1. What is a Platonic solid? What is a deltahedron? Give at least one example of a deltahedron that is not a Platonic solid. What is the error Euclid made when he defined a Platonic solid?

Solution: A Platonic solid is a convex polyhedron whose faces are congruent, regular polygons, and where the same number of faces meet at each corner.

A deltahedron is a convex polyhedron whose faces are congruent, regular triangles. There are eight deltahedra, five of which are not Platonic.

Euclid forgot to require that the vertices should be the same, so his definition includes the deltahedra.
2. Prove that there are only five Platonic solids and three Platonic tilings.

Solution: Then interior angle of a regular p-gon is \( \frac{(p-2)}{p} \). In a Platonic solid where q p-gons meet at each vertex, we must have

\[
q \cdot \frac{(p-2)}{p} < 2
\]

This is to ensure that the q p-gons will fit, and that we can fold the corner up into space. This gives

\[
pq - 2q < 2p, \quad (p - 2)(q - 2) < 4.
\]

The only possible solutions are:

<table>
<thead>
<tr>
<th>P</th>
<th>q</th>
<th>Solid</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>Tet</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>Oct</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>Cub</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>Ico</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>Dod</td>
</tr>
</tbody>
</table>

In a Platonic tiling, we will instead have

\[
q \cdot \frac{(p-2)}{p} = 2, \quad (p - 2)(q - 2) = 4.
\]

The only possible solutions are:

<table>
<thead>
<tr>
<th>P</th>
<th>q</th>
<th>Tile</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>Triangles</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>Squares</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>Hexagons</td>
</tr>
</tbody>
</table>
3. Complete this table for the Platonic solids.

<table>
<thead>
<tr>
<th>Name of solid</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
<th>Sides of each face</th>
<th>Faces at each vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Octahedron</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Solution:
4. How many pentagons and hexagons are there on a football? What is the mathematical name of the shape of a football?

Solution: A football is a truncated icosahedron. Each of the 12 vertices on the icosahedron becomes a pentagon, while the 20 triangles become hexagons.
5. What is the definition of the golden ratio? Derive the numerical value. (If you don't have a calculator, you can leave the answer in exact form using roots.)

Solution: The point C divides the line AB in the golden ratio if AC/CB = AB/AC.

```
A   C   E
```

If AC/CB = AB/AC = x, then

\[ AB = AC + CB \]
\[ x \cdot AC = AC + CB \]
\[ x^2 \cdot CB = x \cdot CB + CB \]
\[ x^2 - x - 1 = 0 \]

The positive root is \((1+\sqrt{5})/2 = 1.6\ldots\)

Solution:

Let M be the midpoint of AB. Suppose AB = a. Then MC = sqrt(5) a/2 and BE = sqrt(5) a/2 - a/2, so AE/AB = (a + sqrt(5) a/2 - a/2)/a = (1 + sqrt(5))/2.
7. Use the chart in Attachment 1 to determine the symmetry type of the frieze patterns below.

Solution:
1. p111
2. p1a1
3. pm11
4. p112
5. p1m1
6. pma2
7. pmm2
8. Use the chart in Attachment 2 to determine the symmetry type of the wallpaper patterns below.

Pattern: 

Pattern: 

Pattern: 

Pattern: 

Pattern: 

Pattern: 

Solution: 1: cmm, 2: p4m, 3: p6m, 4: p6, 5: pmm, 6: p31m, 7: p3m1.
9. Draw the (3.6.3.6) Archimedean tiling.

Solution:
10. Show how to determine the dimensions of A4 paper. (If you don't have a calculator, you can leave the answer in exact form using roots.)

Solution: Suppose a piece of An paper has height = h and width = w. Then A(n+1) is similar to An and has height = w and width = h/2. We also know that A0 has area 1m^2.

If An is similar to A(n+1), then h/w = w/(h/2) or (h/w)^2 = 2 so A0 has dimensions (w, sqrt(2)w) and its area equals sqrt(2) w^2 =1, so w = 2^(-1/4).

It follows that

A0: (2^(-1/4), 2^(1/4)),
A1: (2^(-3/4), 2^(-1/4)),
A2: (2^(-5/4), 2^(-3/4)),
A3: (2^(-7/4), 2^(-5/4)),
A4: (2^(-9/4), 2^(-7/4)) = (0.210, 0.297).
11. Show how to construct a curve with constant width, other than the circle. (Hint: Draw a Reuleaux triangle.) Give one example of why such curves are useful.

Solution:

A Reuleaux triangle is obtained by drawing circular arcs between adjacent vertices with centre at the opposite vertex.

This can be done for any odd-sided polygon, and is useful for making coins that vending machines can identify.
12. What is the difference between a unicursal labyrinth, a simply connected multicursal maze and a multiply connected multicursal maze? Label the three pictures. For which type will the hand-on-the-wall method always work?

Solution: The picture on the right is a unicursal labyrinth. There is only one path, with no branches or dead ends. The two other pictures are multicursal mazes, where there branches and dead ends.

The left one is simply connected, in that the walls form a connected set. In that case, the hand-on-the-wall method will always take you to the goal.

The middle one is multiply connected. The goal is located within an "island", that is a component of the wall that is not connected to the outer wall. In that case, the hand-on-the-wall will not work. If the goal is within a part of the wall that is connected to the outer wall, the hand-on-the-wall will still work, even if the maze is multiply connected.
Attachment 1: Flow chart for frieze patterns.

Is there a vertical reflection?

- yes
  - Is there a horizontal reflection?
    - yes
      - Is there a rotation of 180°?
        - yes
          - p112
        - no
          - p111
    - no
      - pma2
  - no
    - pm11

Is there a horizontal reflection or glide reflection?

- yes
  - Is there a horizontal reflection?
    - yes
      - Is there a rotation of 180°?
        - yes
          - p1m1
        - no
          - p1a1
    - no
      - pm11
- no
  - pmm2
Attachment 2: Flow chart for wallpaper patterns.

- **Is there a reflection?**
  - **Yes:** cm
  - **No:** pm

- **Is there a glide reflection?**
  - **Yes:** pg
  - **No:** p1

- **Are there reflections in two directions?**
  - **Yes:** pmg
  - **No:** p1

- **Are all rotation centres on reflection axis?**
  - **Yes:** pmm
  - **No:** cmm

- **What is the smallest rotation?**
  - **180°**
    - **Is there a reflection?**
      - **Yes:** cm
      - **No:** pm
  
  - **90°**
    - **Is there a reflection?**
      - **Yes:** cm
      - **No:** pm

  - **120°**
    - **Is there a reflection?**
      - **Yes:** cm
      - **No:** pm

  - **60°**
    - **Is there a reflection?**
      - **Yes:** cm
      - **No:** pm