# Undergraduate Research Opportunity Programme in Science

# The Game of Kalah

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# Chapter 1

# KALAH, THE SOWING GAME

## 1.1 Introduction to Kalah

In 1905, William J.Champion, a graduate of Yale University, came across an article concerning the ancient game of Kalah, a game which closely complies with the modern approach to mathematics.

According to an article which appeared in Time Magazine, the issue of June 14, 1963, "Champion began tracing its (Kalah) migrations and permutations. He found an urn painting of Ajax and Achilles playing it during the siege of Troy; he found African chieftains playing for stakes of female slaves, and maharajahs using rubies and star sapphires as counters. He finally traced it back some 7,000 years to the ancient Sumerians, who evolved the 6-12-60 system of keeping numerical records."

He later set up the "The Kalah Game Company" in the 1950s to produce the game commercially.

The term 'mancala' is used to indicate a large group of related games that are played almost all over the world. Mancala games are played on a board that contains 2, 3 or 4 rows of holes. Usually, 2 players play the games, although one-player and three-player variants are also known. Mancala games are played with a large set of counters, which are distributed in a certain configuration (usually an equal number of seeds per hole). A move is made by selecting one of the holes, removing all the counters from it and putting back the counters one by one in adjacent holes in certain direction. This is called sowing. The hole in which the last counter is put determines what happens next. Sometimes a capture takes place and the turn is over, sometimes the sowing continues, and other times the player is allowed to make another move. However, the goal of the game is always to capture as many counters as possible.

There is a variety of board sizes for mancala games and there are even more variations in the rules. In one group of mancala games, a capture is allowed if the last counter is put in an opponent's hole that contains 1 or 2 counters. These are mostly African games. In another group of games, a capture is allowed of the last counter is put in an empty hole on the player's side. This group is mainly played in South East Asia, but Kalah belongs to this group.

The fundamental processes in math come into play during the game of Kalah and the use of reason is vital to victory. The element of chance is absent and the game involves nothing but skill.

## 1.2 Terminology Used

In this investigation, Kalah involving different numbers of seeds and holes are considered. Therefore, we will use the notation Kalah$(m, n)$ to indicate Kalah with $m$ holes per side and initially $n$ counters per hole.

For the case of Kalah$(m, 1)$, further notation is needed, and it is presented on the next page.

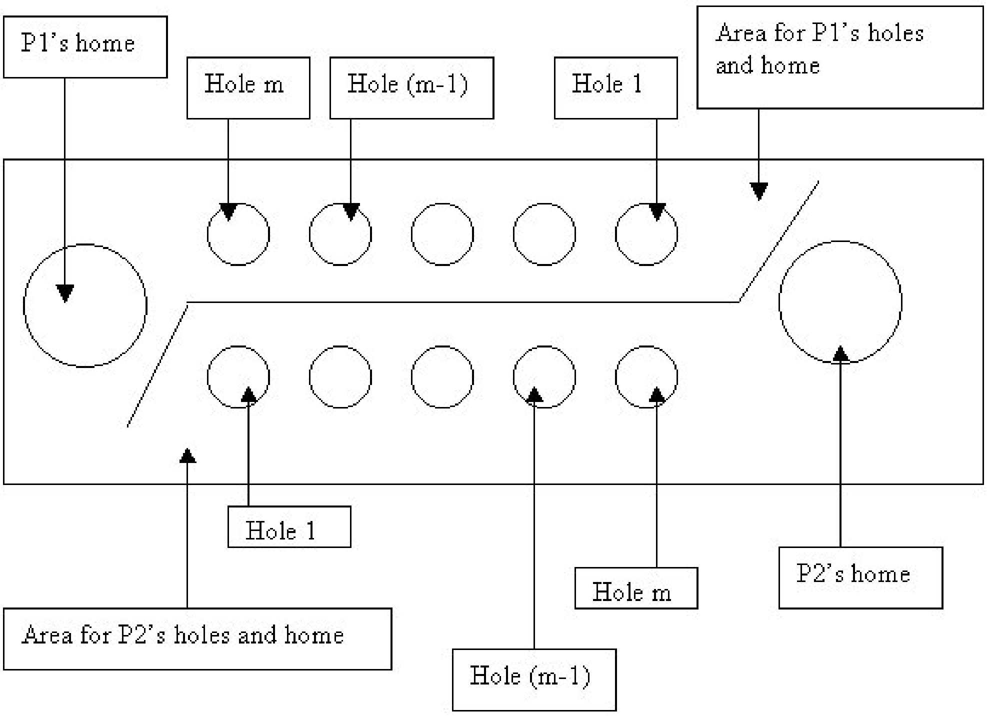


Figure 1: Notation for Kalah$(m, n)$ when $m \neq 1$

# Chapter 2

# KALAH

## 2.1 Introduction and rules for Kalah

Kalah is played on a board with two rows of $m$ holes and two stores. The two players P1 and P2 sit at each side of the board. P1 will always start the game. A move is made by selecting a non-empty hole at the player's side of the board. The counters are lifted from this hole and sown in an anti-clockwise manner. The player's own home is included in the sowing, but the opponent's home is not. Note that the captured counters never re-enter the game.

The three possible outcomes of a move in Kalah are as follows:

(1) If the last counter is put into the player's own home, the player moves again.

(2) If the last counter is put into an empty hole on the player's side of the board, a capture takes place: all stones in the opponent's pit opposite and the last stone of the sowing are put into the player's home and the opponent moves next.

(3) If the last counter is put anywhere else, it is now the opponent's turn.

The game ends when a move leaves no counters on one player's side, in which case the other player captures all remaining counters. The player who collects more counters wins.

## 2.2 Results

### 2.2.1 Analysis of $N = \sum_{x=0}^{\infty} 3^x$ for Kalah$(1, n)$

**Theorem 1** *Using ternary notation, for the Kalah$(1, n)$, all games starting with 11...1 seeds for each player results in a loss for P1, $\forall n > 1, n \in Z$.*

Table 1: End configurations in decimal representation

| Number of seeds | P2's seeds | P1's seeds |
|---|---|---|
| 4 | 5 | 3 |
| 13 | 18 | 8 |
| 40 | 58 | 22 |
| 121 | 179 | 63 |
| 364 | 543 | 185 |

Table 2: End configurations in ternary representation

| Number of seeds | P2's seeds | P1's seeds |
|---|---|---|
| 11 | 11+1 = 12 | 22-12=10 |
| 111 | 111+11+1 = 200 | 222-200=22 |
| 1111 | 1111+111+11+1=2111 | 2222-2111=111 |
| 11111 | 11111+1111+111+11+1 = 20222 | 22222-20222=2000 |
| 111111 | 111111+11111+1111+111+11+1 = 202110 | 222222-202110=20112 |

Proof:

Table 3: Relationship between the last digit in ternary representation and Landing position of last seed

| Last digit in ternary representation | Where the last seed lands |
|---|---|
| 0 | Player's hole |
| 1 | Player's home |
| 2 | Opponent's hole |

Observe that P1 holds the turn throughout the game as the last seed always lands in P1's home. At each sowing, the number of seeds distributed to each hole involved in the sowing is $\frac{n}{3}$ or $\frac{n}{3}+1$, $n$ being the number of seeds being sowed. By observation, both holes receive $\frac{n}{3}$ seeds while P1's home receives $\frac{n}{3}+1$ seeds in this case.

As the number of seeds received by P2 can be expressed as $11\ldots1 + \ldots + 111 + 11 + 1$, the number of seeds P2 receives in total can be approximated by the infinite sum of a geometric series when $n$ is large.

Series is n $+ \frac{n}{3} + \frac{n}{9} + \ldots = \frac{n}{3^0} + \frac{n}{3^1} + \frac{n}{3^2} + \ldots$

Hence, using the formula for summation of infinite geometric series, P2 will have approximately $\frac{a}{1-r} = \frac{n}{1-\frac{n}{3}} = \frac{n}{\frac{2}{3}} = \frac{3n}{2}$ at the end of the game for n large.

This means that P1 will have approximately $2n - \frac{3n}{2} = \frac{n}{2}$ seeds at the end of the game for n large.

# Chapter 3

# KALAHALT

## 3.1 Introduction and rules for KalahAlt

KalahAlt is a variation of Kalah developed in this project.

KalahAlt is played on a board with two rows of $m$ holes and two stores. The two players P1 and P2 sit at each side of the board. P1 will always start the game. A move is made by selecting a non-empty hole at the player's side of the board. The counters are lifted from this hole and sown in an anti-clockwise manner. The player's own home is included in the sowing, but the opponent's home is not. Note that captured counters never re-enter the game.

In this variation of Kalah, regardless of the result of the move made by the player, the opponent moves next, with each player taking strictly alternate turns. However, a player captures the opponent's seeds if the last counter is put into an empty hole on the player's side of the board: all stones in the opposite opponent's pit and the last stone of the sowing are put into the player's home and the opponent moves next.

The game ends when a move leaves no counters on one player's side, in which case the other player captures all remaining counters. The player who collects more counters wins.

## 3.2 Results

### 3.2.1 Analysis of KalahAlt $(m, 1)$

**Theorem 1** *P1 will always be able to obtain at least a draw if he moves in the following manner in order of priority:*
*(i) P1 moves the seed in hole m*
*(ii)*

- *(1) Attempt to capture.*
- *(2) Move the seed nearest home towards home such that the move makes no capture or defence.*

Note that P1 will always be able to make at least one of the above moves if P1 has at least one seed. If P1 is unable to capture, but still has at least one seed, P1 can move the seed nearer to his home.

Observe that P2 has up to 4 countermoves to each of P1's moves.

- (I) Attack an open hole.
- (II) Defend an open hole.
- (III) Move towards home.
- (IV) Random move.(ie none of the previous moves)

Note that as these represent all possible moves by P2, P2 will be able to make at least one of the 4 moves stated above if he has at least one seed left.

Let a capture be defined as when a player makes a move such that he captures at least one of the opponent's seeds.
Let an open hole be defined as one where a capture is possible.

Proof: This shall be proven by induction.
Let $C_1$ = number of captures by P1
Let $C_2$ = number of captures by P2
Let $O_{P1}$ = number of P1's holes open for capture
Let $O_{P2}$ = number of P2's holes open for capture
Let $f(k) = C_1 - C_2 - O_{P1} + O_{P2}$ after the $k$ move.
Let S be the statement that $f(k) \geq 0 \forall k$

Hence, when $k$ is odd, P1's turn is just over. When $k$ is even, P2's turn is just over.

Note that P2 can at most capture at most one of P1's seeds at any turn as P1's strategy does not allow P1 to defend. Hence, it is not possible for P1 to accumulate seeds in any one hole. Hence, we can consider the number of captures and number of open holes instead of the number of seeds captured and the number of seeds open to capture respectively. Also, P1 can move at most one seed per turn.

Let $s$ represent the move number.
$s = 1$ case

By strategy, P1 moved into home. (0 captures)
$\therefore f(1) = 1$ as P2 now has a hole open for capture by P1.

P2 now has 3 moves open to it.

(1) Mirror P1's move. (0 captures)

This results in $\Delta O_{P1} = +1$ and $\Delta O_{P2} = +0$

$\therefore \Delta f(1) = -1$

Hence, $f(2) = 0$

(2) Defend an open hole. (0 captures)

This results in $\Delta O_{P1} = +0$ and $\Delta O_{P2} = -1$

$\therefore \Delta f(1) = -1$

Hence, $f(2) = 0$

(3) Make a random move (0 captures)

This results in $\Delta O_{P1} = +1$ and $\Delta O_{P2} = +0$

$\therefore \Delta f(1) = -1$

Hence, $f(2) = 0$

Hence, $s = 1$ case is true.

Let $s = 2k + 1$ be true. We want to show that $s = 2k + 2$ is true.

<u>(1) P1 captures($C_1 = +1$)</u>

Consider 2 cases:

Case (a)

$$\begin{matrix} \ldots & 0 & \underline{1} & \ldots \\ \ldots & 1 & 1 & \ldots \end{matrix} \quad \rightarrow \quad \begin{matrix} \ldots & 0 & 0 & \ldots \\ \ldots & 0 & 1 & \ldots \end{matrix}$$

$\Delta C_1 = +1$

$\Delta O_{P1} = +0$

$\Delta O_{P2} = +0$

$\therefore \Delta f = 1$

Hence, $f(2k + 1) = f(2k) + 1 \geq 0$

Case (b)

$$\begin{array}{cccccccccccc} \dots & 0 & \underline{1} & \dots & \rightarrow & \dots & 0 & 0 & \dots \\ \dots & 1 & 0 & \dots & & \dots & 0 & 0 & \dots \end{array}$$

$\Delta C_1 = +1$

$\Delta O_{P1} = -1$

$\Delta O_{P2} = -1$

$\therefore \Delta f = 1$

Hence, $f(2k+1) \geq 0$

<u>(i) P2 counter-captures</u>

Consider 4 cases:

Case (a)

$$\begin{array}{cccccccccccc} \dots & 1 & 1 & \dots & \rightarrow & \dots & 1 & 0 & \dots \\ \dots & \underline{1} & 0 & \dots & & \dots & 0 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} = +0$

$\Delta O_{P2} = +0$

$\therefore \Delta f = -1$

$\therefore f(2k+2) = f(2k) + 1 - 1 \geq 0$

Case (b)

$$\begin{array}{cccccccccccc} \dots & 0 & 1 & \dots & \rightarrow & \dots & 0 & 0 & \dots \\ \dots & \underline{1} & 0 & \dots & & \dots & 0 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} = -1$

$\Delta O_{P2} = -1$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case (c)

Example:

$$\begin{array}{cccccccccccccc} \dots & 0 & 1 & 1 & 1 & \dots & \rightarrow & \dots & 0 & 1 & 1 & 0 & \dots \\ \dots & \underline{3} & & & 0 & \dots & & \dots & 0 & 1 & 1 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} \leq -1$

$\Delta O_{P2} = -1$

$\therefore \Delta f \geq -1$

$\therefore f(2k+2) =\geq 0$

Case (d)

Example:

$$\begin{array}{ccccccccccccccc} \dots & 1 & 1 & 1 & 1 & \dots & \rightarrow & \dots & 1 & 1 & 1 & 0 & \dots & & \Delta C_2 = +1 \\ \dots & \underline{3} & & & 0 & \dots & & \dots & 0 & 1 & 1 & 0 & \dots & & \Delta O_{P1} \leq +0 \\ & & & & & & & & & & & & & & \Delta O_{P2} = +0 \\ & & & & & & & & & & & & & & \therefore \Delta f \geq -1 \\ & & & & & & & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

<u>(ii) P2 defends</u>

Consider 2 cases:

Case(a)

$$\begin{array}{ccccccccccc} \dots & 0 & 1 & \dots & \rightarrow & \dots & 0 & 1 & \dots & \Delta C_2 = +0 \\ \dots & \underline{1} & 1 & \dots & & \dots & 0 & 2 & \dots & \Delta O_{P1} = +0 \\ & & & & & & & & & \Delta O_{P2} = -1 \\ & & & & & & & & & \therefore \Delta f = -1 \\ & & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

Case(b)

Example:

$$\begin{array}{ccccccccccccc} \dots & 0 & 1 & 1 & \dots & \rightarrow & \dots & 0 & 1 & 1 & \dots & \Delta C_2 = +0 \\ \dots & \underline{2} & & 1 & \dots & & \dots & 0 & & 2 & \dots & \Delta O_{P1} \leq +0 \\ & & & & & & & & & & & \Delta O_{P2} = -1 \\ & & & & & & & & & & & \therefore \Delta f \geq -1 \\ & & & & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

<u>(iii) P2 moves home</u>

Consider 3 cases

Case(a)

$$\begin{array}{ccccccccc} \dots & 1 & 0 \dots & \rightarrow & \dots & 1 & 0 & \dots & \Delta C_2 = +0 \\ \dots & \underline{1} & 0 \dots & & \dots & 0 & 0 & \dots & \Delta O_{P1} = +1 \\ & & & & & & & & \Delta O_{P2} = +0 \\ & & & & & & & & \therefore \Delta f = -1 \\ & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

Case(b)

$$\begin{array}{cccccccccc} \dots & 0 & 0 & \dots & \rightarrow & \dots & 0 & 0 & \dots \\ \dots & \underline{1} & 0 & \dots & & \dots & 0 & 0 & \dots \end{array}$$

$\Delta C_2 = +0$

$\Delta O_{P1} = +0$

$\Delta O_{P2} = -1$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case(c)

Example:

$$\begin{array}{cccccccccc} \dots & 0 & 1 & \dots & \rightarrow & \dots & 0 & 1 & \dots \\ \dots & \underline{2} & 0 & \dots & & \dots & 0 & 1 & \dots \end{array}$$

$\Delta C_2 = +0$

$\Delta O_{P1} \leq -1$

$\Delta O_{P2} = -1$

$\therefore \Delta f \geq 0$

$\therefore f(2k+2) \geq 1$

<u>(iv) P2 makes a random move</u>

Consider 4 cases

Case(a)

$$\begin{array}{cccccccccc} \dots & 1 & 1 & \dots & \rightarrow & \dots & 1 & 1 & \dots \\ \dots & \underline{1} & 1 & \dots & & \dots & 0 & 2 & \dots \end{array}$$

$\Delta C_2 = +0$

$\Delta O_{P1} = +1$

$\Delta O_{P2} = +0$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case(b)

$$\begin{array}{cccccccccc} \dots & 1 & 0 & \dots & \rightarrow & \dots & 1 & 0 & \dots \\ \dots & \underline{1} & 1 & \dots & & \dots & 0 & 2 & \dots \end{array}$$

$\Delta C_2 = +0$

$\Delta O_{P1} = +1$

$\Delta O_{P2} \leq +0$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case(c)

Example:

$$\begin{array}{ccccccccccc} \ldots & 1 & 1 & 1 & \ldots & \rightarrow & \ldots & 1 & 1 & 1 & \ldots \\ \ldots & \underline{2} & 1 & 1 & \ldots & & \ldots & 0 & 2 & 2 & \ldots \end{array}$$

$\Delta C_2 = +0$

$\Delta O_{P1} = +1$

$\Delta O_{P2} = +0$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case(d)

Example:

$$\begin{array}{ccccccccccc} \ldots & 1 & 0 & 1 & \ldots & \rightarrow & \ldots & 1 & 0 & 1 & \ldots \\ \ldots & \underline{2} & 0 & 1 & \ldots & & \ldots & 0 & 1 & 2 & \ldots \end{array}$$

$\Delta C_2 = +0$

$\Delta O_{P1} = +1$

$\Delta O_{P2} = +1$

$\therefore \Delta f = 0$

$\therefore f(2k+2) \geq 1$

<u>(2) P1 moves into home ($C_1 = +0$)</u>

Consider 2 cases

Case (a)

$$\begin{array}{ccccccccc} \ldots & 0 & \underline{1} & \ldots & \rightarrow & \ldots & 0 & 0 & \ldots \\ \ldots & 0 & 0 & \ldots & & \ldots & 0 & 0 & \ldots \end{array}$$

$\Delta C_1 = +0$

$\Delta O_{P1} = -1$

$\Delta O_{P2} = +0$

$\therefore \Delta f = 1$

Hence, $f(2k+1) \geq 0$

Case (b)

$$\begin{array}{ccccccccc} \ldots & 0 & \underline{1} & \ldots & \rightarrow & \ldots & 0 & 0 & \ldots \\ \ldots & 0 & 1 & \ldots & & \ldots & 0 & 1 & \ldots \end{array}$$

$\Delta C_1 = +0$

$\Delta O_{P1} = +0$

$\Delta O_{P2} = +1$

$\therefore \Delta f = 1$

Hence, $f(2k+1) \geq 0$

<u>(i) P2 captures</u>

Consider 4 cases:

Case (a)

$$\begin{array}{ccccccccc} \dots & 1 & 1 & \dots & \rightarrow & \dots & 1 & 0 & \dots \\ \dots & \underline{1} & 0 & \dots & & \dots & 0 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} = +0$

$\Delta O_{P2} = +0$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case (b)

$$\begin{array}{ccccccccc} \dots & 0 & 1 & \dots & \rightarrow & \dots & 0 & 0 & \dots \\ \dots & \underline{1} & 0 & \dots & & \dots & 0 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} = -1$

$\Delta O_{P2} = -1$

$\therefore \Delta f = -1$

$\therefore f(2k+2) \geq 0$

Case (c)

Example:

$$\begin{array}{ccccccccccccc} \dots & 0 & 1 & 1 & 1 & \dots & \rightarrow & \dots & 0 & 1 & 1 & 0 & \dots \\ \dots & \underline{3} & 0 & 0 & 0 & \dots & & \dots & 0 & 1 & 1 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} \leq -1$

$\Delta O_{P2} = -1$

$\therefore \Delta f \geq -1$

$\therefore f(2k+2) \geq 0$

Case (d)

Example:

$$\begin{array}{ccccccccccccc} \dots & 1 & 1 & 1 & 1 & \dots & \rightarrow & \dots & 1 & 1 & 1 & 0 & \dots \\ \dots & \underline{3} & 0 & 0 & 0 & \dots & & \dots & 0 & 1 & 1 & 0 & \dots \end{array}$$

$\Delta C_2 = +1$

$\Delta O_{P1} \leq +0$

$\Delta O_{P2} = +0$

$\therefore \Delta f \geq -1$

$\therefore f(2k+2) \geq 0$

<u>(ii) P2 defends</u>

Consider 2 cases:

Case(a)

$$\begin{array}{ccccccccccc} \ldots & 0 & 1 & \ldots & \rightarrow & \ldots & 0 & 1 & \ldots & \Delta C_2 = +0 \\ \ldots & \underline{1} & 1 & \ldots & & \ldots & 0 & 2 & \ldots & \Delta O_{P1} = +0 \\ & & & & & & & & & \Delta O_{P2} = -1 \\ & & & & & & & & & \therefore \Delta f = -1 \\ & & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

Case(b)

Example:

$$\begin{array}{ccccccccccccc} \ldots & 1 & 1 & 0 & \ldots & \rightarrow & \ldots & 1 & 1 & 0 & \ldots & \Delta C_2 = +0 \\ \ldots & \underline{2} & 1 & 1 & \ldots & & \ldots & 0 & 2 & 2 & \ldots & \Delta O_{P1} \leq +1 \\ & & & & & & & & & & & \Delta O_{P2} = +0 \\ & & & & & & & & & & & \therefore \Delta f \geq -1 \\ & & & & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

<u>(iii) P2 moves home</u>

Consider 3 cases

Case(a)

$$\begin{array}{cccccccccc} \ldots & 1 & 0 \ldots & \rightarrow & \ldots & 1 & 0 & \ldots & \Delta C_2 = +0 \\ \ldots & \underline{1} & 0 \ldots & & \ldots & 0 & 0 & \ldots & \Delta O_{P1} = +1 \\ & & & & & & & & \Delta O_{P2} = +0 \\ & & & & & & & & \therefore \Delta f = -1 \\ & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

Case(b)

$$\begin{array}{ccccccccccc} \ldots & 0 & 0 & \ldots & \rightarrow & \ldots & 0 & 0 & \ldots & \Delta C_2 = +0 \\ \ldots & \underline{1} & 0 & \ldots & & \ldots & 0 & 0 & \ldots & \Delta O_{P1} = +0 \\ & & & & & & & & & \Delta O_{P2} = -1 \\ & & & & & & & & & \therefore \Delta f = -1 \\ & & & & & & & & & \therefore f(2k+2) \geq 0 \end{array}$$

Case(c)

Example:

$$\begin{array}{cccccccccl} \dots & 0 & 1 & \dots & \rightarrow & \dots & 0 & 1 & \dots & \Delta C_2 = +0 \\ \dots & \underline{2} & 0 & \dots & & \dots & 0 & 1 & \dots & \Delta O_{P1} \leq -1 \\ &&&&&&&&& \Delta O_{P2} = -1 \\ &&&&&&&&& \therefore \Delta f \geq 0 \\ &&&&&&&&& \therefore f(2k+2) \geq 1 \end{array}$$

(iv) P2 makes a random move

Consider 4 cases

Case(a)

$$\begin{array}{cccccccccl} \dots & 1 & 1 & \dots & \rightarrow & \dots & 1 & 1 & \dots & \Delta C_2 = +0 \\ \dots & \underline{1} & 1 & \dots & & \dots & 0 & 2 & \dots & \Delta O_{P1} = +1 \\ &&&&&&&&& \Delta O_{P2} = +0 \\ &&&&&&&&& \therefore \Delta f = -1 \\ &&&&&&&&& \therefore f(2k+2) \geq 0 \end{array}$$

Case(b)

$$\begin{array}{cccccccccl} \dots & 1 & 0 & \dots & \rightarrow & \dots & 1 & 0 & \dots & \Delta C_2 = +0 \\ \dots & \underline{1} & 1 & \dots & & \dots & 0 & 2 & \dots & \Delta O_{P1} = +1 \\ &&&&&&&&& \Delta O_{P2} = +0 \\ &&&&&&&&& \therefore \Delta f = -1 \\ &&&&&&&&& \therefore f(2k+2) \geq 0 \end{array}$$

Case(c)

Example:

$$\begin{array}{ccccccccccccl} \dots & 1 & 1 & 1 & \dots & \rightarrow & \dots & 1 & 1 & 1 & \dots & \Delta C_2 = +0 \\ \dots & \underline{2} & 1 & 1 & \dots & & \dots & 0 & 2 & 2 & \dots & \Delta O_{P1} = +1 \\ &&&&&&&&&&& \Delta O_{P2} = +0 \\ &&&&&&&&&&& \therefore \Delta f = -1 \\ &&&&&&&&&&& \therefore f(2k+2) \geq 0 \end{array}$$

Case(d)

Example:

$$\begin{array}{ccccccccccccc} \ldots & 1 & 0 & 1 & \ldots & \rightarrow & \ldots & 1 & 0 & 1 & \ldots & \Delta C_2 = +0 \\ \ldots & \underline{2} & 0 & 1 & \ldots & & \ldots & 0 & 1 & 2 & \ldots & \Delta O_{P1} = +1 \\ & & & & & & & & & & & \Delta O_{P2} = +1 \\ & & & & & & & & & & & \therefore \Delta f = 0 \\ & & & & & & & & & & & \therefore f(2k+2) \geq 1 \end{array}$$

Hence $k + 1$ case is true.

Therefore, by mathematical induction, P is true for all real and positive integer $k$.

As $C_1 - C_2 - O_{P1} + O_{P2} \geq 0 \; \forall \; k$, P1 will be ahead of P2 at every move in terms of the potential to capture at the next move and the number of captures already executed, taken together. Hence, P1 will always obtain at least a draw for this variant of Kalah, KalahAlt.

# Chapter 4

# SUGGESTIONS FOR FUTURE RESEARCH

Further investigations of similar nature can be done with other variants of Kalah. Some modifications that may yield interesting results are listed as follows.

(1) Allow sowing to take place in either direction.

This would mean that the $m - j - k$, $1 \leq k \leq j$ holes are no longer 'safe' and hence the strategy would have to be changed.

(2) Allow multiple captures, in which all preceding holes satisfying the criteria for capture are allowed to capture.

The addition of this rule would result in more holes to be protected in any turn. The player sowing the seeds also will have to determine which bin to sow, not solely based on which of his opponent's holes has the most seeds, but to take into account the total number that can be obtained through the multiple captures.

(3) Vary the rule of capture.

For example, only when the last seed lands in a hole with already one seed is the player allowed to capture the opponent's seeds, or when the seeds land in the player's territory. Another possibility is that the player captures all the seeds of the hole the last seed lands in.

(4) Allowing each hole to contain different number of seeds.

For example, let both players decide the initial configuration of seeds given each player has the same number of seeds. What would be the most advantageous configuration using the rules of Kalah?

APPENDIX A: PROGRAM USED TO GENERATE DATA FOR KALAH$(1, N)$

## A1 Documentation

1$n$ ver. 6

Introduction

The game of kalah$(1, n)$, where $n$ indicates the number of seeds available to each player at the beginning of the game. This program uses the BASIC programming language.

Rules

(1) There is only one hole and home for each player.

(2) Player 1 always starts the game and sows anti-clockwise from his own hole, skipping only the opponent's home, putting one seed into every hole until there are no seeds left.

(3) If the last seed lands in the player's own empty hole, the player captures all seeds except those in the opponent's home.

(4) If the last counter lands in the player's home, the player makes another move.

Possible improvements.

(1) Allow the program to find the number of steps needed to complete the game.

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Last modified on 16 May 2001

## A2 Source Code

REM to find the game values for kalah(1,n)

```
5 CLS
10 PRINT “Kalah (1,n) Ver 6”
15 PRINT “Created by Pok Ai Ling, Irene. Last modification on 16th May 2001”
20 PRINT
25 PRINT
30 INPUT “what values of n? (n must be a positive integer of value at least 1)”; n
35 IF n <> 3 THEN GOTO 40
40 IF n = 3 THEN GOTO 420
45 IF n = 2 THEN GOTO 435
50 IF n = 1 THEN GOTO 445
55 LET home1 = 0
60 LET hole1 = n
65 LET home2 = 0
70 LET hole2 = n
75 PRINT “hole1=”; hole1
80 PRINT “hole2=”; hole2
85 PRINT “home1=”; home1
90 PRINT “home2=”; home2
95 INPUT “Hit ’Enter’ to continue”; cont
100 IF hole1 = 0 THEN GOTO 385 ELSE GOTO 105
105 IF hole2 = 0 THEN GOTO 400 ELSE GOTO 110
110 LET remainder = hole1 MOD 3
115 LET hole1 = hole1  3
120 IF remainder = 0 AND hole1 = 1 THEN GOTO 140
125 IF remainder = 0 THEN GOTO 185
130 IF remainder = 1 THEN GOTO 225
135 IF remainder = 2 THEN GOTO 265
```

```
140 LET home1 = 2 * n - home2
145 LET hole1 = 0
150 LET hole2 = 0
155 PRINT “hole1=”; hole1
160 PRINT “hole2=”; hole2
165 PRINT “home1=”; home1
170 PRINT “home2=”; home2
175 INPUT “Hit ’Enter’ to continue”; cont
180 GOTO 415
185 LET home1 = hole1 + home1
190 LET hole2 = hole1 + hole2
195 PRINT “hole1=”; hole1
200 PRINT “hole2=”; hole2
205 PRINT “home1=”; home1
210 PRINT “home2=”; home2
215 INPUT “Hit ’Enter’ to continue”; cont
220 GOTO 280
225 LET home1 = hole1 + home1 + 1
230 LET hole2 = hole1 + hole2
235 PRINT “hole1=”; hole1
240 PRINT “hole2=”; hole2
245 PRINT “home1=”; home1
250 PRINT “home2=”; home2
255 INPUT “Hit ’Enter’ to continue”; cont
260 GOTO 100
265 LET home1 = home1 + hole1 + 1
270 LET hole2 = hole2 + hole1 + 1
275 GOTO 195
280 IF hole1 = 0 THEN GOTO 385
285 IF hole2 = 0 THEN GOTO 400
290 LET remainder = hole2 MOD 3
```

```
295 LET hole2 = hole2  3
300 IF remainder = 0 AND hole2 = 1 THEN GOTO 320
305 IF remainder = 0 THEN GOTO 340
310 IF remainder = 1 THEN GOTO 355
315 IF remainder = 2 THEN GOTO 370
320 LET home2 = 2 * n - home1
325 LET hole1 = 0
330 LET hole2 = 0
335 GOTO 155
340 LET home2 = hole2 + home2
345 LET hole1 = hole2 + hole1
350 GOTO 235
355 LET home2 = home2 + hole2 + 1
360 LET hole1 = hole2 + hole1
365 GOTO 195
370 LET home2 = home2 + hole2 + 1
375 LET hole1 = hole1 + hole2 + 1
380 GOTO 235
385 LET home2 = home2 + hole2
390 LET hole2 = 0
395 GOTO 155
400 LET home1 = home1 + hole1
405 LET hole1 = 0
410 GOTO 155
415 IF home1 > home2 THEN PRINT “Player 1 wins!’ ’ ELSE GOTO 420
420 IF home1 < home2 THEN PRINT “Player 1 loses!” ELSE GOTO 430
430 PRINT “It’s a draw!”
435 INPUT “Go again? Press ’0’ to exit”; again
440 IF again = 0 THEN END ELSE GOTO 5
```

## A3 Tabulated Results for $1 \leq n \leq 200$

Table 4: Table for $(1, n)$ for $1 \leq n \leq 200$

| N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | D | 1 | 1 | 26 | L | 20 | 32 | 51 | L | 39 | 63 |
| 2 | L | 1 | 3 | 27 | W | 29 | 25 | 52 | L | 46 | 58 |
| 3 | W | 6 | 0 | 28 | W | 34 | 22 | 53 | L | 51 | 55 |
| 4 | L | 3 | 5 | 29 | L | 22 | 36 | 54 | D | 54 | 54 |
| 5 | W | 6 | 4 | 30 | W | 38 | 22 | 55 | W | 62 | 48 |
| 6 | D | 6 | 6 | 31 | W | 62 | 0 | 56 | W | 60 | 52 |
| 7 | L | 4 | 10 | 32 | L | 30 | 34 | 57 | W | 59 | 55 |
| 8 | D | 8 | 8 | 33 | D | 33 | 33 | 58 | W | 64 | 52 |
| 9 | L | 7 | 11 | 34 | L | 25 | 43 | 59 | L | 52 | 66 |
| 10 | W | 20 | 0 | 35 | W | 41 | 29 | 60 | W | 66 | 54 |
| 11 | D | 11 | 11 | 36 | D | 36 | 36 | 61 | D | 61 | 61 |
| 12 | L | 10 | 14 | 37 | L | 29 | 45 | 62 | W | 64 | 60 |
| 13 | L | 8 | 18 | 38 | D | 38 | 38 | 63 | W | 65 | 61 |
| 14 | D | 14 | 14 | 39 | W | 44 | 34 | 64 | W | 77 | 71 |
| 15 | L | 9 | 21 | 40 | L | 22 | 58 | 65 | W | 70 | 60 |
| 16 | L | 11 | 21 | 41 | W | 45 | 37 | 66 | L | 60 | 72 |
| 17 | L | 14 | 20 | 42 | D | 42 | 42 | 67 | L | 35 | 99 |
| 18 | W | 20 | 16 | 43 | L | 32 | 54 | 68 | L | 44 | 92 |
| 19 | W | 21 | 17 | 44 | D | 44 | 44 | 69 | L | 57 | 81 |
| 20 | W | 26 | 14 | 45 | L | 39 | 51 | 70 | D | 70 | 70 |
| 21 | D | 21 | 21 | 46 | L | 30 | 62 | 71 | W | 73 | 69 |
| 22 | L | 12 | 32 | 47 | W | 49 | 45 | 72 | D | 72 | 72 |
| 23 | L | 8 | 38 | 48 | W | 55 | 41 | 73 | L | 65 | 81 |
| 24 | W | 26 | 22 | 49 | L | 48 | 50 | 74 | L | 70 | 78 |
| 25 | W | 32 | 18 | 50 | W | 65 | 35 | 75 | W | 76 | 74 |

Table 4: Table for $(1, n)$ for $1 \leq n \leq 200$ (cont'd)

| N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 76 | D | 76 | 76 | 101 | W | 111 | 91 | 126 | W | 133 | 119 |
| 77 | L | 76 | 78 | 102 | W | 143 | 61 | 127 | W | 133 | 121 |
| 78 | D | 78 | 78 | 103 | W | 112 | 94 | 128 | L | 122 | 134 |
| 79 | W | 83 | 75 | 104 | W | 107 | 101 | 129 | D | 129 | 129 |
| 80 | L | 61 | 99 | 105 | D | 105 | 105 | 130 | L | 129 | 131 |
| 81 | L | 70 | 92 | 106 | W | 112 | 100 | 131 | L | 130 | 132 |
| 82 | W | 89 | 75 | 107 | W | 109 | 105 | 132 | W | 133 | 131 |
| 83 | L | 67 | 99 | 108 | L | 105 | 111 | 133 | W | 138 | 128 |
| 84 | L | 55 | 113 | 109 | W | 111 | 107 | 134 | L | 80 | 188 |
| 85 | L | 84 | 86 | 110 | W | 119 | 101 | 135 | L | 115 | 155 |
| 86 | D | 86 | 86 | 111 | W | 114 | 108 | 136 | L | 132 | 140 |
| 87 | W | 89 | 85 | 112 | W | 122 | 102 | 137 | W | 143 | 131 |
| 88 | L | 86 | 90 | 113 | L | 112 | 114 | 138 | W | 139 | 197 |
| 89 | D | 89 | 89 | 114 | D | 114 | 114 | 139 | L | 138 | 140 |
| 90 | W | 116 | 64 | 115 | L | 108 | 122 | 140 | L | 138 | 142 |
| 91 | L | 86 | 96 | 116 | L | 114 | 118 | 141 | L | 136 | 146 |
| 92 | D | 92 | 92 | 117 | W | 119 | 115 | 142 | L | 141 | 143 |
| 93 | L | 91 | 95 | 118 | W | 124 | 112 | 143 | L | 134 | 152 |
| 94 | W | 188 | 0 | 119 | L | 117 | 121 | 144 | W | 156 | 132 |
| 95 | L | 93 | 97 | 120 | L | 117 | 123 | 145 | L | 138 | 152 |
| 96 | W | 114 | 78 | 121 | L | 63 | 179 | 146 | L | 144 | 148 |
| 97 | W | 101 | 93 | 122 | L | 120 | 124 | 147 | W | 149 | 145 |
| 98 | W | 100 | 96 | 123 | W | 125 | 121 | 148 | W | 150 | 146 |
| 99 | L | 92 | 106 | 124 | W | 129 | 119 | 149 | L | 136 | 162 |
| 100 | W | 102 | 98 | 125 | W | 128 | 122 | 150 | D | 150 | 150 |

Table 4: Table for $(1, n)$ for $1 \leq n \leq 200$ (cont'd)

| N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 151 | L | 146 | 156 | 168 | D | 168 | 168 | 185 | D | 185 | 185 |
| 152 | L | 134 | 170 | 169 | D | 169 | 169 | 186 | W | 193 | 179 |
| 153 | W | 159 | 147 | 170 | W | 189 | 151 | 187 | L | 182 | 192 |
| 154 | L | 153 | 155 | 171 | W | 175 | 167 | 188 | L | 186 | 190 |
| 155 | W | 161 | 149 | 172 | W | 175 | 169 | 189 | L | 188 | 190 |
| 156 | L | 146 | 166 | 173 | L | 167 | 179 | 190 | W | 196 | 184 |
| 157 | L | 155 | 159 | 174 | L | 172 | 176 | 191 | W | 194 | 188 |
| 158 | D | 158 | 158 | 175 | W | 178 | 172 | 192 | L | 184 | 200 |
| 159 | L | 158 | 160 | 176 | L | 175 | 177 | 193 | W | 196 | 190 |
| 160 | W | 243 | 77 | 177 | W | 184 | 170 | 194 | L | 186 | 202 |
| 161 | L | 155 | 167 | 178 | W | 191 | 165 | 195 | W | 196 | 194 |
| 162 | L | 155 | 169 | 179 | L | 171 | 187 | 196 | L | 88 | 304 |
| 163 | W | 186 | 140 | 180 | W | 181 | 179 | 197 | L | 183 | 211 |
| 164 | L | 153 | 175 | 181 | W | 182 | 180 | 198 | W | 201 | 195 |
| 165 | L | 151 | 179 | 182 | W | 187 | 177 | 199 | W | 204 | 194 |
| 166 | L | 158 | 174 | 183 | W | 190 | 176 | 200 | L | 198 | 202 |
| 167 | W | 172 | 162 | 184 | W | 186 | 182 | | | | |

APPENDIX B: DATA USED FOR KALAH($M, 1$)

Table 6: Game Values for Kalah $(m, 1)$

| $M/N$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | D | L | W | L | W | D |
| 2 | W | L | L | L | W | W |
| 3 | D | W | W | W | W | L |
| 4 | W | W | W | | | |
| 5 | D | | | | | |
| 6 | W | | | | | |

This table was adapted from Irving, G., Donkers, J. and Uiterwijk, J.W.H.M. (2000). Solving Kalah. *ICGA Journal,* Vol. 23, No. 3, pp. 139-147, Table 5

## APPENDIX C: PREVIOUS STRATEGIES TRIED FOR KALAH $(M, 1)$

Result: P1 can obtain at least a draw for Kalah$(m, 1)$

**Attempt 1:**

Strategy

(1) Move any hole that results in the last seed in home

(2) Move any seed(s) that can be used for capture.

Counterexample:

Take the case of (8,1) in Figure 2, with P1 following the strategy strictly, and P2 being free to choose any of the possible moves.

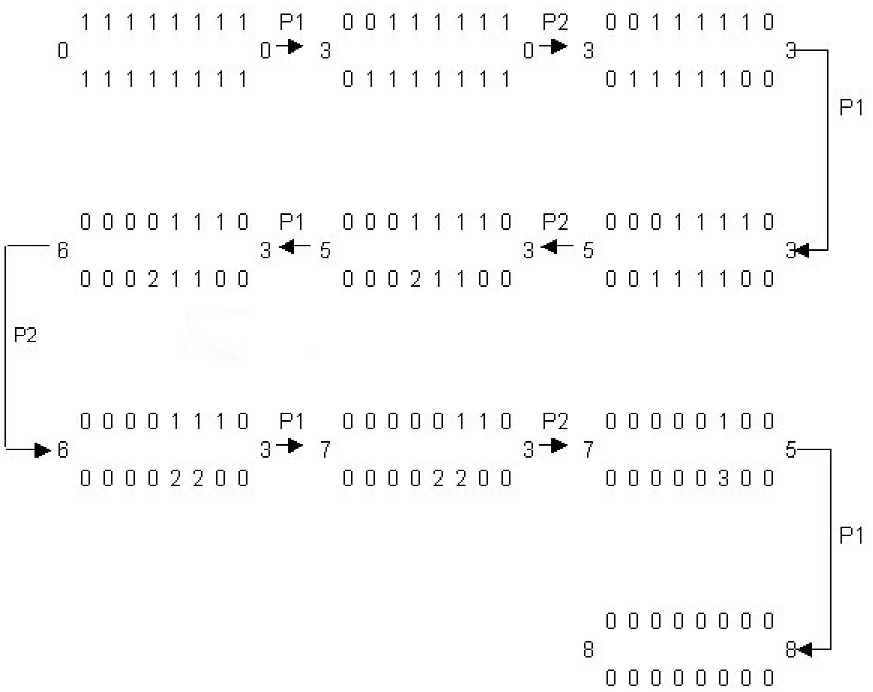


Figure 2: Counterexample to Attempt 1

**Attempt 2:**

Strategy

(1) First move is to sow the seed in hole $m$ then hole $m-1$.

(2) Attempt to capture.

(3) Move the seed nearest home.

Counterexample:

Take the case of (6,1) in Figure 3, with P1 following the strategy strictly, and P2 being free to choose any of the possible moves.

This particular example also illustrates that in order for P2 to reach the configuration to ensure continuous moves into home, P2 has to sacrifice a number of captures initially.

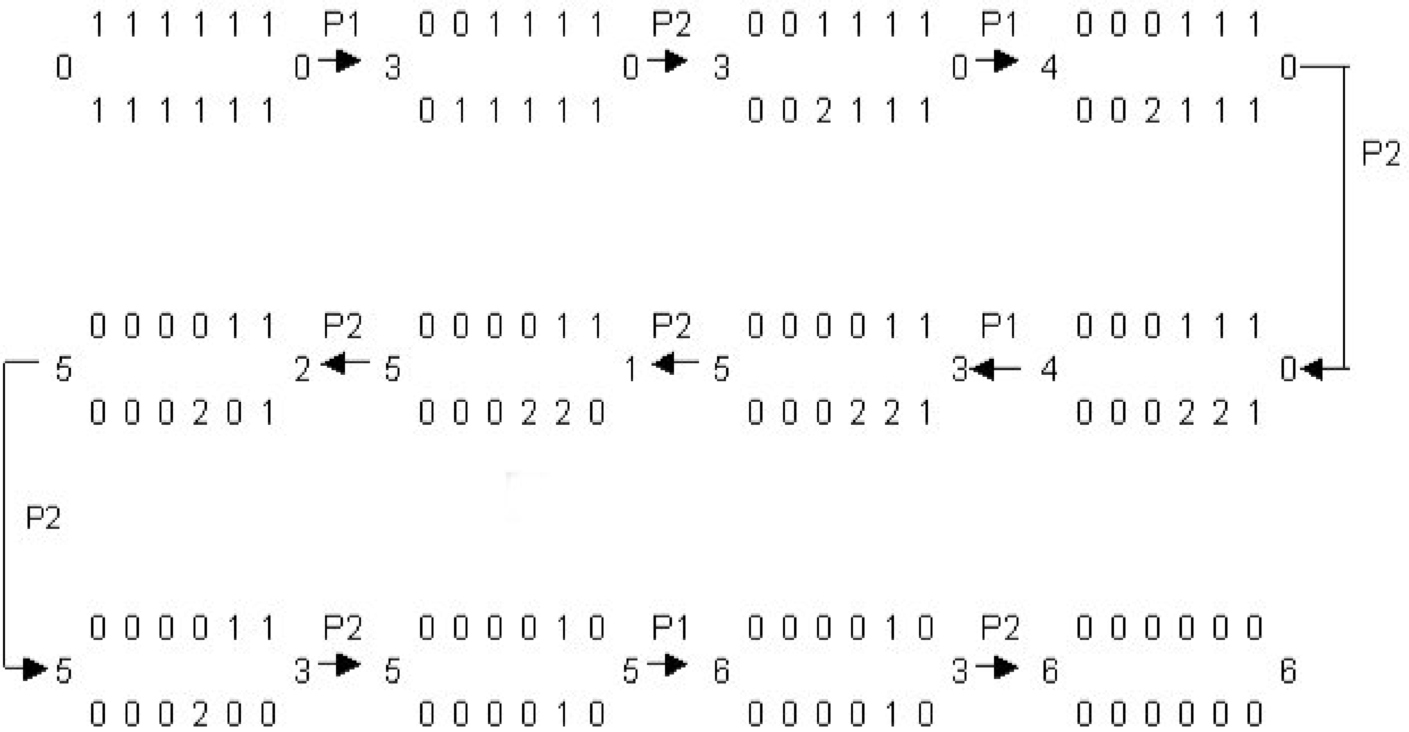


Figure 3: Counterexample to Attempt 2

**Attempt 3:**

Strategy

(1) First move of P1 will be to move the seed in hole $m$ then hole $m-1$.

(2) Mirror P2's moves.

Counterexample:

Take the case of (8,1) in Figure 4, with P1 following the strategy strictly, and P2 being free to choose any of the possible moves.

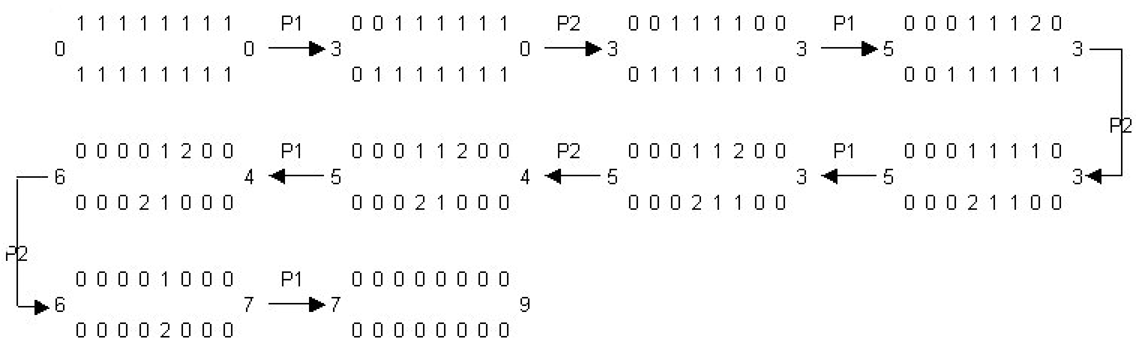


Figure 4: Counterexample to Attempt 3

**Attempt 4:**

Strategy

(I) First move of P1 will be to move seed in hole $m$ and $m-1$

(II) For any move after the previous move, decide in the following manner, in order of priority.

1. Locate all bowls that can be used for capture or that can be captured. Sow the bowl contains the larger number of seeds.

2. Observe the positions of P2's seeds. Let the first hole with a seed nearest P1's home be $j$. Move seed(s) in hole $m-j-1$ on P1's side.

3. Move seeds towards home.

This is the most likely strategy so far, but it is not proven.

Justification:

(I) This set of moves allows P1 to capture the most number of seeds in the first move, yet P2 is only able to equalise for this turn.

(II) 1. This move allows P1 to either defend or attack, depending on which is more advantageous to P1.

2. All seeds in holes between hole 1 and hole $j$ are empty. However, by moving the seed(s) in hole $m-j-1$, we protect the seeds in that hole and opens a hole for capture in the next turn.

3. This move would enable P1 to execute a half-capture, and only occurs when all seeds belonging to P1 are 'safe'.

APPENDIX D: PROGRAM USED TO GENERATE DATA FOR KALAHALT$(1, N)$

## D1 Documentation

ALT ver. 4

Introduction

This is a modified version of Kalah$(1, n)$, where $n$ indicates the number of seeds available to each player at the beginning of the game. This program uses the BASIC programming language.

Rules

(1) There is only one hole and home for each player.

(2) Player 1 always starts the game and sows anti-clockwise from his own hole, skipping only the opponent's home, putting one seed into every hole until there are no seeds left.

(3) If the last seed lands in the player's own empty hole, the player captures all seeds except those in the opponent's home.

(4) Regardless of where the last seed lands, the turn is over and the next move belongs to the opponent.

Possible improvements.

(1) Allow the program to find the number of steps needed to complete the game.

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Last modified on 16 May 2001

## D2 Source Code

REM to find the game values of kalah(1,n) using alternate moves with flow capture

```
5 CLS
10 PRINT “KalahAlt Ver 4”
15 PRINT “Created by Pok Ai Ling, Irene. Last modified on 16th May 2001”
20 PRINT
25 PRINT
30 INPUT “what value of n? (n must be an integer of value 1 or larger)”; n
35 LET home1 = 0
40 LET hole1 = n
45 LET home2 = 0
50 LET hole2 = n
55 PRINT “hole1=”; hole1
60 PRINT “hole2=”; hole2
65 PRINT “home1=”; home1
70 PRINT “home2=”; home2
75 INPUT “Hit ’Enter’ to continue”; cont
80 LET remainder = hole1 MOD 3
85 LET hole1 = hole1  3
90 IF remainder = 0 AND hole1 = 1 THEN GOTO 95 ELSE GOTO 115
95 LET home1 = 2 * n - home2
100 LET hole1 = 0
105 LET hole2 = 0
110 GOTO 340
115 IF remainder = 1 THEN GOTO 120 ELSE GOTO 135
120 LET hole2 = hole1 + hole2
125 LET home1 = hole1 + home1 + 1
130 GOTO 150
135 IF remainder = 2 THEN GOTO 140 ELSE GOTO 155
```

```
140 LET home1 = hole1 + home1 + 1
145 LET hole2 = hole2 + hole1 + 1
150 GOTO 170
155 IF remainder = 0 THEN GOTO 160 ELSE GOTO 195
160 LET home1 = hole1 + home1
165 LET hole2 = hole2 + hole1
170 PRINT "hole1"; hole1
175 PRINT "home1"; home1
180 PRINT "hole2"; hole2
185 PRINT "home2"; home2
190 INPUT "Hit 'Enter' to continue"; cont
195 IF hole1 = 0 THEN GOTO 335
200 IF hole2 = 0 THEN GOTO 380
205 LET remainder = hole2 MOD 3
210 LET hole2 = hole2  3
215 IF remainder = 1 THEN GOTO 220 ELSE GOTO 235
220 LET hole1 = hole1 + hole2
225 LET home2 = hole2 + home2 + 1
230 GOTO 275
235 IF remainder = 0 AND hole2 = 1 THEN GOTO 240 ELSE GOTO 260
240 LET home2 = 2 * n - home1
245 LET hole1 = 0
250 LET hole2 = 0
255 GOTO 275
260 IF remainder = 2 THEN GOTO 265 ELSE GOTO 280
265 LET hole1 = hole1 + hole2 + 1
270 LET home2 = home2 + hole2 + 1
275 GOTO 295
280 IF remainder = 0 THEN GOTO 285 ELSE GOTO 325
285 LET hole1 = hole1 + hole2
290 LET home2 = home2 + hole2
```

295 PRINT “hole1”; hole1

300 PRINT “hole2”; hole2

305 PRINT “home1”; home1

310 PRINT “home2”; home2

315 INPUT “Hit ’Enter’ to continue”; cont

320 IF hole1 = 0 AND hole2 = 0 THEN GOTO 390

325 IF hole1 = 0 THEN GOTO 335 ELSE GOTO 330

330 IF hole2 = 0 THEN GOTO 375 ELSE GOTO 80

335 LET home2 = home2 + hole2

340 LET hole2 = 0

345 PRINT “hole1”; hole1

350 PRINT “hole2”; hole2

355 PRINT “home1”; home1

360 PRINT “home2”; home2

365 INPUT “Hit ’Enter’ to continue”; cont

370 GOTO 390

375 LET home1 = home1 + hole1

380 LET hole1 = 0

385 GOTO 345

390 IF home1 > home2 THEN PRINT “Player 1 wins!” ELSE GOTO 395

395 IF home1 < home2 THEN PRINT “Player 1 loses!” ELSE GOTO 400

400 IF home1 = home2 THEN PRINT “It’s a draw!”

405 INPUT “Go again? Press ’0’ to exit‘”; again

410 IF again = 0 THEN END ELSE GOTO 5

## D3 Tabulated Results for $1 \leq n \leq 200$

Table 5: Table for $(1, n)$ for $1 \leq n \leq 200$

| N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | D | 1 | 1 | 26 | W | 27 | 25 | 51 | L | 50 | 52 |
| 2 | L | 1 | 3 | 27 | W | 28 | 26 | 52 | D | 52 | 52 |
| 3 | W | 6 | 0 | 28 | W | 29 | 27 | 53 | L | 52 | 54 |
| 4 | W | 6 | 2 | 29 | L | 28 | 30 | 54 | L | 53 | 55 |
| 5 | W | 7 | 3 | 30 | L | 28 | 32 | 55 | L | 54 | 56 |
| 6 | L | 5 | 7 | 31 | L | 30 | 32 | 56 | L | 55 | 57 |
| 7 | W | 9 | 5 | 32 | L | 30 | 34 | 57 | L | 55 | 59 |
| 8 | W | 9 | 7 | 33 | L | 32 | 34 | 58 | L | 57 | 59 |
| 9 | L | 8 | 10 | 34 | L | 33 | 35 | 59 | L | 57 | 61 |
| 10 | L | 9 | 11 | 35 | L | 34 | 36 | 60 | L | 59 | 61 |
| 11 | L | 10 | 12 | 36 | L | 35 | 37 | 61 | L | 60 | 62 |
| 12 | W | 13 | 11 | 37 | L | 36 | 38 | 62 | W | 63 | 61 |
| 13 | W | 15 | 11 | 38 | W | 39 | 37 | 63 | W | 65 | 61 |
| 14 | W | 15 | 13 | 39 | W | 40 | 38 | 64 | W | 66 | 62 |
| 15 | W | 16 | 14 | 40 | W | 42 | 38 | 65 | W | 66 | 64 |
| 16 | L | 15 | 17 | 41 | W | 42 | 40 | 66 | W | 67 | 65 |
| 17 | L | 16 | 18 | 42 | W | 43 | 41 | 67 | W | 69 | 65 |
| 18 | L | 16 | 20 | 43 | W | 45 | 41 | 68 | W | 69 | 67 |
| 19 | L | 17 | 21 | 44 | W | 46 | 42 | 69 | W | 70 | 68 |
| 20 | L | 19 | 21 | 45 | W | 47 | 43 | 70 | W | 71 | 69 |
| 21 | L | 20 | 22 | 46 | W | 48 | 44 | 71 | W | 72 | 70 |
| 22 | W | 24 | 20 | 47 | W | 48 | 46 | 72 | W | 73 | 71 |
| 23 | W | 24 | 22 | 48 | W | 49 | 47 | 73 | W | 74 | 72 |
| 24 | W | 25 | 23 | 49 | W | 50 | 48 | 74 | W | 75 | 73 |
| 25 | W | 27 | 23 | 50 | L | 48 | 52 | 75 | W | 76 | 74 |

Table 5: Table for $(1, n)$ for $1 \leq n \leq 200$ (cont'd)

| N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 76 | W | 78 | 74 | 101 | L | 99 | 103 | 126 | W | 127 | 125 |
| 77 | W | 79 | 75 | 102 | L | 100 | 104 | 127 | W | 128 | 126 |
| 78 | W | 80 | 76 | 103 | L | 102 | 104 | 128 | W | 130 | 126 |
| 79 | W | 81 | 77 | 104 | L | 102 | 106 | 129 | W | 131 | 127 |
| 80 | W | 81 | 79 | 105 | L | 104 | 106 | 130 | W | 132 | 128 |
| 81 | L | 80 | 82 | 106 | D | 106 | 106 | 131 | W | 133 | 129 |
| 82 | L | 81 | 83 | 107 | W | 109 | 105 | 132 | W | 134 | 130 |
| 83 | L | 82 | 84 | 108 | W | 109 | 107 | 133 | W | 135 | 131 |
| 84 | L | 82 | 86 | 109 | W | 110 | 108 | 134 | W | 135 | 133 |
| 85 | L | 84 | 86 | 110 | W | 111 | 109 | 135 | W | 137 | 133 |
| 86 | L | 84 | 88 | 111 | W | 112 | 110 | 136 | W | 138 | 134 |
| 87 | L | 85 | 89 | 112 | W | 114 | 110 | 137 | W | 139 | 135 |
| 88 | L | 87 | 89 | 113 | W | 114 | 112 | 138 | W | 139 | 137 |
| 89 | L | 88 | 90 | 114 | W | 115 | 113 | 139 | W | 140 | 138 |
| 90 | L | 89 | 91 | 115 | W | 116 | 114 | 140 | L | 138 | 142 |
| 91 | L | 90 | 92 | 116 | W | 117 | 115 | 141 | L | 140 | 142 |
| 92 | L | 91 | 93 | 117 | W | 118 | 116 | 142 | L | 141 | 143 |
| 93 | L | 92 | 94 | 118 | W | 119 | 117 | 143 | L | 142 | 144 |
| 94 | D | 94 | 94 | 119 | W | 120 | 118 | 144 | L | 143 | 145 |
| 95 | L | 94 | 96 | 120 | D | 120 | 120 | 145 | L | 144 | 146 |
| 96 | L | 95 | 97 | 121 | W | 122 | 120 | 146 | L | 144 | 148 |
| 97 | L | 96 | 98 | 122 | D | 122 | 122 | 147 | L | 145 | 149 |
| 98 | L | 97 | 99 | 123 | W | 124 | 122 | 148 | L | 147 | 149 |
| 99 | L | 97 | 101 | 124 | W | 125 | 123 | 149 | L | 148 | 150 |
| 100 | L | 98 | 102 | 125 | W | 126 | 124 | 150 | L | 149 | 151 |

Table 5: Table for $(1, n)$ for $1 \leq n \leq 200$ (cont'd)

| N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 | N | D/W/L | P1 | P2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 151 | L | 150 | 152 | 168 | L | 167 | 169 | 185 | D | 185 | 185 |
| 152 | L | 151 | 153 | 169 | L | 168 | 170 | 186 | W | 187 | 185 |
| 153 | L | 152 | 154 | 170 | L | 168 | 172 | 187 | W | 189 | 185 |
| 154 | L | 153 | 155 | 171 | L | 170 | 172 | 188 | W | 189 | 187 |
| 155 | L | 154 | 156 | 172 | L | 171 | 173 | 189 | W | 190 | 188 |
| 156 | L | 155 | 157 | 173 | L | 172 | 174 | 191 | W | 193 | 189 |
| 157 | D | 157 | 157 | 174 | L | 172 | 176 | 190 | W | 191 | 189 |
| 158 | L | 157 | 159 | 175 | L | 174 | 176 | 192 | W | 193 | 191 |
| 159 | L | 158 | 160 | 176 | L | 174 | 178 | 193 | W | 195 | 191 |
| 160 | L | 159 | 161 | 177 | L | 175 | 179 | 194 | W | 195 | 193 |
| 161 | L | 160 | 162 | 178 | L | 177 | 179 | 195 | W | 197 | 193 |
| 162 | L | 161 | 163 | 179 | L | 178 | 180 | 196 | W | 198 | 194 |
| 163 | L | 162 | 164 | 180 | L | 179 | 181 | 197 | W | 199 | 195 |
| 164 | L | 163 | 165 | 181 | L | 180 | 182 | 198 | W | 200 | 196 |
| 165 | L | 164 | 166 | 182 | W | 183 | 181 | 199 | W | 201 | 197 |
| 166 | D | 166 | 166 | 183 | W | 184 | 182 | 200 | W | 201 | 199 |
| 167 | L | 166 | 168 | 184 | W | 185 | 183 | | | | |

APPENDIX E: DATA USED FOR KALAHALT $(M, 1)$

Shown below are some game trees for KalahAlt $(m, 1)$, in the case where P1 follows the strategy below and P2 is allowed to make any move. All game values shown (D=Draw, W=Win, L=Lose) are with respect to Player 1.

Observation:

P1 will always be able to obtain at least a draw if he moves in the following manner:

(i) P1 moves the seed in hole $m$

(ii)

(1) Attempt to capture.

(2) Move towards home.

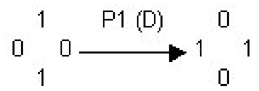


Figure 5: Game tree of KalahAlt(1,1)

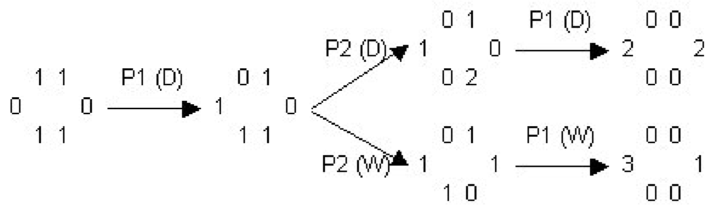


Figure 6: Game tree of KalahAlt(2,1)

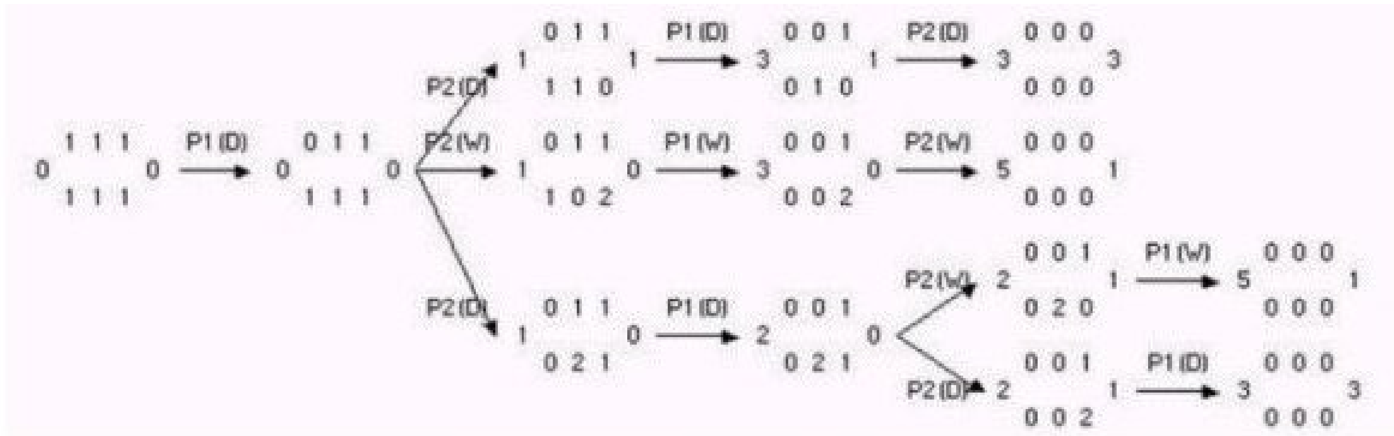


Figure 7: Game Tree for KalahAlt(3,1)

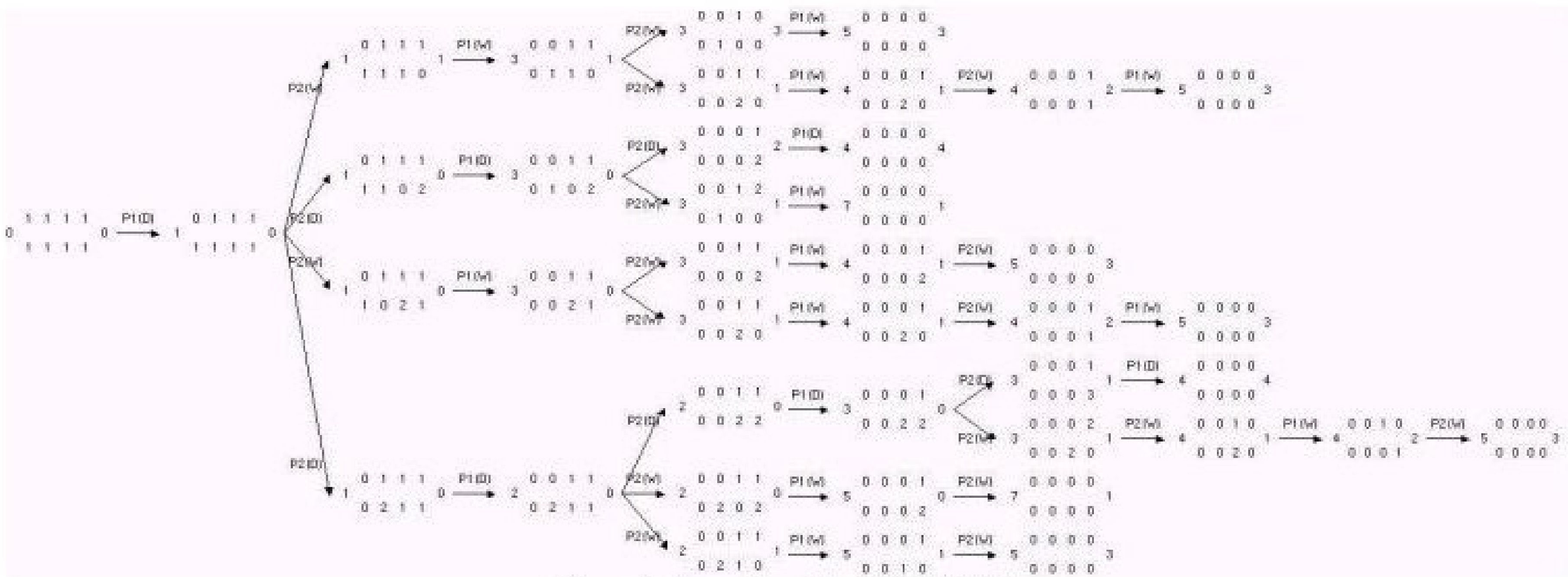

Figure 8: Game Tree for KalahAlt(5,1)

APPENDIX F: REFERENCES

[1] Irving, G., Donkers, J. and Uiterwijk, J.W.H.M.(2000): *Solving Kalah*, *ICGA Journal,* Vol. 23, No. 3, pp. 139-147.