1. Suppose that in the cascade of two tanks: external input flows into tank 1, solution in tank 1 flows into tank 2 and output flows out of tank 2. Suppose tank 1 initially contains $V_1 = 100$ (gal) and tank 2 initially contains $V_2 = 200$ (gal) the volume of brine, respectively. Each tank also initially contains 50 lb of salt. The three flow rates are each 5 gal/min, with pure water flowing into tank 1.

(i) Derive a mathematical model for the amounts $x(t)$ and $y(t)$ of salt in the two tanks at time $t \geq 0$.

(ii) Solve the model and find the analytical solution.

(iii) Find the maximum amount of salt in tank 2.

2. A predator-prey model for herbivore(H)-plankton(P) interaction is

\[
\frac{dP(t)}{dt} = rP \left[ (K - P) - \frac{BH}{C + P} \right],
\]
\[
\frac{dH(t)}{dt} = DH \left[ \frac{P}{C + P} - AH \right], \quad t > 0,
\]

where $r$, $K$, $A$, $B$, $C$ and $H$ are positive constants. Briefly explain the ecological assumptions in the model. Nondimensionalize the system so that it can be written in the form

\[
\frac{dp(\tau)}{d\tau} = p \left[ (k - p) - \frac{h}{1 + p} \right],
\]
\[
\frac{dh(\tau)}{d\tau} = dh \left[ \frac{p}{1 + p} - a h \right], \quad \tau > 0,
\]

where $k$, $d$ and $a$ are constants. Determine the equilibrium and investigate their stability. [Hint: You need only investigate the stability of the equilibrium: (i) $p = h = 0$; and (ii) $p = k$, $h = 0$. For the others, you need only say that they are solutions of a cubic algebraic equation and no need to discuss their stability.

3. Consider a model for the long-term dining behavior of the students at NUS with three canteens as Science Canteen, Arts Canteen and Engineering Canteen. Based on a student survey, it is found that

(i) For those students who eat currently at the Science Canteen, 50% will return to eat there, 25% will go to Engineering Canteen and 25% will go to Arts Canteen.

(ii) For those students who eat currently at the Engineering Canteen, 60% will return to eat there, 30% will go to Science Canteen and 10% will go to Arts Canteen.

(iii) For those students who eat currently at the Arts Canteen, 80% will return to eat there, 15% will go to Science Canteen and 5% will go to Engineering Canteen.

Formulate a model to solve for the long-term percentage of students eating at each canteen.

4. Consider the following economic model: Let $P$ be the price of a single item on the market. Let $Q$ be the quantity of the item available on the market. Both $P$ and $Q$ are
functions of time. If we consider price and quantity as two interacting species, the following model might be proposed:

\[
\begin{align*}
\frac{dP(t)}{dt} &= a P \left( \frac{b}{Q} - P \right), \\
\frac{dQ(t)}{dt} &= c Q (f P - Q), \quad t > 0,
\end{align*}
\]

where \(a, b, c\) and \(f\) are positive constants. Justify and discuss the adequacy of the model.

(i) If \(a = 1\), \(b = 20,000\), \(c = 1\), and \(f = 30\), find the equilibrium points of the system. Find their stability.

(ii) Find the equilibrium and discuss their stability when \(a, b, c\) and \(f\) are positive constants.

(iii) Find the phase trajectory.

5. In Jules Verne’s original problem, the projectile launched from the surface of the Earth is attracted by both the Earth and the Moon, so its distance \(r(t)\) from the center of the Earth satisfies the initial value problem

\[
\frac{d^2r(t)}{dt^2} = -\frac{GM_c}{r^2} + \frac{GM_m}{(S-r)^2}, \quad r(0) = R, \quad r'(0) = v_0,
\]

where \(M_c = 5.975 \times 10^{24}\) kg and \(M_m = 7.35 \times 10^{22}\) kg denote the masses of the Earth and the Moon, respectively; \(R = 6378\) km is the radius of the Earth and \(S = 384000\) km is the distance between the centers of the Earth and the Moon. To reach the Moon, the projectile must only just pass the point between the Moon and the Earth where its net acceleration vanishes. Therefore it is “under the control” of the Moon, and falls from there to the lunar surface. Find the **minimal** launch velocity \(v_0\) that suffices for the projectile to make it “From the Earth to the Moon”. 