Degenerate Linear BVP (Lecture 41)

Recall that it is possible for a homogeneous BVP with \( l_1(y) = l_2(y) = 0 \) to have infinite number of solutions (See Theorem 39.1 or the Remark after the theorem). However, let us note that the situation is quite different in the case of nonhomogeneous BVP. For example,

\[
y'' + 2y' + 2y = 5e^x, \quad y(0) = 0, y(\pi) = 0.
\]

Note that \( y_1 = e^{-x} \cos x \) and \( y_2 = e^{-x} \sin x \) are two linearly independent solutions of the homogeneous equation. Moreover, it is easy to see that \( e^x \) is a particular solution of the unhomogeneous equation. Hence the general solution of the unhomogeneous equation is

\[
y = e^{-x} (C_1 \cos x + C_2 \sin x) + e^x.
\]

But there are no constants \( C_1, C_2 \) such that

\[
C_1 + 1 = 0 = -C_1 e^{-\pi} + e^\pi.
\]

Thus the nonhomogeneous BVP has no solution.

However, if we change the BVP to

\[
y'' + 2y' + 2y = -e^{-x} + \frac{4}{1 + e^x}, \quad y(0) = 0, y(\pi) = 0.
\]

We actually have infinitely many solutions.
Sturm-Liouville Problems  (Lecture 42)

Self-adjoint differential equation: (for more information on self-adjoint, you could refer to Lecture 29.)

\[(p(x)y')' + q(x)y + \lambda r(x)y = P_2[y] + \lambda r(x)y = 0, \quad (42.1)\]

\[a_0 y(a) + a_1 y'(a) = b_0 y(b) + b_1 y'(b) = 0 \quad (a_0^2 + a_1^2 \neq 0, b_0^2 + b_1^2 \neq 0).\]

Note that \(\lambda\) is a parameter. Such a problem is called a Sturm-Liouville Problem. We say that the problem is regular if

\[p, r > 0 \text{ on } [a, b], \ p, q, r \text{ are continuous and } p \text{ is continuously differentiable.}\]

Solving such a problem means finding values of \(\lambda\) (eigenvalues) and the corresponding nontrivial solutions \(\phi_\lambda\) (eigenfunctions). The set of all eigenvalues of a regular problem is called its spectrum.

First, let us look at a very simple example.

Example 1

\[y'' + \lambda y = 0, \ y(0) = y(\pi) = 0.\]

Check that \(p(x) = 1, q(x) = 0\) and \(r(x) = 1.\) Clearly \(p(x), r(x) > 0\) and \(p(x)\) is continuously differentiable. Hence this is a regular Sturm-Liouville problem.
First, if $\lambda = 0$, the only solution is the trivial solution. However, we mention that trivial solution cannot be an eigenfunction. We now consider the other two cases:

Case (i) $\lambda < 0$: again, check that the only solution is the trivial solution.

Case (ii) $\lambda > 0$: for simplicity, let $\lambda = \beta^2$, $\beta > 0$.

Then $y(x) = C_1 \cos \beta x + C_2 \sin \beta x$. However, $y(0) = 0 \implies C_1 = 0$. Hence

$$0 = y(\pi) = C_2 \sin \beta \pi \implies \beta = k \in \mathbb{N} \text{ if we want } C_2 \neq 0.$$

Thus when $\lambda = k^2, k \in \mathbb{N}$, then it has an eigenfunction $\phi_\lambda(x) = \sin kx$. 
Example 2

\[ y'' + \lambda y = 0, \quad y(1) = y'(2) = 0. \]

First of all, note that it is easier to look at the problem

\[ y'' + \lambda y = 0, \quad y(0) = y'(1) = 0. \]

It can be done by changing of variable: \( t = x - 1 \).

It is again obvious that when \( \lambda = 0 \), the only solution is the trivial solution. Next, it is also easy to check that if \( \lambda < 0 \), then it still has no nontrivial solution.

Finally, if \( \lambda = \beta^2, \beta > 0 \), again, \( y(0) = 0 \) \( \implies \) \( C_1 = 0 \). Next,

\[ 0 = y'(1) = \beta C_2 \cos \beta \implies \beta = \frac{2k - 1}{2} \pi, k \in \mathbb{N}. \]

Thus \( \lambda = \left( \frac{2k - 1}{2} \pi \right)^2 \) is an eigenvalue and its eigenfunction is \( \phi_\lambda(x) = \sin \frac{2k - 1}{2} \pi (x - 1) \).
[Exercise] Find the eigenvalues and their corresponding eigenfunctions of the following Sturm-Liouville problem:
\[ y'' + \lambda y = 0, \ y'(0) = y'(\pi) = 0. \]
Let us now make some observations from the above examples.

(1) If $\psi_1(x)$ and $\psi_2(x)$ are both eigenfunctions with respect to the same eigenvalue, then $\psi_1(x) = k\psi_2(x)$, $k$ is a nonzero constant.

(2) If $\phi_1(x)$ and $\phi_2(x)$ are eigenfunctions correspond to eigenvalues $\lambda_1$ and $\lambda_2$ respectively, $\lambda_1 \neq \lambda_2$, then
\[
\int_{0}^{\pi} \phi_1(x)\phi_2(x)dx = 0.
\]
Moreover, $\phi_1(x)$ and $\phi_2(x)$ are linearly independent.
(3) All eigenvalues are real.

(4) There are infinite number of eigenvalues. Moreover, it can be arranged in an increasing order \( \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \) such that \( \lim_{n \to \infty} \lambda_n = \infty \).

It turns out that all the above observations are true for any regular Sturm-Liouville problem.

(1) If \( \psi_1(x) \) and \( \psi_2(x) \) are both eigenfunctions with respect to the same eigenvalue, then \( \psi_1(x) = k\psi_2(x) \), \( k \) is a nonzero constant.

(2) If \( \phi_1(x) \) and \( \phi_2(x) \) are eigenfunctions correspond to eigenvalues \( \lambda_1 \) and \( \lambda_2 \) respectively, \( \lambda_1 \neq \lambda_2 \), then

\[
\int_a^b \phi_1(x)\phi_2(x)r(x)dx = 0.
\]

Moreover, \( \phi_1(x) \) and \( \phi_2(x) \) are linearly independent.

(3) All eigenvalues are real.

(4) There are infinite number of eigenvalues. Moreover, it can be arranged in an increasing order \( \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \) such that \( \lim_{n \to \infty} \lambda_n = \infty \).
Proof.
Next, let us note that the idea of eigenvalues and eigenfunctions can be used for other linear BVP: $L[y] + \lambda r(x)y = 0$, $l_1(y) = l_2(y) = 0$. For example,

$$p_0(x)y'' + p_1(x)y' + q(x)y + \lambda r(x)y = 0, \; l_1(y) = 0, l_2(y) = 0.$$  

In particular, we can consider

$$y'' + 2y' + \lambda y = 0, \; y(0) = y(1) = 0.$$
Finally, let us look at some examples of singular Sturm-Liouville problems. We say that the self-adjoint BVP: \((p(x)y')' + q(x)y + \lambda r(x)y = 0\) is singular if it is not regular. For example, if \(p(x) = 0\) for some \(x \in [a, b]\), or \(a_0 = a_1 = 0\), or \(r(x) \leq 0\) for some \(x \in [a, b]\). Let us look at some examples:

**Example 43.1** \(y'' + \lambda y = 0\), \(y(0) = 0\), \(|y(x)| \leq M < \infty\) for all \(x \in (0, \infty)\).

Note that any \(\lambda \in (0, \infty)\) is an eigenvalue with eigenfunction \(\sin \sqrt{\lambda}x\). Note that it is now impossible to arrange all the eigenvalues as \(\{\lambda_1, \lambda_2, \cdots\}\), such that \(\lambda_1 < \lambda_2 < \cdots\).

**Example 43.3** \(((1 - x^2)y')' + \lambda y = 0\) (Legendre’s differential equations)

\[
\lim_{x \to -1} y(x) < \infty, \quad \lim_{x \to 1} y(x) < \infty.
\]

The eigenvalues of this problem are \(\lambda_n = n(n - 1); n = 1, 2, \cdots\), and the corresponding eigenfunctions are the Legendre’s polynomials \(P_{n-1}\) (we will discuss this later).
Remark (an application)

Example  \( y'' + y = f(x) = \begin{cases} 1, & \text{if } 0 < x < \pi/2 \\ -1, & \text{if } \pi/2 < x < \pi \end{cases} \quad y(0) = y(\pi) = 0 \)

Solution: Consider the problem: \( y'' + y + \lambda y = 0, \quad y(0) = y(\pi) = 0 \). Recall that the eigenvalues and eigenfunctions are \( k^2 - 1 \) and \( \phi_k(x) = \sin kx \).

Note that \( \phi_k''(x) + \phi_k(x) = (1 - k^2)\phi_k(x) \). We will only need to find \( c_k, k \in \mathbb{N} \) such that

\[
(\sum_{k=1}^{\infty} c_k \phi_k(x))'' + \sum_{k=1}^{\infty} c_k \phi_k(x) = \sum_{k=1}^{\infty} (1 - k^2)c_k \phi_k(x) = f(x).
\]
Orthogonal functions and polynomials  (Lecture 38)

Definition 38.1 A set of functions \( \{ \phi_n(x) : n = 0, 1, 2, \ldots \} \) is said to be orthogonal in \([a, b]\) (finite or infinite interval) with respect to a nonnegative function \( r(x) \) (weight function) if
\[
\int_a^b \phi_m(x)\phi_n(x)r(x)dx = 0 \text{ for all } m \neq n.
\]
Note that in most cases, \( r \) is continuous, \( r(x) > 0 \) for all \( x \) except finitely many points and \( \phi_m \) are usually continuous for all \( m \). Furthermore we usually do not include zero functions in the family.

Example 1. First, let us recall some families of eigenfunctions from examples in Lecture 42.
\[
\{ \sin kx : k \in \mathbb{N} \}, \ \{ \sin(2k - 1)x/2 : k \in \mathbb{N} \}, \ {\cos kx : k = 0, 1, 2, \ldots}.
\]
Note that these are orthogonal families of functions in \([0, \pi]\) with respect to the weight function 1.
Example 2. (Fourier series)

\[ \{1, \cos kx, \sin kx : k \in \mathbb{N}\} \]

is an orthogonal family of functions in \([-\pi, \pi]\) with respect to the weight function 1.

Example 3. If \(\{\phi_n : n \in \mathbb{N}\}\) is the collection of eigenfunctions (with respect to eigenvalues \(\lambda_n, \lambda_1 < \lambda_2 < \cdots\)) of a regular Sturm-Liouville problem (42.1), then it is orthogonal in \([a, b]\) with respect to the weight function \(r(x)\).
As we are going to use the terms piecewise continuous and piecewise smooth, let us define them here.

**Definition**  A function $f$ is said to be piecewise continuous on $[a, b]$ if there exist finitely many points $\{x_1, x_2, \cdots, x_n\} \subset (a, b)$ such that

(1) $f$ is continuous on $(a, b)$ except $\{x_1, x_2, \cdots, x_n\}$,

(2) both $f(a^+), f(b^-)$ exist, and

(3) $f(x_i^+), f(x_i^-)$ exist for all $i = 1, \cdots, n$.

[Example]

(1) Continuous functions on $[a, b]$ are piecewise continuous on $[a, b]$.

(2) Continuous functions on $(a, b)$ need not be piecewise continuous on $[a, b]$.

(3) Let $a = x_1 < x_2 < \cdots < x_n = b$. If $f$ is continuous on $[x_i, x_{i+1}]$ for all $i$, then $f$ is piecewise continuous. However, note that not all piecewise continuous functions are of this form.
Remark: occasionally, a piecewise continuous function on \([a, b]\) may not be defined at points that it is not continuous. However, in most cases, we will try to define them.

More examples:

[Exercise] Which of the following functions are piecewise continuous on \([0, \pi]\)?

1. \(\sqrt{x} \sin(1/x)\)
2. \(\cos(1/x)\)
3. the greatest integer function
4. \(f(x) = \begin{cases} 1, & \text{if } 0 < x < \pi/2 \\ -1, & \text{if } \pi/2 < x < \pi \end{cases}\)

Indeed, a function is piecewise continuous on \([a, b]\) if and only if there exist \(a = x_1 < x_2 < \cdots < x_n = b\) such that for each \(i\) there exists a continuous function \(g_i\) on \([x_i, x_{i+1}]\) such that \(f\) is equal to \(g_i\) on \((x_i, x_{i+1})\).