6. Formulate the following barrier option pricing problem as a partial differential equation with suitable boundary and final conditions: The option has barriers levels $S_u$ and $S_d$, above and below the initial asset price, respectively. If the asset touches both barriers before expiry, then the option has a final payoff $(S - X)^+$. Otherwise the option does not pay out.

Solution: Let $V(S, t)$ denote the value of the option before any of barriers is triggered. When either of the barriers is triggered, the option becomes a standard knock-in option. That is, if the barrier level $S_u$ is touched, then the rebate is $C_{di}(S_u, t; X, S_d)$; if the barrier level $S_d$ is touched, then the rebate is $C_{ui}(S_d, t; X, S_u)$. Here $C_{di}(\cdot, \cdot; X, H)$ and $C_{ui}(\cdot, \cdot; X, H)$ represent the down-in and down-out options with strike $X$ and barrier $H$, respectively. Then the pricing model is

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \text{ for } S_u < S < S_d
$$

$$
V(S_u, t) = C_{di}(S_u, t; X, S_d)
$$

$$
V(S_d, t) = C_{ui}(S_d, t; X, S_u)
$$

$$
V(S, T) = 0
$$