Ordering Trees via Immanants - An Update

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Abstract. Let $\mathcal{T}_n$ be the set of trees on n vertices. It was proven in [9] that amongst the trees T in $\mathcal{T}_n$, the star $S(n)$ and the path $P(n)$ are the "smallest" and "largest" trees respectively, in the sense that for all immanants d_λ indexed by the partitions λ of n, we have $d_\lambda(L(S(n))) \leq d_\lambda(L(T)) \leq d_\lambda(L(P(n)))$, where $L(T)$ denotes the Laplacian matrix of T. In this project, we sharpen this result by proving that the above inequality is strict whenever $\lambda \neq (1^n), (2, 1^{n-2})$, and provide a simpler proof using the notion of vertex orientations. We also make use of the new tools to prove a similar ordering concerning the double stars and the 1-brooms.

1 Introduction

Let n be a positive integer. If $f : S_n \to \mathbb{C}$ is a function from the symmetric group S_n into the field of complex numbers $\mathbb{C}$, then the *generalized matrix function* d_f defined on the domain $M_n(\mathbb{C})$ of all $n \times n$ complex matrices is given by

$$d_f(A) := \sum_{\sigma \in S_n} f(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij}) \in M_n(\mathbb{C})$. For an irreducible character χ_λ of S_n associated with the partition λ of n, the function d_{χ_λ} is known as an *immanant*, written simply as d_λ for convenience. In particular,

$$d_\lambda(A) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Familiar examples of immanants include the determinant and permanent, which correspond to the alternating character $\chi_{(1^n)}$ and the trivial character $\chi_{(n)}$ respectively. The normalized immanant $\overline{d}_\lambda(A)$ is defined by $\overline{d}_\lambda(A) := d_\lambda(A)/\chi_\lambda(\mathrm{id})$, where id denotes the identity permutation in S_n. For hook partitions $\lambda = (k, 1^{n-k})$, the corresponding immanant $d_{(k,1^{n-k})}$ is known as a *hook immanant*, which we shall denote by d_k.

In 1917, I. Schur proved that for all irreducible characters χ_λ of S_n and $n \times n$ positive semi-definite (psd) matrices A, we have

$$0 = \det A \leq \overline{d}_\lambda(A).$$

Related to Schur's result is the famous *permanental-dominance* conjecture which asserts that for all irreducible characters χ_λ and psd A, we have

$$\overline{d}_\lambda(A) \leq \operatorname{per} A.$$

Let G be a graph on n vertices with vertex set $\{v_1, v_2, ..., v_n\}$. Its Laplacian matrix $L(G) = (l_{ij})$ is defined by

$$l_{ij} = \begin{cases} \deg_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j, \text{ and } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

The restriction of immanants to Laplacian matrices of graphs, which are a class of psd matrices, is particularly interesting as it entails a symbiotic interplay between the algebraic properties of the irreducible characters of S_n and the topological features of graphs. For instance, it was shown in [9] that for all immanants d_λ, where λ is a partition of n,

$$d_\lambda(L(S(n))) \leq d_\lambda(L(T)) \leq d_\lambda(L(P(n))), \tag{1}$$

where $S(n)$ denotes the star, $P(n)$ denotes the path, and T denotes an arbitrary tree on n vertices respectively.

In this project, we will restrict ourselves to Laplacian matrices of trees. Even in this reduced subset of graphs, interesting results abound. For example, it was shown in [2, 3] that for a given tree T on n vertices,

$$\overline{d}_{k-1}(L(T)) \leq \frac{k-2}{k-1}\overline{d}_k(L(T)),$$

for $k = 2, 3, ..., n$ - a refinement of the famous result by Heyfron (see [6]) that for all $n \times n$ psd matrices A,

$$\det A = \overline{d}_1(A) \leq \overline{d}_2(A) \leq ... \leq \overline{d}_n(A) = \operatorname{per} A.$$

We define an equivalence relation $\sim$ on the set of trees with n vertices, which we denote by $\mathcal{T}_n$, as follows:

$$T_1 \sim T_2 \Leftrightarrow \overline{d}_\lambda(L(T_1)) = \overline{d}_\lambda(L(T_2)) \ \forall \text{ immanants } d_\lambda.$$

A partial order $\leq$ on the equivalence classes $\mathcal{T}_n/\sim$ is defined by

$$T_1 \leq T_2 \Leftrightarrow \overline{d}_\lambda(L(T_1)) \leq \overline{d}_\lambda(L(T_2)) \ \forall \text{ immanants } d_\lambda.$$

(Note that there exist nonisomorphic trees on n vertices whose Laplacian matrix share the same immanant values for all immanants - see for example [9] - and so the partial order $\leq$ cannot be defined on $\mathcal{T}_n$ itself.) In this project when we speak of a tree $T \in \mathcal{T}_n$, we identify the tree T as a representative of the equivalence class of $\mathcal{T}_n/\sim$ to which it belongs. If for two trees T and T', we have $T \leq T'$, we say that tree T' *dominates* T.

Equation (1) is then equivalent to saying that for trees T with n vertices, $S(n) \leq T \leq P(n)$.

It is known that for any graph G on n vertices, $d_{(2,1^{n-2})}(L(G)) = 2e(T)\tau(G)$, where $e(G)$ is the number of edges in G and $\tau(G)$ is the number of spanning trees in G. (See [3, Theorem 4.1], [7, Theorem 2].) If $G = T$ is a tree on n vertices, then it is clear that $\overline{d}_{(2,1^{n-2})}(L(T)) = 2$, a constant. Furthermore it is also clear that $\overline{d}_{(1^n)}(L(T)) = \det L(T) = 0$. Hence in the ordering of trees via immanants, the immanants d_λ arising from the partitions $\lambda = (1^n)$ and $(2, 1^{n-2})$ need not be considered.

2 Vertex Orientations and Immanants of Laplacians of Trees

Let T be a tree with n vertices. Let $E(T)$ denote the edge set of T and $V(T) = \{v_1, v_2, ..., v_n\}$ be the vertices. A vertex orientation of T is obtained by assigning to each vertex an arrow pointing along one of its incident edges. Some edges will have two arrows on them, while it is possible that others might have none. A vertex orientation in which there are exactly j vertices with two arrows on them is known as a j-vertex orientation.

For $j = 0, 1, 2, ..., [n/2]$, we let $a_T(j)$ denote the number of j-vertex orientations that can be assigned to the tree T. In a j-vertex orientation, the edges that have two arrows on them form a j-matching.

Proposition 2.1. *For any tree T with n vertices, where $n > 1$, we have $a_T(0) = 0$ and $a_T(1) = n - 1$.*

Proof. Let $n > 1$. Since a tree T with n vertices has $n-1$ edges, so by the Pigeonhole Principle, at least one edge must have two arrows on it. Hence $a_T(0) = 0$.

Suppose T is assigned a 1-vertex orientation. Let e be the edge with two arrows. Then all the edges other than e have one arrow, and hence all the vertices must be directed towards the edge e. Since there are $n - 1$ possibilities for the choice of e, we have $a_T(1) = n - 1$. ☐

We now show that for a tree T, every immanant of its Laplacian matrix $L(T) = (l_{ij})$ can be expressed as a linear combination of the $a_T(j)$'s. By definition,

$$d_\lambda(L(T)) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} l_{i\sigma(i)}.$$

Each term $l_{i\sigma(i)}$ is non-zero if and only if $\sigma(i) = i$ or v_i and $v_{\sigma(i)}$ are adjacent. Since T is acyclic, we see that the term $\prod_{i=1}^{n} l_{i\sigma(i)}$ is zero if σ is a permutation containing cycles of length > 2 in its cycle decomposition. Hence we only need to consider the sum over those $\sigma \in S_n$ whose cycle decomposition contains only transpositions and fixed points.

Let M be a j-matching of T. For a vertex $v \in V(T)$, we use the notation $v \in M$ to mean that v is incident with one of the edges in M, and $v \notin M$ to mean otherwise. For $j = 0, 1, 2, ..., [n/2]$, the *weighted j-matching number* of T, denoted by $m_T(j)$, is defined to be

$$m_T(j) := \sum_{M \subseteq E(T)} \prod_{v \notin M} \deg_T(v),$$

where the sum is taken over all possible j-matchings M of T. Note that

$$m_T(0) = \prod_{v \in V(T)} \deg_T(v),$$

and if T has a perfect matching, then we define $m_T(n/2) := 1$. For convenience, we define $m_T(k) := 0$ for $k > [n/2]$.

For $j = 0, 1, 2, ..., [n/2]$, we set $\chi_\lambda(j) := \chi_\lambda(\sigma)$ if the permutation σ has cycle structure $(2^j, 1^{n-2j})$, that is, comprising exactly j transpositions and $n - 2j$ fixed points. In particular $\chi_\lambda(0) = \chi_\lambda(\mathrm{id})$. Using this notation, it is easy to check that

$$d_\lambda(L(T)) = \sum_{j=0}^{[n/2]} \chi_\lambda(j) m_T(j). \tag{2}$$

In order to express $d_\lambda(L(T))$ as a linear combination of the $a_T(j)$'s, we need the following lemma.

Lemma 2.2. *Let T be a tree with n vertices. Then*

$$m_T(j) = \sum_{i=j}^{[n/2]} \binom{i}{j} a_T(i) \tag{3}$$

for every $j = 0, 1, 2, ..., [n/2]$.

Proof. Let $Q_j = \{(M, O) \mid M$ is a j-matching and O is an i-vertex orientation with $i \geq j$ such that every edge in M has two arrows in the orientation $O\}$. We prove (3) by counting $|Q_j|$ in two different ways.

Given a j-matching M, we can form a vertex orientation O by directing the vertices $v \in M$ along the edges in M and orientating the remaining vertices arbitrarily. Clearly $(M, O) \in Q_j$. Since the vertices $v \notin M$ can be orientated in a total of $\prod_{v \notin M} \deg_T(v)$ possible ways,

$$|Q_j| = \sum_{M \subseteq E(T)} \prod_{v \notin M} \deg_T(v) = m_T(j),$$

where the sum is taken over all j-matchings M.

Given an i-vertex orientation O, where $j \leq i \leq [n/2]$, we can form a matching M by choosing any j of the i independent edges in O. There are $\binom{i}{j}$ ways of so doing, and so

$$|Q_j| = \sum_{i=j}^{[n/2]} \binom{i}{j} a_T(i),$$

and the result follows. □

Equation (3) allows us to express $d_\lambda(L(T))$ in terms of the $a_T(i)$'s. We have

$$\begin{aligned} d_\lambda(L(T)) &= \sum_{j=0}^{[n/2]} \chi_\lambda(j) m_T(j) \\ &= \sum_{j=0}^{[n/2]} \chi_\lambda(j) \sum_{i=j}^{[n/2]} \binom{i}{j} a_T(i) \\ &= \sum_{i=1}^{[n/2]} a_T(i) \sum_{j=0}^{i} \chi_\lambda(j) \binom{i}{j}. \end{aligned} \tag{4}$$

3 Binomial Sums of Characters of S_m

In light of equation (4), it is natural to examine the properties of sums of the form

$$\sum_{j=0}^{n} \chi_\lambda(j) \binom{n}{j}.$$

It turns out that these sums are always nonnegative. This will be a crucial result in establishing (1).

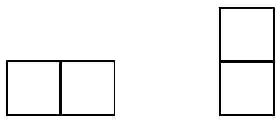
We will adopt the convention that $\binom{k}{j} = 0$ whenever $j < 0$ or $k < j$; and $\binom{0}{0} = 1$.

Theorem 3.1 (see [1, Lemma 3.1]). *Let χ_λ be an irreducible character of S_m, where $m \geq 2n$ and n is a positive integer. Define $l(\lambda)$ to be the number of parts in the partition λ. Then*

$$\sum_{j=0}^{n} \chi_\lambda(j) \binom{n}{j} \geq 0, \tag{5}$$

equality holding if and only if $m - n < l(\lambda)$.

Proof. In the language of Young Tableaux, let T_I and T_J denote rim hooks of the form indicated below.



T_I T_J

Let $I(\lambda)$, $J(\lambda)$ and $K(\lambda)$ denote the sets of partitions of $m-2$ that are obtained from λ by removing a rim hook of type T_I, a rim hook of type T_J, and two nonadjacent rim hooks of size 1, respectively, from the Young Tableau associated with λ. Using the Murnaghan-Nakayama Rule (see for instance [5, 8]), we have, for $0 \leq j \leq n-1$,

$$\begin{aligned} \chi_\lambda(j) &= \sum_{\mu \in I(\lambda)} \chi_\mu(j) + \sum_{\mu \in J(\lambda)} \chi_\mu(j) + 2 \sum_{\mu \in K(\lambda)} \chi_\mu(j) \ , \quad \text{and} \\ \chi_\lambda(j+1) &= \sum_{\mu \in I(\lambda)} \chi_\mu(j) - \sum_{\mu \in J(\lambda)} \chi_\mu(j). \end{aligned}$$

Therefore

$$\chi_\lambda(j) + \chi_\lambda(j+1) = 2 \sum_{\mu \in I(\lambda)} \chi_\mu(j) + 2 \sum_{\mu \in K(\lambda)} \chi_\mu(j). \tag{6}$$

For $n = 1$, the sum in inequality (5) is $\chi_\lambda(0) + \chi_\lambda(1)$. We readily observe from equation (6) that

$$\chi_\lambda(0) + \chi_\lambda(1) = 2 \sum_{\mu \in I(\lambda)} \chi_\mu(0) + 2 \sum_{\mu \in K(\lambda)} \chi_\mu(0) \ \geq \ 0.$$

Furthermore,

$$\begin{aligned} \chi_\lambda(0) + \chi_\lambda(1) = 0 \ &\Leftrightarrow \ \sum_{\mu \in I(\lambda)} \chi_\mu(0) = \sum_{\mu \in K(\lambda)} \chi_\mu(0) = 0 \\ &\Leftrightarrow \ I(\lambda) = K(\lambda) = \emptyset \\ &\Leftrightarrow \ \lambda = (1^m) \end{aligned}$$

Since $\lambda = (1^m)$ is the only partition with $l(\lambda) = m$, the stated condition is satisfied.

For larger values of n, we observe that

$$\begin{aligned}
\sum_{j=0}^{n} \chi_\lambda(j)\binom{n}{j} &= \sum_{j=0}^{n} \chi_\lambda(j)\left(\binom{n-1}{j-1} + \binom{n-1}{j}\right) \\
&= \sum_{j=0}^{n-1} (\chi_\lambda(j) + \chi_\lambda(j+1))\binom{n-1}{j} \\
&= \sum_{j=0}^{n-1} \left(2\sum_{\mu\in I(\lambda)} \chi_\mu(j) + 2\sum_{\mu\in K(\lambda)} \chi_\mu(j)\right)\binom{n-1}{j} \\
&= 2\sum_{\mu\in I(\lambda)}\sum_{j=0}^{n-1} \chi_\mu(j)\binom{n-1}{j} + 2\sum_{\mu\in K(\lambda)}\sum_{j=0}^{n-1} \chi_\mu(j)\binom{n-1}{j}.
\end{aligned} \tag{7}$$

Since all the irreducible characters χ_μ that appear in the final expression are characters of S_{m-2}, and $m-2 \geq 2(n-1)$, the conclusion

$$\sum_{j=0}^{n} \chi_\lambda(j)\binom{n}{j} \geq 0$$

follows inductively.

For the latter part of the theorem, we note that for all $\mu \in I(\lambda)$ and $K(\lambda)$, $l(\mu) = l(\lambda)$ or $l(\lambda) - 1$. If λ satisfies $m - n < l(\lambda)$, then $(m-2) - (n-1) < l(\mu)$. By inductive hypothesis, $\sum_{j=0}^{n-1} \chi_\mu(j)\binom{n-1}{j} = 0$ for all $\mu \in I(\lambda) \cup K(\lambda)$. It therefore follows from (7) that $\sum_{j=0}^{n} \chi_\lambda(j)\binom{n}{j} = 0$ for all λ satisfying $m - n < l(\lambda)$.

If λ satisfies $m - n \geq l(\lambda)$, then we claim that we can find some $\mu \in I(\lambda) \cup K(\lambda)$ such that $m - n - 1 \geq l(\mu)$. This is clearly true whenever $m - n - 1 \geq l(\lambda)$. If $l(\lambda) = m - n$, then $l(\lambda) \geq m/2$, and hence, the smallest part of λ is of size 1 or 2. This means there is some $\mu \in I(\lambda) \cup K(\lambda)$ such that $l(\mu) = l(\lambda) - 1 = m - n - 1$, and so our claim is proven. We now make the inductive assertion that such μ satisfies $\sum_{j=0}^{n-1} \chi_\mu(j)\binom{n-1}{j} > 0$, and hence the result follows. □

We state, as an important corollary, the case $n = 2$.

Corollary 3.2. *Let χ_λ be an irreducible character of S_m, where $m \geq 4$. Then*

$$\chi_\lambda(0) + 2\chi_\lambda(1) + \chi_\lambda(2) \geq 0,$$

equality holding if and only if $\lambda = (1^m)$ or $(2, 1^{m-2})$.

4 The Star is the Smallest Tree

In this section, we show that for all trees with n vertices, the star achieves the smallest immanant for all immanants d_λ, where λ is a partition of n.

Theorem 4.1. *Let $S(n)$ denote the star on n vertices, where $n \geq 4$, and T be an arbitrary tree on n vertices that is not the star. Then for all partitions λ of n other than (1^n) and $(2, 1^{n-2})$, we have*

$$d_\lambda(L(T)) > d_\lambda(L(S(n))).$$

Proof. By Proposition 2.1, $a_{S(n)}(1) = a_T(1) = n-1$. It is easy to see that for a tree T on n vertices, $a_T(2) > 0$ if and only if T is not the star $S(n)$. For partitions λ other than (1^n) and $(2, 1^{n-2})$, $\sum_{j=0}^{2} \chi_\lambda(j)\binom{2}{j} > 0$ by Corollary 3.2. Therefore using equation (4), we obtain

$$\begin{aligned} d_\lambda(L(T)) &= \sum_{k=1}^{[n/2]} a_T(k) \sum_{j=0}^{k} \chi_\lambda(j)\binom{k}{j} \\ &> (n-1)(\chi_\lambda(0) + \chi_\lambda(1)) \\ &= d_\lambda(L(S(n))). \end{aligned}$$

for $\lambda \neq (1^n), (2, 1^{n-2})$. □

This tells us that $S(n) \leq T$ for any $n \geq 4$, and in fact we have the strict inequality

$$d_\lambda(L(S(n))) < d_\lambda(L(T)) \;\; \forall \;\text{ partitions } \lambda \text{ of } n \neq (1^n), (2, 1^{n-2}).$$

5 Two Useful Results

Lemma 5.1 (see [9, Lemma 3.21]). *Let T and T' be trees on n vertices. If*

$$\prod_{v \in V(T)} \deg_T(v) > \prod_{v \in V(T')} \deg_{T'}(v),$$

then there exists some immanant d_λ for which $d_\lambda(L(T)) > d_\lambda(L(T'))$.

Proof. Let χ_{reg} denote the character arising from the regular representation of $G = S_n$. It is well known that

$$\chi_{reg} = \sum_\lambda \chi_\lambda(0)\chi_\lambda,$$

where the sum is taken over all partitions λ of n (recall that the complete set of irreducible characters χ_λ of S_n can be indexed by the partitions λ of n). It is easy to check that

$$d_{\chi_{reg}} = \sum_\lambda \chi_\lambda(0) d_\lambda. \tag{8}$$

Since for each $\sigma \in S_n$, $\chi_{reg}(\sigma)$ is given by

$$\chi_{reg}(\sigma) = \begin{cases} |S_n| = n! & \text{if } \sigma = \text{id}; \\ 0 & \text{otherwise}, \end{cases}$$

so we have, for each $A = (a_{ij}) \in M_n(\mathbb{C})$,

$$d_{\chi_{reg}}(A) = n! \prod_{i=1}^{n} a_{ii}.$$

In particular, for a tree T on n vertices with vertex set $\{v_1, v_2, ..., v_n\}$, we have

$$d_{\chi_{reg}}(L(T)) = n! \prod_{i=1}^{n} \deg_T(v_i). \tag{9}$$

It now follows from (8) and (9) that if $d_\lambda(L(T)) \leq d_\lambda(L(T'))$ for every λ, then

$$\prod_{v \in V(T)} \deg_T(v) \leq \prod_{v \in V(T')} \deg_{T'}(v).$$

We are done. □

Lemma 5.1 provides an expedient means of testing whether it is possible that one tree dominates another. In particular, if $\prod_{v\in V(T)} \deg_T(v) > \prod_{v\in V(T')} \deg_{T'}(v)$, then T' cannot dominate T.

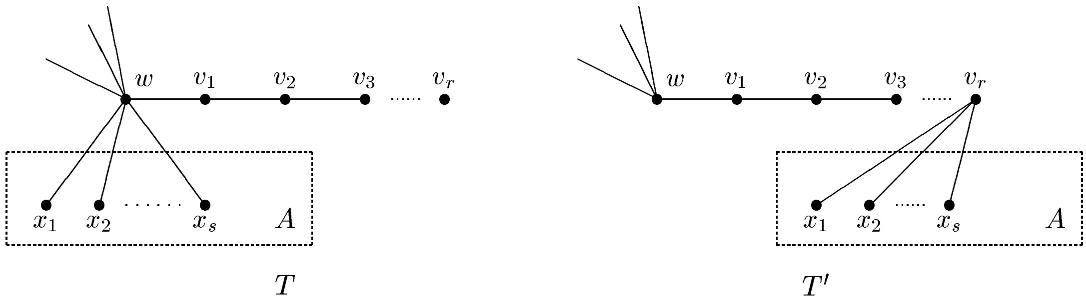
Theorem 5.2. *Let T be a tree on n vertices ($n \geq 4$) and let w be a vertex of T. Suppose w is adjacent to the vertices $x_1, x_2, ..., x_s$, $s \geq 1$. Let A be the set of all vertices v each of which is the end-point of some path of the form $\{w, x_i, ..., v\}$ for some $1 \leq i \leq s$, together with the vertices $x_1, x_2, ..., x_s$.*

Suppose further that $v_1, v_2, ..., v_r$, $r \geq 1$, are vertices of T with

$$\deg(v_1) = \deg(v_2) = ... = \deg(v_{r-1}) = 2, \ \deg(v_r) = 1,$$

$\{w, v_1, v_2, ..., v_r\}$ is a path in T, and all the v_i's are disjoint from A.

We construct a tree T' be removing the edges $\{w, x_1\}, \{w, x_2\}, ..., \{w, x_s\}$, and replacing with the edges $\{v_r, x_1\}, \{v_r, x_2\}, ..., \{v_r, x_s\}$.



Then we have

$$T \leq T'.$$

Furthermore, if $\deg_T(w) \geq s + 2$, then there is some immanant d_λ such that

$$d_\lambda(L(T)) < d_\lambda(L(T')).$$

Proof. Fix any integer j satisfying $1 \leq j \leq [n/2]$. To show that $T \leq T'$, it suffices, by equation (4) and Theorem 3.1, to show that $a_T(j) \leq a_{T'}(j)$. We prove this by establishing an injective map

$$\{ \ j\text{-vertex orientations of } T \ \} \longrightarrow \{ \ j\text{-vertex orientations of } T' \ \}.$$

Let M be any given j-vertex orientation of the tree T. We transform it to a j-vertex orientation of T' in the following manner.

Case 1. The vertex w does **not** point along **any** of the edges $\{w, x_1\}, \{w, x_2\}, ..., \{w, x_s\}$:

For all $1 \leq i \leq s$, if in M, x_i was pointing towards w, we now let it point towards v_r; otherwise we do not alter its orientation.

Case 2. The vertex w points along the edge $\{w, x_t\}$ for some $1 \leq t \leq s$:

In the new vertex orientation of T', we now let w point towards v_1. For the vertices $v_1, v_2, ..., v_{r-1}$, we "flip" their orientations by letting them point in the opposite direction. We let v_r point towards x_t. For all $1 \leq i \leq s$, if in M, x_i was pointing towards w, we now let it point towards v_r; otherwise we do not alter its orientation.

It is an easy exercise to check that our construction provides the necessary injective map, and so we have established that $T \leq T'$.

Suppose $\deg_T(w) = s + k$, where $k \geq 2$. To prove the second part of the lemma, we only have to note that

$$\frac{\prod_{v\in V(T)} \deg_T(v)}{\prod_{v\in V(T')} \deg_{T'}(v)} = \frac{s+k}{k(s+1)} < 1$$

and apply Lemma 5.1. □

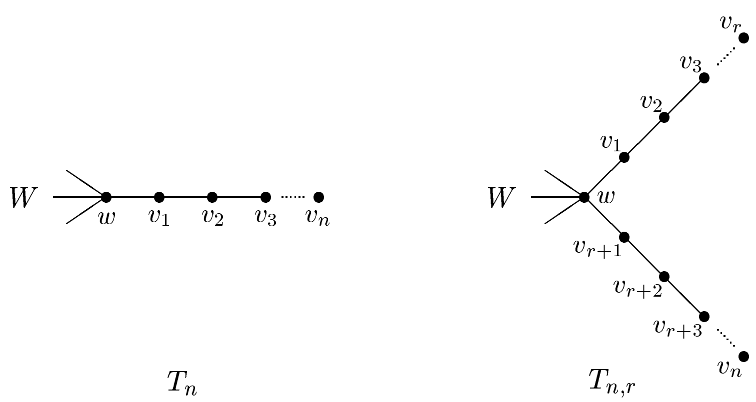
6 The Path is the Largest Tree

In this section, we show that for all trees with n vertices, the path $P(n)$ achieves the largest immanant for all immanants d_λ, where λ is a partition of n. In addition, the inequality is strict in the sense that given any tree T on n vertices, we can always find some immanant d_λ such that $d_\lambda(L(T)) < d_\lambda(L(P(n)))$.

Lemma 6.1. *Let W be an arbitrary tree of order ≥ 2 and w be a vertex of W. For integers n and r satisfying $1 \leq r < n$, we define the trees T_n and $T_{n,r}$ with vertex and edge sets as follows:*

$$\begin{aligned} V(T_n) = V(T_{n,r}) &= V(W) \ \sqcup \ \{v_1, v_2, ..., v_n\}; \\ E(T_n) &= E(W) \ \sqcup \ \{\{w, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n-1}, v_n\}\}; \\ E(T_{n,r}) &= E(W) \ \sqcup \ \{\{w, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{r-1}, v_r\}\} \\ &\qquad \sqcup \ \{\{w, v_{r+1}\}, \{v_{r+1}, v_{r+2}\}, \{v_{r+2}, v_{r+3}\}, ..., \{v_{n-1}, v_n\}\}. \end{aligned}$$

where the symbol $\sqcup$ means disjoint union of sets. In other words, T_n is obtained by appending a path of n vertices to w and $T_{n,r}$ is obtained by appending a path of r vertices and a path of $n-r$ vertices to w.



We have the following result:

$$T_{n,r} \leq T_n.$$

In addition, we can always find some immanant d_λ such that $d_\lambda(L(T_{n,r})) < d_\lambda(L(T_n))$.

Proof. This is but a special case of Theorem 5.2. □

If, in a tree T, there exist paths $\{w, v_1, v_2, ..., v_r\}$ and $\{w, v_{r+1}, v_{r+2}, ..., v_n\}$ for some $1 \leq r < n$ such that $\deg(v_i) \leq 2$ for every i and $\deg(v_r) = \deg(v_n) = 1$, let us call the tree T' obtained from T by replacing the edge $\{w, v_{r+1}\}$ with the edge $\{v_r, v_{r+1}\}$ a *majorant* of T. Lemma 6.1 tells us that $T \leq T'$.

Lemma 6.2. *Every tree T on n vertices ($n \geq 4$) that is not isomorphic to the path has a majorant.*

Proof. If w and w' are adjacent vertices of T, the set of all vertices v each of which is the end-point of some path of the form $\{w, w', ..., v\}$, together with w and w', generates a subtree of T in which w is an end-vertex. We shall denote this subtree by $T(w, w')$.

Since T is not isomorphic to the path $P(n)$, it admits a vertex of degree 3 or greater, say v. If for each vertex w adjacent to v, the subtree $T(v, w)$ admits only vertices of degree ≤ 2, we are done, for then there would exist two paths that start at v. If not, there would exist a vertex w adjacent to v such that $T(v,w)$ contains a vertex v' of degree ≥ 3. We may choose v' so that $d(v,v')$ is maximum. For each w' adjacent to v', if $T(v',w')$ does not contain v, then it is a subtree of $T(v,w)$, and hence by the maximality of $d(v,v')$, all its vertices are of degree ≤ 2. Since $\deg_T(v') \geq 3$, this shows there are two paths starting at v'. The proof is complete. □

Theorem 6.3. *Let $n \geq 4$, $P(n)$ denote the path on n vertices and T be an arbitrary tree on n vertices that is not the path. Then $T \leq P(n)$. In addition, we can always find some immanant d_λ such that $d_\lambda(L(T)) < d_\lambda(L(P(n)))$.*

Proof. By Lemma 6.2, T has a majorant, say T_1. If T_1 is isomorphic to $P(n)$, we are done, else we can find a majorant T_2 of T_1. This process must eventually terminate, so we get an ordered sequence of trees

$$T \leq T_1 \leq T_2 \leq ... \leq T_r \cong P(n).$$

By the second part of Lemma 6.1, we can find some immanant d_λ such that $d_\lambda(L(T_r)) < d_\lambda(L(P(n)))$. Since $T \leq T_r$, it also follows that $d_\lambda(L(T)) < d_\lambda(L(P(n)))$. We are done. □

7 Some Refinements and Observations

For the purpose of this section we derive an explicit formula for $\sum_{j=0}^{i} \chi_\lambda(j)\binom{i}{j}$ when λ is restricted to a hook partition.

Lemma 7.1 (see [2, Lemma 3.1]). *Let $\lambda = (k, 1^{n-k})$ be a hook partition of n, and let i be an integer with $0 \leq i \leq [n/2]$. Then*

$$\sum_{j=0}^{i} \chi_\lambda(j)\binom{i}{j} = 2^i \binom{n-i-1}{k-i-1}. \tag{10}$$

Proof. We first verify the expression for small values of i and k, then proceed by induction on i. For $k = 1$, $\chi_{(1^n)}(j) = (-1)^j$. Therefore we have

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{if } i \geq 1, \end{cases}$$

and so (10) holds. For $k = 2$ we use the well known formula

$$\chi_{(2,1^{n-2})}(j) = (-1)^j (n - 2j - 1)$$

(see for instance [8]) to obtain

$$\sum_{j=0}^{i} \chi_{(2,1^{n-2})}(j)\binom{i}{j} = \sum_{j=0}^{i} (-1)^j (n-2j-1)\binom{i}{j} = \begin{cases} n-1 & \text{if } i = 0; \\ 2 & \text{if } i = 1; \\ 0 & \text{if } i \geq 2, \end{cases}$$

and (10) holds.

Consider $3 \le k \le n$. For $i = 0$, the left-hand side of (10) becomes

$$\chi_{(k,1^{n-k})}(0) = \binom{n-1}{k-1},$$

which is the degree of the irreducible character $\chi_{(k,1^{n-k})}$. Consider $i = 1$. Using the Murnaghan-Nakayama Rule, we observe that

$$\begin{aligned}\chi_{(k,1^{n-k})}(1) &= \chi_{(k-2,1^{n-k})}(0) - \chi_{(k,1^{n-k-2})}(0) \\ &= \binom{n-3}{k-3} - \binom{n-3}{k-1}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\sum_{j=0}^{1} \chi_{(k,1^{n-k})}(j)\binom{1}{j} &= \chi_{(k,1^{n-k})}(0) + \chi_{(k,1^{n-k})}(1) \\ &= \binom{n-1}{k-1} + \binom{n-3}{k-3} - \binom{n-3}{k-1} \\ &= 2\binom{n-2}{k-2},\end{aligned}$$

agreeing with the right-hand side of (10). We now apply induction on i to obtain

$$\begin{aligned}\sum_{j=0}^{i} \chi_{(k,1^{n-k})}(j)\binom{i}{j} &= \sum_{j=0}^{i-1} \chi_{(k,1^{n-k})}(j)\binom{i-1}{j} + \sum_{j=1}^{i} \chi_{(k,1^{n-k})}(j)\binom{i-1}{j-1} \\ &= 2^{i-1}\binom{n-i}{k-i} + \sum_{j=1}^{i}\left[\chi_{(k-2,1^{n-k})}(j-1) - \chi_{(k,1^{n-k-2})}(j-1)\right]\binom{i-1}{j-1} \\ &= 2^{i-1}\binom{n-i}{k-i} + \sum_{j=0}^{i-1}\left[\chi_{(k-2,1^{n-k})}(j) - \chi_{(k,1^{n-k-2})}(j)\right]\binom{i-1}{j} \\ &= 2^{i-1}\left[\binom{n-i}{k-i} + \binom{n-i-2}{k-i-2} - \binom{n-i-2}{k-i}\right] \\ &= 2^{i}\binom{n-i-1}{k-i-1}.\end{aligned}$$

The proof is complete. □

We can now express the hook immanant $d_k(L(T))$ of a tree T strictly in terms of the $a_T(i)$'s and binomial terms. Using (4) and (10), we have

$$\begin{aligned}d_k(L(T)) &= \sum_{i=1}^{[n/2]} a_T(i)\left[\sum_{j=0}^{i} \chi_{(k,1^{n-k})}(j)\binom{i}{j}\right] \\ &= \sum_{i=1}^{[n/2]}\left[a_T(i)2^i\binom{n-i-1}{k-i-1}\right].\end{aligned} \tag{11}$$

The case $k = 3$ is of importance of us. We have

$$\begin{aligned}d_3(L(T)) &= a_T(1)\cdot 2\cdot\binom{n-2}{1} + a_T(2)\cdot 2^2\cdot\binom{n-3}{0} \\ &= 2(n-1)(n-2) + 4a_T(2).\end{aligned} \tag{12}$$

It was proven in [4] that

$$d_3(L(T)) = 4\left[\sum_{u,v\in V(T),\ u\neq v}(d(u,v)-1)\right]$$
$$= 4\left[\sum_{u,v\in V(T),\ u\neq v} d(u,v)\right] - 4\binom{n}{2} \qquad (13)$$
$$= 4\left[\sum_{u,v\in V(T),\ u\neq v} d(u,v)\right] - 2n(n-1).$$

Therefore equations (12) and (13) tells us that

$$2(n-1)(n-2) + 4a_T(2) = 4\left[\sum_{u,v\in V(T),\ u\neq v} d(u,v)\right] - 2n(n-1),$$

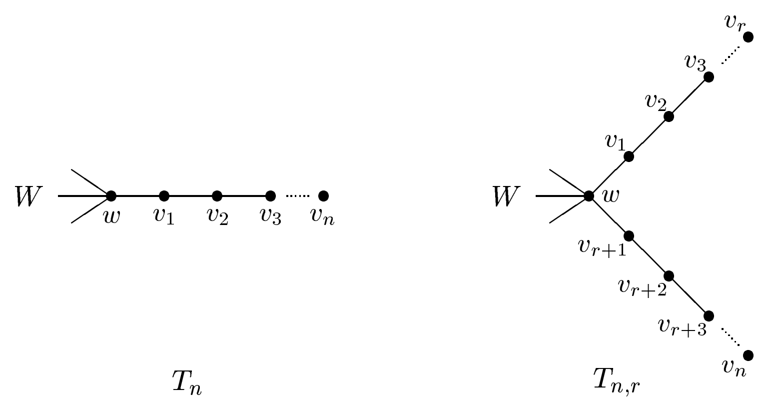
meaning that

$$\sum_{u,v\in V(T),\ u\neq v} d(u,v) = a_T(2) + (n-1)^2. \qquad (14)$$

Lemma 7.2. *Let W be a tree of order ≥ 2, w be a vertex of W, and T_n, $T_{n,r}$ be trees as defined in Lemma 6.1 with $1 \leq r < n$. Then*

$$a_{T_{n,r}}(2) < a_{T_n}(2).$$

Proof. We make a labelling of the vertices similar to that in Lemma 6.1:



From equation (14), it suffices to show that

$$A \ := \sum_{u,v\in V(T_{n,r}),\ u\neq v} d_{T_{n,r}}(u,v) \ < \sum_{u,v\in V(T_n),\ u\neq v} d_{T_n}(u,v) \ := \ B.$$

If u and v are vertices in W, then their contribution to the quantities A and B are equal. Also, if we set $S := \{w, v_1, v_2, ..., v_n\}$, then it follows by symmetry that

$$\sum_{u,v\in S,\ u\neq v} d_{T_{n,r}}(u,v) \ = \sum_{u,v\in S,\ u\neq v} d_{T_n}(u,v).$$

Finally we consider, without loss of generality, $u \in V(W) \setminus \{w\}$ and $v \in S \setminus \{w\}$. If $v \in \{v_1, v_2, ..., v_r\}$, then their contribution to A and B are equal. If $v \in \{v_{r+1}, v_{r+2}, ..., v_n\}$, then it is clear that

$$d_{T_{n,r}}(u, v) < d_{T_n}(u, v).$$

The result follows. □

Let T be a tree on n vertices, and let $T_1, T_2, ..., T_r$ be a sequence of trees such that T_{i+1} is a majorant of T_i for $1 \leq i \leq r-1$, and

$$T \cong T_1 \leq T_2 \leq ... \leq T_r \cong P(n).$$

By Lemma 7.2, $a_{T_i}(2) < a_{T_{i+1}}(2)$ for each $1 \leq i \leq r-1$, and so

$$a_T(2) < a_{P(n)}(2). \tag{15}$$

Examining the proof of Theorem 5.2 reveals that for all $1 \leq j \leq [n/2]$,

$$a_T(j) \leq a_{P(n)}(j). \tag{16}$$

Therefore we have the following result:

Theorem 7.3. *Let T be a tree on n vertices ($n \geq 4$) and λ be a partition of n other than (1^n) and $(2, 1^{n-2})$. Then*

$$d_\lambda(L(T)) < d_\lambda(L(P(n))).$$

Proof. Follows from equation (4), Theorem 3.1, equations (15) and (16). □

This is an improvement over Theorem 6.3.

Let p and q be positive integers with $p \leq q$. Take the star $S(p+1)$ and append $q-1$ leaves to one of the end-vertices. Call this a *double star* and denote it by $D(p, q)$. We have the following result:

Theorem 7.4. *For positive integers p and q ($p \leq q$) satisfying $p + q = n$, the double stars $D(p, q)$ are linearly ordered with respect to the ordering $\leq$. Furthermore,*

$$D(1, n-1) \leq D(2, n-2) \leq D(3, n-3) \leq ... \leq D([\tfrac{n}{2}], [\tfrac{n+1}{2}]).$$

Proof. It is easy to check that $a_{D(p,q)}(2) = (p-1)(q-1)$ and $a_{D(p,q)}(j) = 0$ for all $j \geq 3$. Furthermore, $a_{D(p,q)}(2)$ is a quadratic function of p that attains its maximum when $p = [n/2]$. The result is immediate. □

Let p, q and r be positive integers with $p \leq q$. Take the path $P(r+2)$ and append $p-1$ leaves to one of the end-vertices, and $q-1$ leaves to the other. Call this an r*-broom* and denote it by $B(r; p, q)$. We have the following result:

Theorem 7.5. *For positive integers p and q ($p \leq q$) satisfying $p + q + 1 = n$, the 1-brooms $B(1; p, q)$ are linearly ordered with respect to the ordering $\leq$. Furthermore,*

$$B(1;1,n-2) \leq B(1;2,n-3) \leq B(1;3,n-4) \leq ... \leq B(1;[\tfrac{n-1}{2}],[\tfrac{n}{2}]).$$

Proof. It is easy to check that $a_{B(1;p,q)}(2) = 2(p-1)(q-1)+p+q-2$ and $a_{B(1;p,q)}(j) = 0$ for all $j \geq 3$. Similar to Theorem 7.4, we have only to observe that $a_{B(1;p,q)}(2)$ is a quadratic function of p that attains its maximum when $p = [\frac{n-1}{2}]$. □

8 Open Problems

To conclude, we propose some interesting problems.

Problem 8.1. For any fixed r, are the r-brooms $B(r;p,q)$ $(p+q+r=n)$ linearly ordered with

$$B(r;1,n-r-1) \leq B(r;2,n-r-2) \leq B(r;3,n-r-3) \leq ... \leq B(r;[\tfrac{n-r}{2}],[\tfrac{n-r+1}{2}])?$$

Problem 8.2. Let T_1 and T_2 be trees with the same number of vertices. Suppose T_1 is a tree with bridge e and let A_{T_1} and B_{T_1} be the components of T_1 obtained by removing the edge e. Suppose T_2 is a tree with bridge f and let A_{T_2} and B_{T_2} be the components of T_2 obtained by removing the edge f.

Let T_1^n and T_2^n denote the trees obtained from T_1 (respectively T_2) by replacing the edge e (respectively f) with a path of length n. What are the necessary and sufficient conditions that A_{T_i} and B_{T_i} $(i=1,2)$ must satisfy in order that $T_1 \leq T_2$ implies $T_1^n \leq T_2^n$ for all n? For example it is known (see for example [9, Proposition 3.26]) that it is sufficient that $A_{T_1} \cong A_{T_2}$ and $B_{T_1} = B_{T_2} = \{v\}$, a single vertex.

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