads to errors in the measurements of the length of the noon shadows of the gnomon.

To counter these problems, Guo and his co-workers came up with a device called the shadow definer (影符). According the Yuan Shi, it consisted of a small sheet of copper, with a hole about 2mm pierced through its centre. The

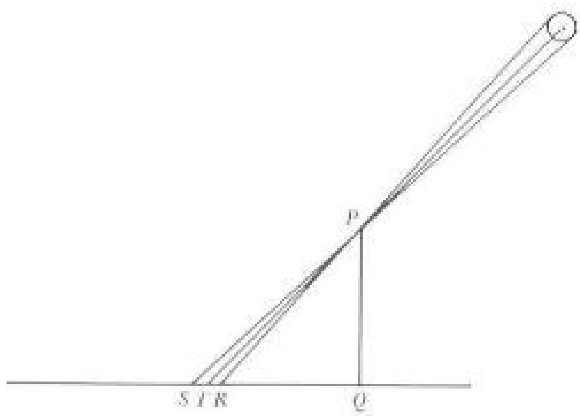

Figure 1: Determining point T, with QT being the actual length of the noon shadow of the gnomon

copper sheet was supported by a square framework, and was mounted on a pivot so that it could be freely turned to the angle where it was perpendicular to the sun's rays. The whole instrument was set on the Sky Measuring Scale in such a way that it could be moved along the scale (See Fig 2). To measure the noon shadows of the gnomon, the shadow definer was moved until the sun's rays shined through the hole of the copper sheets, producing a spot of light in the middle of the shadow of the copper sheet on the scale. The shadow definer was then slowly moved back and forth until the hole was in line with the crossbar, which was at the top of the gnomon, and the sun. When this was achieved, a small sharp image of the crossbar could be seen running across the spot of light that was on the scale. If the hole was slightly out of position, the image of the crossbar would cut the spot into two unequal parts. Then the position of the shadow definer was slowly adjusted on the scale until the image of the crossbar cut the spot of light exactly in half. This precise shadow on the spot of light would then point out on the scale where the exact tip of the noon shadows of the gnomon was. With this method, only the image of the crossbar was needed to measure the full length of the noon shadows of the 40 chi gnomon, and therefore the astronomers would not have to worry about light and ill-defined shadows created by the long gnomon. Moreover, by making use of the principle of the pin-hole to achieve the effect of focusing with the shadow definer, Guo and his co-workers were able to get a sharp distinct image of the crossbar which accurately pointed out the actual length of the noon shadows of the gnomon on the scale. Therefore, using the shadow definer, Guo and his co-workers were able to preserve the advantages of having a long gnomon and eliminating the disadvantages of it.

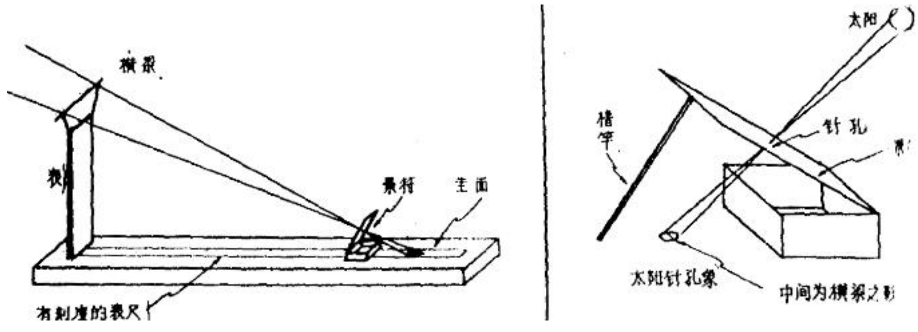

Figure 2: The Shadow Definer

Besides modifying the gnomon, Guo and his co-workers also brought modifications to the armillary sphere, another tool that was indispensable to Chinese astronomers. The armillary sphere consists of a number of rings that correspond to the various great circles of the celestial sphere, for example, the celestial equator, the ecliptic and the horizon. It is an instrument that was used by Chinese astronomers to determine the celestial positions of heavenly bodies. There are basically two types of armillary spheres: the equatorial armillary sphere and the ecliptic armillary sphere, which use different coordinate systems to determine the positions of the heavenly bodies. The equatorial armillary sphere, as its name implies, uses declination and right ascension which determine the positions of the heavenly bodies with respect to the celestial equator, whereas the ecliptic armillary sphere uses latitude and longitude which determine the positions of the heavenly bodies with respect to the ecliptic (the path that the sun moves across the sky as seen from the earth).

The equatorial armillary sphere basically contains three rings, with another additional ring to support the whole structure (see Fig 3). The axis CD is made to point towards the north celestial pole, which will in turn make the plane that ring PR lies on parallel to the plane of the celestial equator. The axis can be made to point towards the north celestial pole by making use of the north pole star. By pointing this axis directly towards this north pole star, the axis will then point towards the north celestial pole. Ring KLMN is mobile, which makes it possible to move about the axis that it is mounted to. It also has the sight Q on it which astronomers line up with the star under observation and the sight A on the axis PR. In this way, the proper bearing of the star can be achieved. Both ring KLMN and PR contain graduations, and thus when the required heavenly body is targeted using the sights, astronomers can then read off the north polar distance (the Chinese form of declination) of the heavenly body from the graduations on ring KLMN and the position of the heavenly body in a xiu (宿 , the Chinese form of right ascension) from the graduations on ring PR. Thus the positions of the heavenly bodies with respect to the equatorial coordinates can be determined. Later an ecliptic ring is added to the equatorial armillary sphere, but its function has always being secondary, mainly for demonstrational purposes.

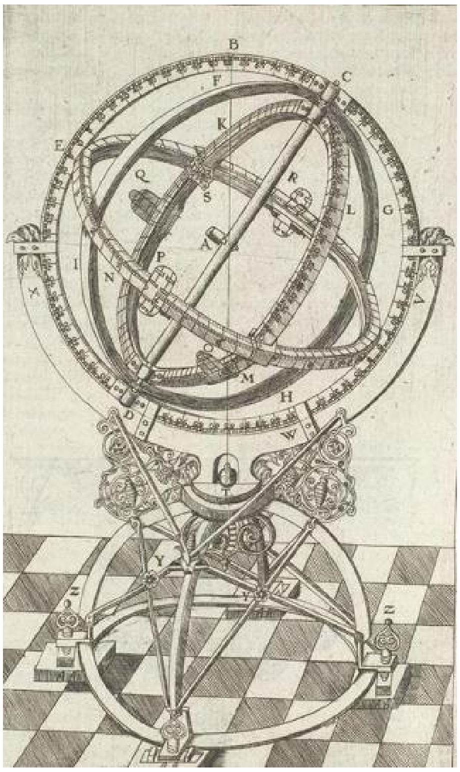

Figure 3: The Equtorial Armillary

The ecliptic armillary sphere, on the other hand, has a slightly different structure from that of the equatorial armillary sphere (see Fig 4 and 5). Ring B and C contain graduations, and ring B itself contains an inner mobile ring E (see Fig) which can be turned around ring B. It also has two sights S and T, which astronomers can use to target at the required heavenly body. The axis HK is still made to point towards the north celestial pole, and ring A and C are pivoted round the axis HK (note that ring A and C are connected together, which make up the inner framework). The angle between the axis HK and PQ is made to be the same as the angle between the ecliptic and the equator, and thus it is possible for ring C to swing into the plane of the ecliptic. To fix the position of this particular armillary sphere, the inner framework of ring A and C is turned about HK until the shadow of one half of ring C falls on the other half. This will make ring C parallel to the ecliptic. Then the ring B, together with its inner ring E, are individually moved until the sights are lined up with the targeted heavenly bodies. Once the required heavenly body is targeted, the sights will point out the latitude of the heavenly body on the graduations on ring B, and the reading of ring B itself on the graduations on ring C will give the particular heavenly body's longitude. The latitude and longitude of the heavenly body determine its position with respect to the ecliptic. Note that to differentiate between the two types of armillary spheres, we just need to see the number of axis they have. The equatorial armillary sphere has only one axis that the various rings move about, while the ecliptic armillary sphere has two axis that the various rings can move about.

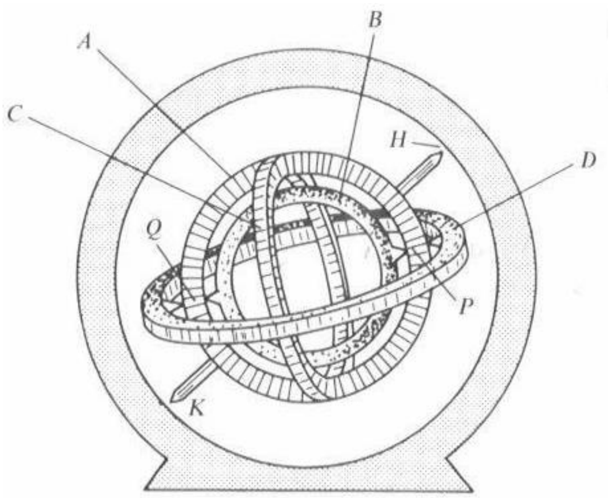

Figure 4: The Ecliptic Armillary

Chinese astronomers had always used equatorial armillary spheres because of their loyalty towards the use of the equatorial coordinates, which happens to be the standard coordinate system that is used by astronomers today around the world. European astronomers, however, started out with the ecliptic armillary spheres and used the ecliptic coordinates to determine the positions of the heavenly bodies, but later modified them into equatorial armillary spheres and switched to the equatorial coordinates. One of the reasons that led to this modification was that more accurate readings can achieved using the equatorial coordinates as compared to the ecliptic

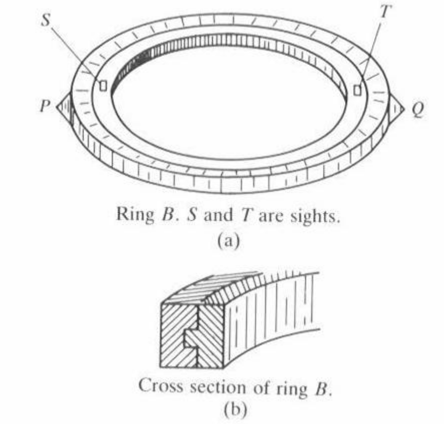

Figure 5: A Close-up look of Ring B

coordinates. The developments of the armillary spheres also included the introduction of various rings that correspond to other great circles of the celestial sphere. Guo and his co-workers built an equatorial armillary sphere which had rings representing many of the great circles of the celestial sphere (See Fig 6).

However, the main contribution that Guo and his co-workers had on the development of the armillary sphere was not to build such a complicated armillary sphere, but to modify its structure into a more simplified form. The new form of armillary "sphere" that Guo and his co-workers had built is shown in Fig 7 and 8. It was called the equatorial torquetum (簡儀). Clearly, it was no longer in a spherical form, however the equatorial torquetum still contains some familiar features that the equatorial armillary sphere have. Ring f is the mobile declination ring or meridian double circle (四游雙環), which graduates on both sides in degrees and minutes. It carries a sighting-tube, tube i, which can be moved around ring f, and is used for the determination of the declination of the required heavenly body. The sighting tube is pointed at one end to clearly mark out the reading on the ring f. The stretchers g, g' (直距), and the double brace h (横) are built in to prevent the deformation of the declination ring f. Ring j is the mobile equatorial circle (赤道環), which is also graduated in degrees and minutes, and has markings that mark out the boundaries of the 28 xiu. Like the declination ring f, it is also strengthened by cross-stretchers. The right ascension of the heavenly body is read off from this equatorial circle, and there are two independently movable radial pointers k, k', which have pointed ends to mark off the boundaries of the xiu that the heavenly body is in on the equatorial circle.

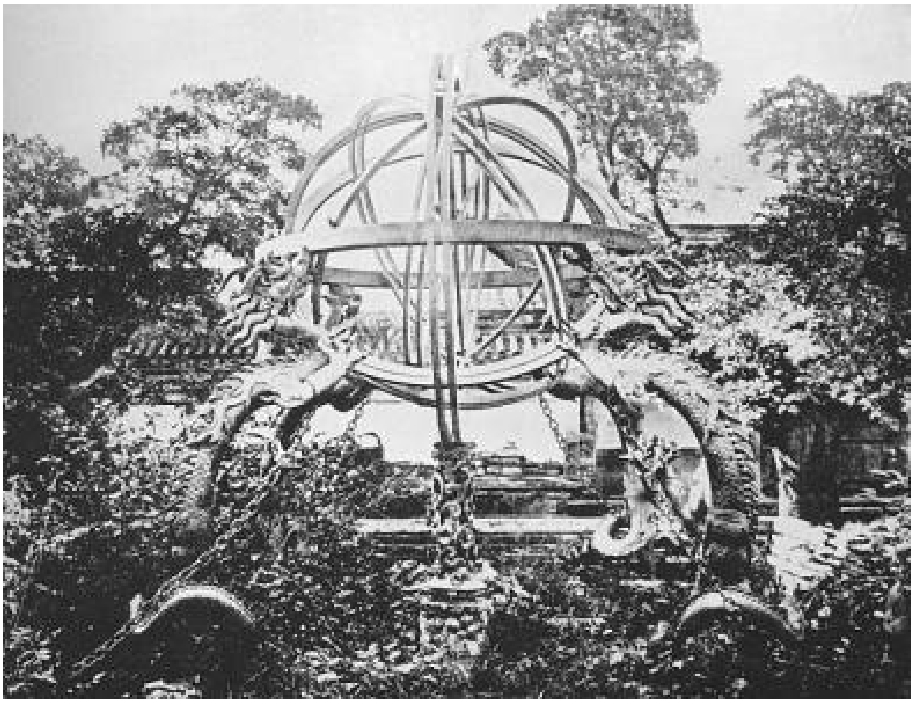

Figure 6: Guo Shoujing's Armillary Sphere

Apart from these usual features of the equatorial armillary sphere, this modified armillary contains some new features. The ring l is the pole determining circle (定极環), which has a cross-piece inside it with a central hole. Observation of the pole star is effected through a small hole in a bronze plate attached to the south pole cloud frame standards d, d' (南极云架]). The function of this pole determining circle is to determine the moment of culmination of the pole star. Also, it has the fixed terrestrial coordinate azimuth circle, m (陰緯環), and the revolving vertical circle n (立運環), with alidades, for the measurement of altitudes. These two rings will give the positions of the heavenly bodies with respect to the horizon. Thus, the equatorial torquetum is able to provide two systems of coordinates (the equatorial coordinates and the coordinates with respect to the horizon) to determine the positions of the heavenly bodies.

The advantage that this equatorial torquetum has over an equatorial armillary sphere containing all the rings that this torquetum have is that it separates the various rings apart, and thus avoiding confusion over the functions of the various rings. Also, the readings on the graduations of the rings can be taken easily as compared to that of the armillary sphere. Fur-

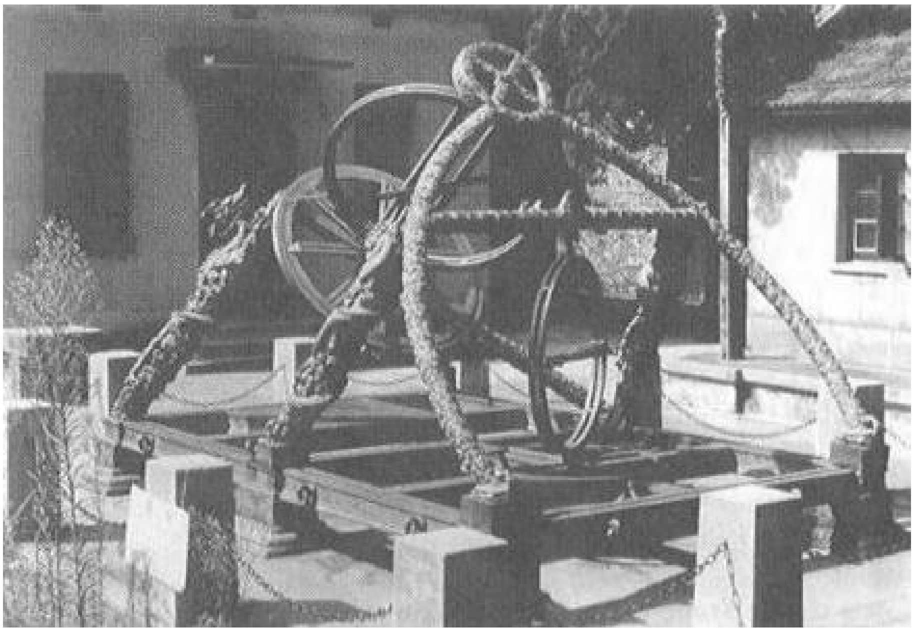

Figure 7: Guo Shoujing's Equatorial Torquetum

thermore, the construction of this equatorial torquetum make it more stable than the equatorial armillary sphere and resist deformation better, which is important since deformation in the structure of these instruments will lead to inaccuracy in the readings of the heavenly bodies' positions.

On the whole, Guo Shoujing and his co-workers were able to innovate when it came to the building of their instruments for measurements, and this brought about an increase in the accuracy of the observational results. Combining with the new mathematical methods that they had derived, more accurate calculations could be made based on these results.

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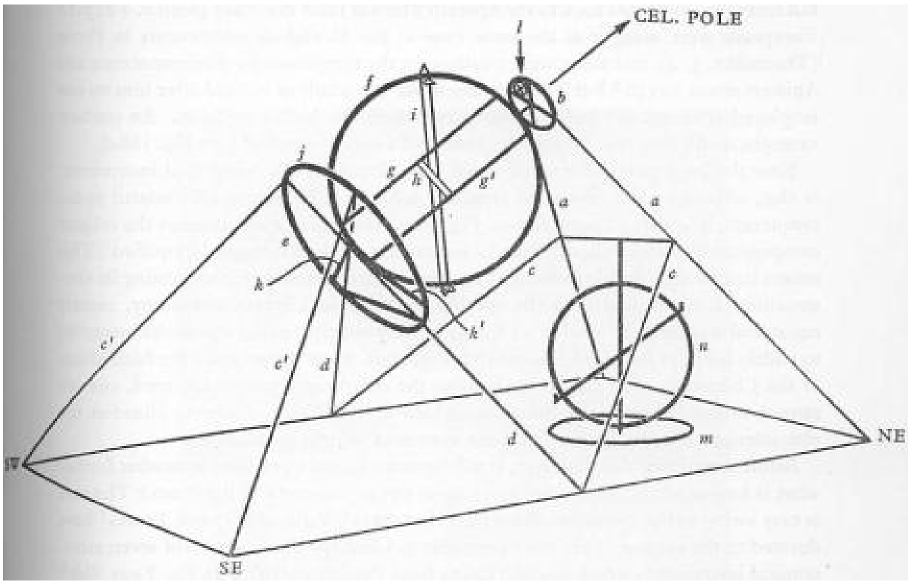

Figure 8: The Structure of the Equtorial Torquetum

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