Undergraduate Research Opportunities

Programme in Science

ANALYSIS OF KALAH

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ANALYSIS OF KALAH

1. ABSTRACT

The aim for this research will be to suggest possible winning strategy for the first player and possible counter attack for the second player on two cases of Kalah. The two cases of the game that will be analyzed are Kalah (1, m) and Kalah (n, 1). Kalah (1, m) is the case where there is only one pit and varying number of seeds in each pit. Kalah (n, 1) is the case where there is one seed and varying number of pits. Other than suggesting possible winning strategy and counter attack, any observations together with its proof and or trends of the two cases will be noted down as well.

2. INTRODUCTION

2.1 Naming of the Game

Depending upon where Kalah game is played and the culture in which it took root, the game has been given many names. These names refer to the manner of winning, the mode of play, the board used, or the seeds used. Kalah bears the meaning of loses in Indonesia.

The English name it as *Count and Capture*. It refers to the fact that there is no chance factors in the game. Player strategy is dependent upon ability to reason and count. Winning is based upon a player's ability to claim or capture an opponent's game seeds.

Islamic cultures name the game according to the physical action that takes place during the game. Thus, it has the name *Mancala* that is an Arabic word meaning in English "to move" or "to transfer".

In certain West African dialects, the depressions or cups in the board are referred to as *wari* or *awari* that bears the meaning of “houses” in English. Therefore, some cultures name the game as *Wari*.

Often, seeds used in the game are pebbles, ivory balls, coins, seeds, etc. this game is known as *Adi* in Nigeria due to the reason that the seeds used are the seeds from the *Adi* plant.

2.2 Brief Introduction of Kalah

Kalah is a modern, commercial variant of Mancala, introduced and made available to the public in 1950s by a firm of the name " The Kalah Game Company". In 1960, the first computerised version of the game was produced and many followed. Kalah has a relatively long history in Artificial Intelligence: Bell (1968) already used Kalah to demonstrate game played by computer, and Slagle and Dixon (1970) used Kalah to illustrate their M and N search algorithm. Nowadays, Kalah is often used as an example game in computer-science.

2.3 Distribution of Kalah

2.3.1 From Then Till Now

Kalah games are an extensive class of board games played today primarily in Africa, the Middle East, Asia, and the Caribbean area. Its origins are rooted in ancient Egypt and it can be traced to the Empire Age - about the 15th to 11th centuries B.C. The game spread from Egypt to many parts of Africa and then to the Middle East. As Muslim culture spread in the early A.D. centuries, it carried the game with it to India, Ceylon, Malaya, Indonesia, and the Philippines. It was carried from many Black African cultures during the period of the slave trade to the Caribbean area. Until recent times, the game was unknown in the non-Islamic parts of Europe and The Americans.

2.3.2Commercial Distribution of the Game

Nokia Hand phone – Bantumi - Bantumi is derived from "Mancala" which originated in Africa, thousands of years ago. This Nokia variant has four seeds in each pit at the start of the game. The objective of the game is to move your seeds from the pits at the bottom of the screen into your score pit on the right hand side.

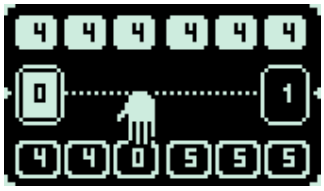

Figure 1. Bantumi. A classic start

3. ABOUT KALAH

The game usually involves two players (North and South), a board with two rows of six pits and two score pits called Kalahas, and 48 seeds. The board is placed between the two players. Each of the players takes 24 seeds and puts four pieces in each of the six pits on their side of the board. In this research, the case of Kalah (6,4) will not be analyzed. Analysis will be carried out mainly on Kalah (1, m) and Kalah (n, 1).

The two larger pits to the right of the player at each end of the board are known as the Kalaha. The Kalaha are left empty at the beginning of the game. The first player scoops up all the pieces from any non-empty pit on his side of the board and sows in anti-clockwise direction, starting with the next pit.

3.1 Rules

i. The player can start his move from any non-empty pit from his side of the board.

ii. The player cannot start his move using the pieces on the opponent's side of the board.

iii. The player's Kalaha, the player's pits and the opponent's pits are included in the sowing.

iv. The player cannot sow any of his seeds into the opponent's Kalaha.

v. If a seed lands in the player's Kalaha, his score increases by 1 and he retain the right to continue playing.

vi. If the last seed does not end up in the player's Kalaha, the player loses his turn.

vii. The seeds that are being captured or the seeds that has entered both players' Kalaha do not re-enter the game. The game value of this game thus depends on the configuration of the active seeds or seeds that are not captured.

3.2 Possible Outcomes of a Move

i. Home or Kalah-move - If the last seed lands in his score pit, the player will have another turn.

ii. Capture - If the last seed lands in an empty pit on the player's side of the board, it is added to his total and any seeds in the opposite pit are added as well. The player's turn is then over.

iii. Null Capture – If the last seed played ends in an empty pit on the player's side and the opponent's pit that is directly opposite is also empty. For this case, the last seed is still placed in the player's Kalaha and the player's turn is over.

iv. If the last seed is put anywhere else, the turn is over directly.

v. End Game - If the player is left with no seeds on his side, the game is over. The remaining seeds are added automatically to the opponent's score pit.

3.3 Observations

i. The game value of this game decreases as with each turn that any of the two players get to sow.

Proof:

For every turn that any of the two players get to sow his seeds from any pit that he chooses, the number of seeds that are not captured or has not entered into the players' Kalaha decreases. Since the game value depends on the configuration of the active seeds, game value of the game decreases with each sowing that is done.

ii. The player's score increases by at least 1 for every capture he makes.

Proof:

The rule states that the player can get to capture the opponent's seed or seeds opposite to the pit that his last seed lands if the last seed lands in an empty pit on his side. There are basically two parts to this proof for the cases that are investigated. The cases that will be investigated are Kalah (1, m) and Kalah (n, 1). For both the cases, these two parts of the proof applies:

Assuming that the player's last seed lands in an empty pit on his side

a. The opponent's pit that is opposite to the pit where the player's last seed lands is also empty.

For this case, the player is able to capture his own seed. This proves that if this case is to happen, the player is able to capture one seed. Thus, the observation is valid.

b. The opponent's pit that is opposite to the pit where the player's last seed lands contains at least one seed.

In this case, the player gets to capture his one seed ($i = 1$) and the opponent's seed ($j \geq 1$)

m = number of seeds captured

$= i+j$

$\geq 1+1$

≥ 1

Thus, the observation is valid.

In conclusion, every capture that a player makes will increase the score of the player by at least 1.

iii. If in the case for normal play, player 1 starts by moving the seed that will end up in his home, he will lead by at least one capture. Thus, he can optimise his chances of winning by leading by at least one capture.

iv. There are five types of moves that a player can make:

a. Move to capture – This is a move to capture at least itself (1 seed) if the opponent's pit opposite to the player's pit where his last seed lands is empty. If the opponent's pit is non-empty, the player can capture at least 2 seeds.

b. Move to defend – This is a move to prevent a capture of a seed that is exposed to the opponent in the immediate move that the opponent might make.

c. Mirror the move of the opponent - This is a move where the player imitates the moves made by his opponent.

d. Move to go home – This is a move where the last seed of the player lands in his Kalaha or moving of the seeds that are closer to his Kalaha.

e. Random move – This is a move other than those described above.

3.4 Objective of the Game

Aims to be the player with the highest number of seeds in his Kalaha.

3.5 Confusing Variations

i. The game usually opens with 4 seeds in every pit, but other amounts (2,3,5,6) are possible.

ii. Two players normally play the games but there are variants that allow one or three players to play the game.

iii. Opponent's Kalaha is not skipped while sowing seeds.

iv. The player captures the seeds in the reached pit as well as those previous pits that match the criteria without taking in account the territorial limits or take the limits of the territory into account.

v. Flow Capture - This consists of the capturing of the seeds of some pits during the player's sowing. If allowed, both the player and the opponent can use that technique. Flow capture can capture seeds only into his own territory.

vi. Multiple sowing - As the last seeds are dropped, the player grabs all the seeds in this pit and starts to sow again. This ends when the last dropped seeds set the number of the seeds in the pit to one in any territory.

4. KALAH (1, m)

The rules are stated as above in Section 3. For the case of Kalah (1, m), instead of having six pits on each side of the board and 4 seeds per pit, we have 1 pit on each side of the board and we varies the number of seeds in the pit.

4.1 Normal Play

Table 1. Kalah (1, m) following the rules of Normal Play. $1 \leq m \leq 75$

M	endgame	Status	L/D/W	m	endgame	status	L/D/W	m	endgame	status	L/D/W
	P1	P2			P1	P2			P1	P2	
1	1	1	D	**26**	11	41	L	**51**	39	63	L
2	1	3	L	**27**	29	25	W	**52**	46	58	L
3	6	0	W	**28**	34	22	W	**53**	51	55	L
4	3	5	L	**29**	22	36	L	**54**	53	55	L
5	6	4	W	**30**	38	22	W	**55**	62	48	W
6	6	6	D	**31**	62	0	W	**56**	60	52	W
7	4	10	L	**32**	30	34	L	**57**	59	55	W
8	8	8	D	**33**	40	28	W	**58**	64	52	W
9	7	11	L	**34**	25	43	L	**59**	52	66	L
10	20	0	W	**35**	41	29	W	**60**	66	54	W
11	11	11	D	**36**	36	36	D	**61**	61	61	D
12	7	17	L	**37**	29	45	L	**62**	64	60	W
13	8	17	L	**38**	38	38	D	**63**	65	61	W
14	14	14	D	**39**	44	34	W	**64**	77	51	W
15	9	21	L	**40**	21	58	L	**65**	70	60	W
16	11	21	L	**41**	45	37	W	**66**	60	72	L
17	14	20	L	**42**	42	42	D	**67**	35	99	L
18	20	16	W	**43**	32	54	L	**68**	44	92	L
19	21	17	W	**44**	44	44	D	**69**	57	81	L
20	26	14	W	**45**	39	51	L	**70**	70	70	D
21	26	16	W	**46**	62	30	L	**71**	73	69	W
22	14	32	L	**47**	49	45	W	**72**	72	72	D
23	8	38	L	**48**	55	41	W	**73**	65	81	L
24	18	30	L	**49**	47	51	L	**74**	70	78	L
25	32	18	W	**50**	65	35	W	**75**	76	74	W

Table 2. Kalah (1, m) following the rules of Normal Play. $76 \le m \le 150$

M	endgame status		L/D/W	m	endgame status		L/D/W	m	endgame status		L/D/W
	P1	P2			P1	P2			P1	P2	
76	76	76	D	**101**	111	91	W	**126**	133	119	W
77	76	78	L	**102**	143	61	W	**127**	133	121	W
78	78	78	D	**103**	112	94	W	**128**	122	134	L
79	83	75	W	**104**	107	101	W	**129**	129	129	D
80	61	99	L	**105**	105	105	D	**130**	129	131	L
81	70	92	L	**106**	112	100	W	**131**	130	132	L
82	89	75	W	**107**	109	105	W	**132**	133	131	W
83	67	99	L	**108**	105	111	L	**133**	138	128	W
84	55	113	L	**109**	111	107	W	**134**	80	188	L
85	84	86	L	**110**	119	101	W	**135**	115	155	L
86	86	86	D	**111**	114	108	W	**136**	132	140	L
87	84	90	W	**112**	122	102	W	**137**	143	131	W
88	86	90	L	**113**	112	114	L	**138**	139	137	W
89	89	89	D	**114**	114	114	D	**139**	138	140	L
90	116	64	W	**115**	108	122	L	**140**	138	142	L
91	86	96	L	**116**	114	118	L	**141**	136	146	L
92	92	92	D	**117**	119	115	W	**142**	141	143	L
93	91	95	L	**118**	124	112	W	**143**	134	152	L
94	188	0	W	**119**	117	121	L	**144**	156	132	W
95	93	97	L	**120**	117	123	L	**145**	138	152	L
96	114	78	W	**121**	63	179	L	**146**	144	148	L
97	101	93	W	**122**	120	124	L	**147**	149	145	W
98	100	96	W	**123**	125	121	W	**148**	150	146	W
99	92	106	L	**124**	129	119	W	**149**	136	162	L
100	102	98	W	**125**	128	122	W	**150**	150	150	D

Table 3. Kalah (1, m) following the rules of Normal Play. $151 \leq m \leq 225$

m	endgame		status	m	endgame		status	m	endgame		status
	P1	P2			P1	P2			P1	P2	
151	146	156	L	**176**	175	177	L	**201**	194	208	L
152	134	170	L	**177**	184	170	W	**202**	103	301	L
153	159	147	W	**178**	191	165	W	**203**	206	200	W
154	153	155	L	**179**	171	187	L	**204**	204	204	D
155	161	149	W	**180**	181	179	W	**205**	155	255	L
156	146	166	L	**181**	182	180	W	**206**	208	204	W
157	155	159	L	**182**	187	177	W	**207**	214	200	W
158	158	158	D	**183**	190	176	W	**208**	214	202	W
159	158	160	L	**184**	186	182	W	**209**	212	206	W
160	243	77	W	**185**	185	185	D	**210**	209	211	L
161	155	167	L	**186**	193	179	W	**211**	213	209	W
162	155	169	L	**187**	182	192	L	**212**	71	353	L
163	186	140	W	**188**	186	190	L	**213**	205	221	L
164	153	175	L	**189**	188	190	L	**214**	212	216	L
165	151	179	L	**190**	196	184	W	**215**	215	215	D
166	158	174	L	**191**	194	188	W	**216**	223	209	W
167	172	162	W	**192**	184	200	L	**217**	223	211	W
168	168	168	D	**193**	196	190	W	**218**	262	174	W
169	169	169	D	**194**	186	202	L	**219**	217	221	L
170	189	151	W	**195**	196	194	W	**220**	220	220	D
171	175	167	W	**196**	88	304	L	**221**	222	220	W
172	175	169	W	**197**	183	211	L	**222**	226	218	W
173	167	179	L	**198**	201	195	W	**223**	223	223	D
174	172	176	L	**199**	204	194	W	**224**	222	226	L
175	178	172	W	**200**	198	202	L	**225**	235	215	W

Table 4. Kalah (1, m) following the rules of Normal Play. $226 \leq m \leq 300$

m	endgame	status	L/D/W	m	endgame	status	L/D/W	m	endgame	status	L/D/W
	P1	P2			P1	P2			P1	P2	
226	225	227	L	**251**	249	253	L	**276**	269	283	L
227	223	231	L	**252**	249	252	L	**277**	275	279	L
228	240	216	W	**253**	243	263	L	**278**	285	271	W
229	230	228	W	**254**	253	255	L	**279**	280	278	W
230	231	229	W	**255**	242	268	L	**280**	287	273	W
231	212	250	L	**256**	227	285	L	**281**	290	272	W
232	227	237	L	**257**	260	254	W	**282**	282	282	D
233	208	258	L	**258**	248	268	L	**283**	566	0	W
234	211	257	L	**259**	256	262	L	**284**	282	286	L
235	238	232	W	**260**	190	330	L	**285**	283	287	L
236	228	244	L	**261**	261	261	D	**286**	290	282	W
237	223	251	L	**262**	291	233	W	**287**	274	300	L
238	171	302	L	**263**	262	264	L	**288**	294	282	W
239	238	240	L	**264**	270	258	W	**289**	287	291	L
240	227	253	L	**265**	211	319	L	**290**	309	271	W
241	240	242	L	**266**	267	265	W	**291**	291	291	D
242	237	247	L	**267**	262	272	L	**292**	270	314	L
243	234	252	L	**268**	267	269	L	**293**	295	291	W
244	244	244	D	**269**	269	269	D	**294**	284	304	L
245	239	251	L	**270**	275	265	W	**295**	302	288	W
246	267	225	W	**271**	269	273	L	**296**	291	301	L
247	249	245	W	**272**	273	271	W	**297**	304	290	W
248	243	253	L	**273**	361	185	W	**298**	296	300	L
249	246	252	L	**274**	277	271	W	**299**	301	297	W
250	249	251	L	**275**	275	275	D	**300**	291	309	L

Some of the results are highlighted. The ones that are in red is the case where player one wins all the seeds. The ones in purple are when the difference in the number of seeds between the two players is more than 100. And the ones in blue are those where the game ends up in a draw between the two players.

Verification of Steps for Selected Cases

m = 1 D

0 1 → 0 1 ⇒ 1 0
1 0 0 1 0 1

m = 31 W

0 31 → 0 40 → 0 44 → 0 0
31 0 10 11 3 15 0 62

m = 85 L

0 85 → 0 113 → 0 122 → 0 125 ⇒ 42 41
85 0 28 29 9 39 3 42 45 42

→ 42 56 ⇒ 61 18 → 61 29 → 61 33 ⇒ 72 11
15 57 34 57 11 69 3 73 14 73

→ 72 16 ⇒ 78 5 ⇒ 80 1 → 80 5 ⇒ 82 1
4 78 9 78 11 78 3 82 50 82

→ 82 3 ⇒ 86 0
1 84 0 84

4.1.1Possible Strategies for Player 1

For this case, there exist no possible strategies for player 1. Player 1 can only sow his seeds and lose his turn if the last seed does not end in his Kalaha. Player 1 will regain his turn when player 2 loses turn. This will continue until one of the players does not have any more counter in his pits. The remaining seeds will then go into the opponent's Kalaha. The player with the larger number of seeds is the winner. If both players have the same number of seeds, it is a draw.

4.1.2 Possible Strategies for Player 2 (Counter Attacks)

There is no possible counter attacks available for player 2 in this case.

4.1.3 Observations

It can be observed that

i. If a player starts off with 0 (mod 3) number of seeds, his last seed will end in his pit.

ii. If a player starts off the game with 1 (mod 3) number of seeds, his last seeds will end in his Kalaha.

iii. If a player starts off the game with 2 (mod 3) number of seeds, his last seeds will end up in his opponent's pit.

iv. The result for the game where both players has one seed each is always a draw.

Proof:

In this case, player one can only move his seed back home. Once that move is made, player one will has no more seed in his pit.

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \rightarrow \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \Rightarrow \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

Following the rule for this game, player two gets to bag all the remaining seed. Thus, for this configuration, it is always a draw for both players. This applies for both form of Kalah that is the normal play and the alternate play.

v. The results for the game where the player starts with $3^0 + 3^1 + 3^2 + \ldots + 3^m$ [decimal representation] or 111…1(ternary representation) number of seeds is always a lost for player 1 excluding the case where there is only one seed. The ternary representation for these numbers is of this form: *11…1*which has k number of "1"s, k>1.

Two observations can be seen from the Kalah that started with 11…1 (ternary representation) number of seeds.

a. For this starting configuration of 11…1 that has k number of "1"s, the number of turns that P1 gets to retain is equivalent to the number of "1"s that P1 has.

This is a trend that can be seen easily by working out number of cases of Kalah with a starting configuration of 11…1(ternary configuration).

The following example illustrates that the number of turn that P1 gets to retain is equal to the number of "1"s he has.

For e.g.:

<u>m = 1, n=4, k=2</u>

$$\begin{array}{cccc} 0 & 11 & \rightarrow & 0 \quad 1+11 \\ & 11 \quad 0 & H1 & 1 \quad 1(decimal)+1 \end{array}$$

P1 gets to retain his turn for the first time.

$$\begin{array}{ccc} \rightarrow & 0 & 1+11 \\ H2 & 0 & 1(decimal)+1 \end{array}$$

P1 retains his turn for the second time.

(Unless otherwise stated, the "1"s is of base 3)

<u>Proof:</u>

When k =2, the above example has shown that the number of turns that P1 gets to retain is equivalent to the number of "1"s P1 has.

Assuming that when P1 has a starting configuration of 11…1 that has x number of '1"s, P1 gets to retain his turn for x times. Want to show that when P1 has a starting configuration of 11…1 that has x+1 number of "1"s, he gets to retain x+1 number of turns:

It is not difficult to notice that with each round of sowing that P1 gets, the number of seed that he has in his pit decrease by one "1". As illustrated by the above example, after the first sowing back to home, the number of seeds in P1's pit decreases from 11(ternary representation) to 1(ternary representation). Thus, after one round of sowing, the number of seeds that is in P1's pit is 11…1(ternary representation) where there is x number of "1"s.

We have assumed earlier on that if P1 has a starting configuration of 11…1(ternary representation) that has x number of "1"s, P1 gets to retain his turn for x times. As a result, P1 gets to retain his turn for x+1 times altogether.

By mathematical induction, P1 gets to retain the number of turns that is equivalent to the number of "1"s he has for his

b. P2 always gets to win with such configuration that implies it is always a loss for P1.

Proof:

For such cases, both players always have 1 (mod 3) number of seeds. Notice that all this while, the player always gets another turn to play as his last seed always ends up in his Kalaha and the opponent is unable to make any move.

Though the player gets another turn since the last seed always ends up in his Kalaha, for every round of sowing that he gets, he gains $\frac{1}{3}$ of what he started with and loses $\frac{1}{3}$ to his opponent. Another $\frac{1}{3}$ will end up in his pit. Eventually, he will get to the state where he has 4 seeds in his pit. After this round of sowing, he ends up with an empty pit. The rules state that if a player is left with no seeds, the opponent will capture all the remaining seeds. Thus, in order to proof that the player will always loses, it is sufficient to proof that the number of seeds that the player has in his Kalaha is always smaller than the number of seeds that the opponent has in the opponent's pit.

Assuming the player started the game with k number of "1"s seeds:

The recursive formula for the number of seeds, written in its base 3 or ternary representation, in the player's Kalaha is of the form:

$$1 + 11 + \ldots + \textit{11}\ldots\textit{1} + k \qquad (1)$$

where *11…1* (ternary representation) has k-1 number of "1"s and k is any positive integers of base 10.

The recursive formula for the number of seeds in the opponent's Kalaha will be:

$$1 + 11 + \ldots + \underline{\textit{11}\ldots\textit{1}} \qquad (2)$$

where $\underline{\textit{11}\ldots\textit{1}}$ has k number of "1"s and k is any positive integers of base 10.

The recursive formula for the difference in the number of seeds between the two players is:

$$\underline{\textit{11}\ldots\textit{1}} - k \qquad (3)$$

where $\underline{\textit{11}\ldots\textit{1}}$ has k number of "1"s and k is any positive integers of base 10.

P2 gets to win the game by $\underline{\textit{11}\ldots\textit{1}} - k$.

Since the rules state that the opponent will get to pocket all the seeds when the player no longer can make any more move and we know that the number of seeds in the opponent's pit which eventually equates to the number of seeds in the opponent's Kalaha is always bigger than the number of seeds in the player Kalaha, it is clear that the player will always loses.

Note: With each sowing, the number of 1 for the ternary representation of the number of seeds decreases. For e.g.: if player one starts off with 13 seeds that have the ternary representation of 111. This ternary representation has 3 "1"s. After one sowing, the number of seeds in the player's pit is 4. 4 in terms of ternary representation are 11. This has one "1" less than the ternary representation of 13.

vi. The player who start off with 10111…1 (ternary representation) number of seeds always wins all the seeds that is on the board.

E.g.: 3 have a ternary representation of 10

10 is of the ternary representation of 101

31 is of the ternary representation of 1011

Proof:

With each sowing, the number of "1"s after "10" decreases. As proven in above, with this strings of "1"s, the player gets to retain his turn. This will continue till the player is left with 3 seeds in his pit. The sowing of 3 seeds will result in the last seed landing in his empty pit. Following the rule, the player gets to capture the opponent's seeds that are opposite to this empty pit in which the last seed lands. This result in the player capturing all the seeds that is on the game board.

For e.g.:

0	31		→	0	40		→	0	44		→	0	0						
	31	0			10	11			3	15			0	62					

P1 wins all the seeds that are available on the board.

4.2 Alternate Play

The additional rule is that the player loses his turn after each move even if the move is a move into his Kalaha.

Table 5. Kalah (1, m) following the rules of Alternate Play. $1 \le m \le 75$

m	endgame status		L/D/W	m	endgame status		L/D/W	m	endgame status		L/D/W
	P1	P2			P1	P2			P1	P2	
1	1	1	D	**26**	27	25	W	**51**	50	52	L
2	1	3	L	**27**	28	26	W	**52**	52	52	D
3	6	0	W	**28**	29	27	W	**53**	52	54	L
4	6	2	W	**29**	28	30	L	**54**	53	55	L
5	7	3	W	**30**	28	32	L	**55**	54	56	L
6	5	7	L	**31**	30	32	L	**56**	55	57	L
7	9	5	W	**32**	30	34	L	**57**	55	59	L
8	9	7	W	**33**	32	34	L	**58**	57	59	L
9	8	10	L	**34**	33	35	L	**59**	57	61	L
10	9	11	L	**35**	34	36	L	**60**	59	61	L
11	10	12	L	**36**	35	37	L	**61**	60	62	L
12	13	11	W	**37**	36	38	L	**62**	63	61	W
13	15	11	W	**38**	39	37	W	**63**	65	61	W
14	15	13	W	**39**	40	38	W	**64**	66	62	W
15	16	14	W	**40**	42	38	W	**65**	66	64	W
16	15	17	L	**41**	42	40	W	**66**	67	65	W
17	16	18	L	**42**	43	41	W	**67**	69	65	W
18	16	20	L	**43**	45	41	W	**68**	69	67	W
19	17	21	L	**44**	46	42	W	**69**	70	68	W
20	19	21	L	**45**	47	43	W	**70**	71	69	W
21	20	22	L	**46**	48	44	W	**71**	72	70	W
22	24	20	L	**47**	48	46	W	**72**	73	71	W
23	24	22	W	**48**	49	47	W	**73**	74	72	W
24	25	23	W	**49**	50	48	W	**74**	75	73	W
25	27	23	W	**50**	48	52	L	**75**	76	74	W

Table 6. Kalah (1, m) following the rules of Alternate Play. $76 \leq m \leq 150$

m	endgame status		L/D/W	m	endgame status		L/D/W	M	endgame status		L/D/W
	P1	P2			P1	P2			P1	P2	
76	78	74	W	**101**	99	103	L	126	127	125	W
77	79	75	W	**102**	100	104	L	127	128	126	W
78	80	76	W	**103**	102	104	L	128	130	126	W
79	81	77	W	**104**	102	106	L	129	131	127	W
80	81	79	W	**105**	104	106	L	130	132	128	W
81	80	82	L	**106**	106	106	D	131	133	129	W
82	81	83	L	**107**	109	105	W	132	134	132	W
83	82	84	L	**108**	109	107	W	133	135	131	W
84	82	86	L	**109**	110	108	W	134	135	133	W
85	84	86	L	**110**	111	109	W	135	137	133	W
86	84	88	L	**111**	112	110	W	136	138	134	W
87	85	89	L	**112**	114	110	W	137	139	135	W
88	87	89	L	**113**	114	112	W	138	139	137	W
89	88	90	L	**114**	115	113	W	139	140	138	W
90	89	91	L	**115**	116	114	W	140	138	142	L
91	90	92	L	**116**	117	115	W	141	140	142	L
92	91	93	L	**117**	118	116	W	142	141	143	L
93	92	94	L	**118**	119	117	W	143	142	144	L
94	94	94	D	**119**	120	118	W	144	143	145	L
95	94	96	L	**120**	120	120	D	145	144	146	L
96	95	97	L	**121**	122	120	W	146	144	148	L
97	96	98	L	**122**	122	122	D	147	145	149	L
98	97	99	L	**123**	124	122	W	148	147	149	L
99	97	101	L	**124**	125	123	W	149	148	150	L
100	98	102	L	**125**	126	124	W	150	149	151	L

Table 7. Kalah (1, m) following the rules of Alternate Play. $151 \leq m \leq 225$

m	endgame status		L/D/W	m	endgame status		L/D/W	m	endgame status		L/D/W
	P1	P2			P1	P2			P1	P2	
151	150	152	L	**176**	174	178	L	**201**	201	201	D
152	151	153	L	**177**	175	179	L	**202**	203	201	W
153	152	154	L	**178**	177	179	L	**203**	203	203	D
154	153	155	L	**179**	178	180	L	**204**	205	203	W
155	154	156	L	**180**	179	181	L	**205**	206	204	W
156	155	157	L	**181**	180	182	L	**206**	207	205	W
157	157	157	D	**182**	183	181	W	**207**	208	206	W
158	157	159	L	**183**	184	182	W	**208**	209	207	W
159	158	160	L	**184**	185	183	W	**209**	210	208	W
160	159	161	L	**185**	185	185	D	**210**	211	209	W
161	160	162	L	**186**	187	185	W	**211**	212	210	W
162	161	163	L	**187**	189	185	W	**212**	212	212	D
163	162	164	L	**188**	189	187	W	**213**	214	212	W
164	163	165	L	**189**	190	188	W	**214**	215	213	W
165	164	166	L	**190**	191	189	W	**215**	215	215	D
166	166	166	D	**191**	193	189	W	**216**	217	215	W
167	166	168	L	**192**	193	191	W	**217**	218	216	W
168	167	169	L	**193**	195	191	W	**218**	220	216	W
169	168	170	L	**194**	195	193	W	**219**	220	218	W
170	168	172	L	**195**	197	193	W	**220**	222	218	W
171	170	172	L	**196**	198	194	W	**221**	223	219	W
172	171	173	L	**197**	199	195	W	**222**	224	220	W
173	172	174	L	**198**	200	196	W	**223**	225	221	W
174	172	176	L	**199**	201	197	W	**224**	225	223	W
175	174	176	L	**200**	201	199	W	**225**	227	223	W

Table 8. Kalah (1, m) following the rules of Alternate Play. $226 \le m \le 300$

m	endgame status		L/D/W	m	endgame status		L/D/W	m	endgame status		L/D/W
	P1	P2			P1	P2			P1	P2	
226	228	224	W	**251**	250	252	L	**276**	275	277	L
227	228	226	W	**252**	251	253	L	**277**	277	277	D
228	229	227	W	**253**	252	254	L	**278**	277	279	L
229	231	227	W	**254**	252	256	L	**279**	278	280	L
230	231	229	W	**255**	253	257	L	**280**	279	281	L
231	232	230	W	**256**	255	257	L	**281**	280	282	L
232	233	231	W	**257**	256	258	L	**282**	281	283	L
233	234	232	W	**258**	257	259	L	**283**	283	283	D
234	233	235	L	**259**	258	260	L	**284**	283	285	L
235	234	236	L	**260**	259	261	L	**285**	284	286	L
236	234	238	L	**261**	260	262	L	**286**	285	287	L
237	235	239	L	**262**	261	263	L	**287**	286	288	L
238	237	239	L	**263**	263	263	D	**288**	287	289	L
239	237	241	L	**264**	264	264	D	**289**	288	290	L
240	239	241	L	**265**	265	265	D	**290**	288	292	L
241	240	242	L	**266**	265	267	L	**291**	289	293	L
242	241	243	L	**267**	266	268	L	**292**	290	294	L
243	242	244	L	**268**	268	268	D	**293**	291	295	L
244	243	245	L	**269**	268	270	L	**294**	293	295	L
245	244	246	L	**270**	269	271	L	**295**	295	295	D
246	245	247	L	**271**	270	272	L	**296**	295	297	L
247	247	247	D	**272**	271	273	L	**297**	296	298	L
248	247	249	L	**273**	271	275	L	**298**	297	299	L
249	248	250	L	**274**	273	275	L	**299**	298	300	L
250	249	251	L	**275**	273	277	L	**300**	298	302	L

Table 9. Kalah (1, m) following the rules of Alternate Play. $301 \leq m \leq 350$

m	endgame status		L/D/W	m	endgame status		L/D/W
	P1	P2			P1	P2	
301	300	302	L	**326**	328	324	W
302	300	304	L	**327**	328	326	W
303	301	305	L	**328**	330	326	W
304	302	306	L	**329**	330	328	W
305	303	307	L	**330**	331	329	W
306	305	307	L	**331**	333	329	W
307	306	308	L	**332**	334	330	W
308	307	309	L	**333**	335	331	W
309	310	308	W	**334**	336	332	W
310	312	308	W	**335**	336	334	W
311	312	310	W	**336**	337	335	W
312	313	311	W	**337**	339	335	W
313	314	312	W	**338**	340	336	W
314	315	313	W	**339**	340	338	W
315	316	314	W	**340**	341	339	W
316	317	315	W	**341**	342	340	W
317	316	318	W	**342**	342	342	D
318	319	317	W	**343**	343	343	D
319	321	317	W	**344**	344	344	D
320	321	319	W	**345**	345	345	D
321	322	320	W	**346**	347	345	W
322	323	321	W	**347**	347	347	D
323	324	322	W	**348**	349	347	W
324	326	322	W	**349**	351	347	W
325	327	323	W	**350**	351	349	W

Verification of Steps for Selected Cases

m = 1 D

$$\begin{matrix} 0 & 1 \\ & 1 & 0 \end{matrix} \rightarrow \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \Rightarrow \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

m = 2 L

$$\begin{matrix} 0 & 2 \\ & 2 & 0 \end{matrix} \rightarrow \begin{matrix} 0 & 3 \\ 0 & 1 \end{matrix} \Rightarrow \begin{matrix} 3 & 0 \\ 0 & 1 \end{matrix}$$

m = 3 W

$$\begin{matrix} 0 & 3 \\ & 3 & 0 \end{matrix} \rightarrow \begin{matrix} 0 & 0 \\ 0 & 6 \end{matrix}$$

4.2.1 Possible Strategies for Player 1

Similarly, there is no possible strategy to be suggested to player 1.

4.2.2 Possible Strategies for Player 2 (Counter Attacks)

There exists no counter attacks available for player 2.

4.2.3 Observations

i. Excluding the case where m = 3, there is no other configuration in which player one wins all the seeds unlike the case for normal play. The main reason is that the player does not get to retain his turn even if his last seed lands in his Kalaha till the stage when the player is left with 3 seeds to sow.

ii. The difference between the numbers of seeds that both players have in their respective Kalaha is always an even number and at most 4 or zero for all m excluding the case when m = 3.

5. Kalah (n, 1)

The rules are stated as above in Section 3. For the case of Kalah (n, 1), instead of having six pits on each side of the board and 4 seeds per pit, this case have n pits on each side of the board and one seeds in each of the pit.

5.1 Normal Play

The results as shown in Table 10 is obtained with the help of a C++ program that was provided by Geoffrey Irving.

Table 10. Results for Kalah (n, 1) following the rules for Normal Play

N\1	L/D/W-P1
1	D
2	W
3	D
4	W
5	D
6	W

5.1.1 Possible Strategies for Player 1

The five moves that can be made are capture, home, mirror, defend and random. At the beginning of the Kalah, in order to maximise the chances of winning, the moves to be made at the beginning are as follows:

1. Kalaha – It is advisable for the player to move seed in pit 1 towards Kalaha for the first move since it will guarantee another move for the player that might result in a capture. This ensures that the player can lead by at least 2 seeds.
2. Capture – The following move is to move the seed in pit 2. This move will result in a capture of at least 1 seed.

This is termed as the classic starting strategy.

The following are some of the possible strategies that have been studied and failed.

i. Capture, Kalaha - A hypothesis was constructed from Table10. It is assumed that the first player employs the classic starting strategy as suggested above. After using that strategy, he attempts to capture whenever possible. If capture is not possible, the first player attempts to sow the seed that will land in his Kalaha or move the seed that is nearest to his Kalaha. P2 is free to make any kind of moves. The hypothesis was that whenever n is even, the first player would always win.

This strategy works for the case when n is 2 and 4

n = 2, m = 1

$$0\ 1\ 1 \rightarrow 0\ 1\ 1 \rightarrow 0\ 1\ 0 \rightarrow 1\ 0\ 0$$
$$1\ 1\ 0\ H \quad 1\ 0\ 1\ C \quad 0\ 0\ 3\ H \quad 0\ 0\ 3$$

P1 wins.

n = 4, m = 1

$$0\ 1\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1\ 0 \Rightarrow 1\ 0\ 1\ 1\ 0$$
$$1\ 1\ 1\ 1\ 0\ H \quad 1\ 1\ 1\ 0\ 1\ C \quad 1\ 1\ 0\ 0\ 3\ H \quad 1\ 1\ 0\ 0\ 3$$

$$\Rightarrow 3\ 0\ 0\ 1\ 0 \rightarrow 3\ 0\ 0\ 0\ 0$$
$$C \quad 0\ 1\ 0\ 0\ 3\ C \quad 0\ 0\ 0\ 0\ 5$$

P1 wins.

The strategy fails once it reached the case when n = 6 that is supposed to be a win. It is not true when it is assumed that the first player always move to capture whenever possible, else he has to move the seed that is nearest to his Kalaha.

n = 6, m = 1

$$0\ 1\ 1\ 1\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1\ 1\ 1\ 0$$
$$1\ 1\ 1\ 1\ 1\ 1\ 0\ H \quad 1\ 1\ 1\ 1\ 1\ 0\ 1\ C \quad 1\ 1\ 1\ 1\ 0\ 0\ 3$$

$\Rightarrow$ 1 0 1 1 1 1 0 $\Rightarrow$ 1 0 1 1 2 0 0 $\rightarrow$ 1 0 1 1 2 0 0
H 1 1 1 1 0 0 3 *D* 1 1 1 1 0 0 3 *C* 1 1 1 0 0 0 4

$\Rightarrow$ 1 0 2 2 0 0 0 $\rightarrow$ 1 0 2 2 0 0 0
D 1 1 1 0 0 0 4 *C* 1 1 0 0 0 0 5

$\rightarrow$ 3 0 3 0 0 0 0 $\rightarrow$ 3 0 3 0 0 0 0
C 0 1 0 0 0 0 5 *C* 0 0 0 0 0 0 6

$\rightarrow$ 3 0 2 2 0 0 0 $\rightarrow$ 6 0 0 0 0 0 0
C 0 1 0 0 0 0 5 *C* 0 0 0 0 0 0 6

A draw.

ii. Defend, capture, Kalaha - Assuming that the classic starting strategy is employed. The strategy that is to be used will be to defend if possible, else capture. If it is not possible to defend or capture, then go back to his Kalaha. This strategy is failed again in the case of n = 6:

<u>n = 6, m = 1</u>

0 1 1 1 1 1 1 $\rightarrow$ 0 1 1 1 1 1 1 $\rightarrow$ 0 1 1 1 1 1 0
1 1 1 1 1 1 0 *H* 1 1 1 1 1 0 1 *C* 1 1 1 1 0 0 3

$\Rightarrow$ 1 0 1 1 1 1 0 $\Rightarrow$ 1 0 1 1 2 0 0 $\rightarrow$ 1 0 1 1 2 0 0
H 1 1 1 1 0 0 3 *D* 1 1 1 1 0 0 3 *D* 0 2 1 1 0 0 3

$\Rightarrow$ 1 0 2 0 2 0 0 $\rightarrow$ 1 0 2 0 2 0 0
R 0 2 1 1 0 0 3 *D* 0 2 0 2 0 0 3

$\Rightarrow$ 2 1 0 0 2 0 0 $\Rightarrow$ 3 0 0 0 2 0 0
H 0 2 0 2 0 0 3 *H* 0 2 0 2 0 0 3

$\Rightarrow$ 6 0 0 1 0 0 0 $\rightarrow$ 6 0 0 1 0 0 0
C 0 0 0 2 0 0 3 *C* 0 0 0 0 1 0 4

$\Rightarrow$ 7 0 0 0 0 0 0 $\rightarrow$ 7 0 0 0 0 0 0
C 0 0 0 0 1 0 4 *C* 0 0 0 0 0 0 5

P1 lose.

5.1.2 Possible Strategies for Player 2 (Counter Attacks)

The following are some of the possible strategies that were tested and have failed:

i. Mirror – This is the case where second player always mirror the moves of first player. This strategy failed when it reach the case where n is 6.

n = 6, m = 1

$$0\ 1\ 1\ 1\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1\ 1\ 1\ 0$$
$$1\ 1\ 1\ 1\ 1\ 1\ 0\ H \quad 1\ 1\ 1\ 1\ 1\ 0\ 1\ C \quad 1\ 1\ 1\ 1\ 0\ 0\ 3$$

$$\Rightarrow 1\ 0\ 1\ 1\ 1\ 1\ 0 \Rightarrow 3\ 0\ 0\ 1\ 1\ 1\ 0 \rightarrow 3\ 0\ 0\ 1\ 1\ 0\ 0$$
$$H \quad 1\ 1\ 1\ 1\ 0\ 0\ 3\ C \quad 0\ 1\ 1\ 1\ 0\ 0\ 3\ C \quad 0\ 1\ 1\ 0\ 0\ 0\ 5$$

$$\Rightarrow 5\ 0\ 0\ 0\ 1\ 0\ 0 \rightarrow 5\ 0\ 0\ 0\ 0\ 0\ 0$$
$$C \quad 0\ 0\ 1\ 0\ 0\ 0\ 5\ C \quad 0\ 0\ 0\ 0\ 0\ 0\ 7$$

P2 loses.

ii. Capture, Kalaha - The counter example given for the case when P2 mirrors P1's moves also illustrates the fact that employing the strategy of capturing the seed whenever possible, else sowing the seed that will land in the player's Kalaha or moving the seed that is closest to Kalaha is not a good strategy.

iii. Kalaha – The counter example for the mirror strategy also shows that the strategy for P2 to always move the seed that is closest to P2's Kalaha will not work well.

iv. Defend, capture, Kalaha – P2 is assumed to always defend if possible, else capture. If both defend and capture are not possible, then P2 is to sow the seeds that is nearest to his Kalaha. This strategy failed at the case when n = 4

n = 3, m = 1

$$0\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 0 \Rightarrow 0\ 2\ 0\ 0$$
$$1\ 1\ 1\ 0\ H \quad 1\ 1\ 0\ 1\ C \quad 1\ 0\ 0\ 3\ D \quad 1\ 0\ 0\ 3$$

$$\rightarrow 0\ 2\ 0\ 0 \Rightarrow 2\ 0\ 0\ 0$$
$$C \quad 0\ 0\ 0\ 4\ C \quad 0\ 0\ 0\ 4$$

P2 loses.

5.1.3 Observations

i. Here are some of the configurations that will result in the player being able to retain his turn for a number of times:

1.
$$\begin{array}{cccccc} a & \ldots & b & c & d & \\ & \ldots & 3 & 2 & 1 & f \end{array}$$

For this case, the player can get to retain his turn for at most 5 times.

$$\begin{array}{ccccccccccccccccc} a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d \\ & \ldots & 3 & 2 & 1 & f & H1 & & \ldots & 3 & 2 & 0 & f+1 & H2 & & \ldots & 3 & 0 & 1 & f+2 \end{array}$$

$$\begin{array}{ccccccccccccccc} \rightarrow & a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d \\ H3 & & \ldots & 3 & 0 & 0 & f+3 & H4 & & \ldots & 0 & 1 & 1 & f+4 \end{array}$$

$$\begin{array}{ccccccc} \rightarrow & a & \ldots & b & c & d \\ H5 & & \ldots & 0 & 1 & 0 & f+5 \end{array}$$

2.
$$\begin{array}{cccccc} a & \ldots & b & c & d & \\ & \ldots & 3 & 2 & 0 & f \end{array}$$

Player gets to retain his turn for at most 4 times.

$$\begin{array}{ccccccccccccccccc} a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d \\ & \ldots & 3 & 2 & 0 & f & H1 & & \ldots & 3 & 0 & 1 & f+1 & H2 & & \ldots & 3 & 0 & 0 & f+2 \end{array}$$

$$\begin{array}{ccccccccccccccc} \rightarrow & a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d \\ H3 & & \ldots & 0 & 1 & 1 & f+3 & H4 & & \ldots & 0 & 1 & 0 & f+4 \end{array}$$

3.
$$\begin{array}{cccccc} a & \ldots & b & c & d & \\ & \ldots & 0 & 2 & 1 & f \end{array}$$

This configuration allows the player to retain his turn for at most 3 times.

$$\begin{array}{ccccccccccccccccc} a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d & & \rightarrow & a & \ldots & b & c & d \\ & \ldots & 0 & 2 & 1 & f & H1 & & \ldots & 0 & 2 & 0 & f+1 & H2 & & \ldots & 0 & 0 & 1 & f+2 \end{array}$$

$$\begin{array}{ccccccccccc} \rightarrow & a & \ldots & b & c & d \\ H3 & & \ldots & e & 0 & 0 & f+3 \end{array}$$

ii. It can be easily observed from the above part (i) observation that in order for the player to be able to retain his turn, if a pit is n steps away from his Kalaha, that pit will have to contain n number of seeds.

iii. The game is always a draw (when n is odd) or win (when n is even) for the first player if first player always follow the strategy of moving the seeds nearest to his Kalaha and the second player always mirror his moves.

5.2 Alternate Play

The additional rule will be that the player will have to lose his turn after every move that he has made. The results for Table 11 are obtained by hands.

Table 11. Results of Kalah (n, 1) following the rules for Alternate Play

N\1	L/D/W-P1
1	D
2	D
3	D
4	D

5.2.1 Possible Strategies for Player 1

The first move to be made by P1 is preferably the moving of the seed that is nearest to his Kalaha. Random move is not encouraged. This can be illustrated by the following e.g.:

n = 3, m = 1

$$0\ 1\ 1\ 1 \rightarrow 0\ 1\ 1\ 1 \Rightarrow 1\ 0\ 1\ 1 \rightarrow 0\ 0\ 1\ 2$$
$$1\ 1\ 1\ 0\ R \quad 1\ 0\ 2\ 0\ H \quad 1\ 0\ 2\ 0\ H \quad 1\ 0\ 0\ 1$$

$$\Rightarrow 3\ 0\ 0\ 2 \Rightarrow 5\ 0\ 0\ 0$$
$$C \quad 0\ 0\ 0\ 1\ C \quad 0\ 0\ 0\ 1$$

P1 loses.

Thus, the classic starting move for the alternate play will be to move to Kalaha. It is assumed that the first player always starts the Kalah using the classic starting move. When it is impossible to make a move to Kalaha, it is suggested that P1 moves to capture.

The results that are obtained following this suggested strategy is at least a draw for P1 until the case when n = 6. This is thus a non-losing strategy and guarantees a higher chance of winning.

Proof:

Assuming that P1 started off his move by moving to his Kalaha, he captures 1 seed into his Kalaha. At this time, P1 leads by one capture. P2 then have three types of move that he can make:

1. Capture – In order to capture any seeds, P2 can only go to his Kalaha at the beginning.

   P1 can then counter it by capturing of seeds. By capturing, P1 continue to be in the lead in terms of the number of captures made. Since for every capture, at least 1 seed will go into the Kalaha and P1 is in the lead, the result of the game is at least a draw even if P2 always move to capture.

2. Defend – By choosing the move to defend, P2 is behind P1 by at least one capture that encompasses 1 seed.

   P1 can then respond by capture. If P2 continue to use the defend strategy, the number of captures made by P1 will increase in this case. Unless P2 makes a series moves that can result in capturing of more seeds or equal amount of seeds, P1 wins. Thus, P1 will at least have a draw in this case.

3. Random – This move not only results in P2 being behind P1 by at least one capture of one seed. Unless P2 is able to make a series of moves that can capture more or an equal amount of seeds, P1 wins. Thus, it is at least a draw for P1 again.

5.2.2 Possible Strategies for Player 2 (Counter Attacks)

P1 is assume to use the classic starting move and followed by move to capture whenever the move to his Kalaha is not possible. The following are strategies that were tested and have failed:

i. Mirror – For this case, P2 mirrors the moves of P1.

n = 4, m = 1

0 1 1 1 1 → 0 1 1 1 1 ⇒ 1 0 1 1 1 → 1 0 1 1 0
1 1 1 1 0 *H* 1 1 1 0 1 *H* 1 1 1 0 1 *C* 1 1 0 0 3

⇒ 3 0 0 1 0 → 3 0 0 0 0
C 0 1 0 0 3 *C* 0 0 0 0 5

P2 loses

ii. Random, Kalaha

n = 3, m = 1

0 1 1 1 → 0 1 1 1 ⇒ 0 2 0 1 → 0 2 0 0
1 1 1 0 *H* 1 1 0 1 *R* 1 1 0 1 *C* 1 0 0 3

⇒ 1 0 0 0 → 1 0 0 0
H 2 0 0 3 *C* 0 0 0 5

P2 loses.

iii. Defend, Kalaha

n = 5, m = 1

0 1 1 1 1 1 → 0 1 1 1 1 1 ⇒ 0 1 1 1 2 0
1 1 1 1 1 0 *H* 1 1 1 1 0 1 *D* 1 1 1 1 0 1

→ 0 1 1 1 2 0 ⇒ 0 1 2 2 0 0 → 0 1 2 2 0 0
C 1 1 1 0 0 2 *D* 1 1 1 0 0 2 *C* 1 1 0 0 0 3

⇒ 0 2 3 0 0 0 → 0 2 3 0 0 0 ⇒ 1 3 0 0 0 0
D 1 1 0 0 0 3 *C* 1 0 0 0 0 4 *D* 2 0 0 0 0 4

Overflowing of P2's seeds

→ 1 3 0 0 0 0 ⇒ 2 0 0 0 0 0 → 2 0 0 0 0 0
C 0 1 0 0 0 5 *H* 1 2 0 0 0 5 *C* 0 0 0 0 0 8

P2 loses.

6. Kalah (n, m) where $2 \leq n \leq 6$ **and** $1 \leq m \leq 10$

Table 12. Kalah (n, m) where $2 \leq n \leq 6$ and $1 \leq m \leq 10$

N\m	2	3	4	5	6
1	W	D	W	D	W
2	L	W	W	D	W
3	L	W	W	W	W
4	L	W	W	W	W
5	W	W	W	W	W
6	L	L	D	W	
7	L	L			
8	L	L			
9	L	L			
10	L	L			

Table 13. Kalah (n,m) moves taken by the players

N and m	Moves Made
3 and 1	$12 \Rightarrow 12$
3 and 2	$21 \Rightarrow 31 \rightarrow 2 \Rightarrow 2 \rightarrow 2 \Rightarrow 1$
3 and 3	$32 \Rightarrow 2 \rightarrow 2 \Rightarrow 3 \rightarrow 1 \Rightarrow 1 \rightarrow 13 \Rightarrow 312$
4 and 1	$12 \Rightarrow 12 \rightarrow 3$
4 and 2	$1 \Rightarrow 1 \rightarrow 214 \Rightarrow 213 \rightarrow 3121$
4 and 3	$31 \Rightarrow 3 \rightarrow 412 \Rightarrow 2 \rightarrow 1 \Rightarrow 4 \rightarrow 4235 \Rightarrow 2 \rightarrow 4 \Rightarrow 3$
5 and 1	$12 \Rightarrow 12 \rightarrow 3 \Rightarrow 3$
5 and 2	$24 \Rightarrow 23 \rightarrow 31 \Rightarrow 5 \rightarrow 5 \Rightarrow 24 \rightarrow 2$
5 and 3	$31 \Rightarrow 2 \rightarrow 2 \Rightarrow 53 \rightarrow 14151213 \Rightarrow 24 \rightarrow 2 \Rightarrow 1 \rightarrow 15 \Rightarrow 5 \rightarrow 413121$
6 and 1	$12 \Rightarrow 12 \rightarrow 3 \Rightarrow 3 \rightarrow 4$
6 and 2	$24 \Rightarrow 24 \rightarrow 36 \Rightarrow 1 \rightarrow 5 \Rightarrow 6 \rightarrow 4 \Rightarrow 5$
6 and 3	$2 \Rightarrow 1 \rightarrow 31 \Rightarrow 415 \rightarrow 4 \Rightarrow 512 \rightarrow 215131214 \Rightarrow 13 \rightarrow 3 \Rightarrow 12 \rightarrow 4$

This is the result of Table 12 of Kalah is obtained using C++ program on Linux machine. The moves of the players are shown in Table 13.The strategy involved is basically to move the seed or seeds from the pit in which the last seed will land in the Kalaha whenever possible. When the move to have the last seed lands in the Kalaha is not possible, the player then attempts to capture. The strategy was assumed to be applicable to both players of Kalah.

[The explanation for the numbering of the pits and symbols are provided in Section 11]

7. Computations

Computations of the Kalah (n, m) where $2 \le n \le 6$ and $1 \le m \le 10$ are done on Linux machine with Pentium III Intel Processor and 128.0 MB of RAM. The case for Kalah (1, n) – alternate and normal play, is done using the QBasic Program on Windows machine with Pentium II Intel processor and 128.0 MB of RAM.

8. Discussions and Conclusion

Solving the game from the computational point of view may seem more and more possible as stated in the journal that I have used as my main reference. From that point of view, solving bears the meaning of the ability to predict the endgame status of a given configuration with a few pushes of buttons on the computer keyboard.

From mathematical point of view, solving of the game may bear the meaning of the conjecturing of winning strategies and counter attacks for both of the players. This is a rather challenging process. Thus far that I have investigated, I still see no sure win strategies for both players. I am only able to come up with possible strategies and observations that I have made while analysing the game. This game is an interesting, complicating and challenging game where, I think, even if there exists no sure win strategy, there exists a better strategy that can increase the chances of winning for both of the players and more can be done to make the strategies a better one.

9. Reference

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11. TERMINOLOGY

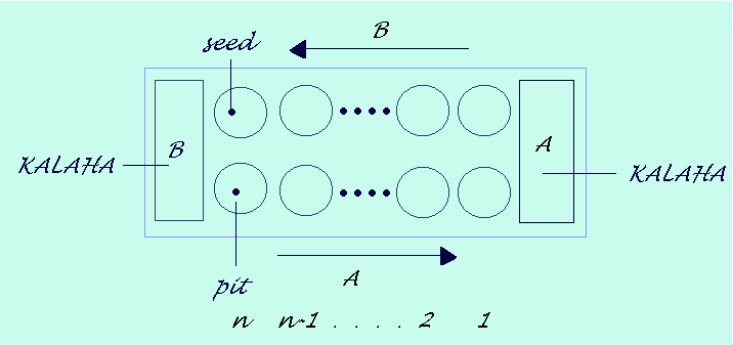

Figure 2. The playing board

1) As seen in the above figure 2, the pits of a player is numbered from his right most pit, excluding his Kalaha, to his left most pit in increasing order of n. n represents natural numbers.
2) Active seed - Seeds that are not captured or has not entered into the players' Kalaha
3) Kalah (m, n)

   where m represents the number of seeds in each pit where m = Z^+

   and n represents the number of pits that is involved in a game where n = Z^+
4) Kalaha - Home or score pit
5) L/D/W –Lose, Draw or Win from the point of view of the first player
6) P1 is the short form for player 1
7) P2 is the short form for player 2
8) Pit – small holes on each side of the board. The holes are divided into North and South territories
9) Player 1 is the player who gets to start the game first
10) Possible strategies are strategies that have been studied.
11) Seeds – Counters

12) $\rightarrow or \Rightarrow$ 0 1 ... 1
$H/D/C/R$ 1 ... 1 0

The Kalaha are located at the extreme right and left. The bottom row represents pits and Kalaha of P1. The top row represents the pits and Kalaha of P2.

H – Moving the seed which is nearest to the player's Kalaha

D – Defend the seed that is exposed to capture

C – Capture at least the seed of the player.

R – Random move

13) $\rightarrow$ Represents the move made by P1

14) $\Rightarrow$ Represents the move made by P2