Chapter 5

Path-dependent options

The contracts we have seen so far are the most basic and important derivative products. In this chapter, we shall discuss some complex contracts, including barrier options, Asian options, lookback options and so on.

5.1 Barrier options

Barrier (or knock-in, knock-out) options are triggered by the action of the underlying hitting a prescribed value at some time before expiry. For example, as long as the asset remains below a pre-determined barrier price during the whole life of the option, the contract will have a call payoff at expiry. However, should the asset reach this level before expiry, then the option becomes worthless because it has “knocked out”. Barrier options are clearly path dependent. A barrier option is cheaper than a similar vanilla option since the former provides the holder with less rights than the latter does.

5.1.1 Different types of barrier options

There are two main types of barrier option:

1. The out option, that only pays off if a level is not reached. If the barrier is reached then the option is said to have knocked out;
2. The in option, that pays off as long as a level is reached before expiry. If the barrier is reached then the option is said to have knocked in.

Then we further characterize the barrier option by the position of the barrier relative to the initial value of the underlying:

1. If the barrier is above the initial asset value, we have an up option.
2. If the barrier is below the initial asset value, we have a down option. Finally, the options can also be classified as call and put according to the payoffs. In addition, if early exercise is permitted, the option is called American-style barrier option. In the following, barrier options always refer to European-style options except for special claim.

5.1.2 In-Out parity

For European style barrier option, the relationship between a ‘in’ barrier option and an ‘out’ barrier option (with same payoff and same barrier level) is very simple:

\[ \text{in} + \text{out} = \text{vanilla}. \]

If the ‘in’ barrier is triggered then so is the ‘out’ barrier, so whether or not the barrier is triggered the portfolio of an ‘in’ and ‘out’ option has the vanilla payoff at expiry.

However, the above in-out parity doesn’t hold for American-style barrier options.

5.1.3 Pricing by Monte-Carlo simulation

To illustrate method, we consider an up-out-call option whose terminal payoff can be written as

\[ (S_T - X)^+ I_{\{S_t < H, \, t \in [0,T]\}} \]

where \( H \) is the barrier level, \( I \) is the indicator function, i.e.

\[ I_{\{S_t < H, \, t \in [0,T]\}} = \begin{cases} 1, & \text{if } S_t < H \text{ for all } t \in [0,T] \\ 0, & \text{otherwise} \end{cases} \]

According to the risk neutral pricing principle, the option value is

\[ e^{-rT} \hat{E}[(S_T - X)^+ I_{\{S_t < H, \, t \in [0,T]\}}] \]

We then use the Monte-Carlo simulation to get an approximate value of the option.

It should be pointed out that the simulation can only apply to European style options.
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5.1.4 Pricing in the PDE framework

Barrier options are weakly path dependent. We only have to know whether or not the barrier has been triggered, we do not need any other information about the path. This is contrast to some of the contracts we will be seeing shortly, such as the Asian and lookback options, that are strongly path dependent.

We can use $V(S, t)$ to denote the value of the barrier contract before the barrier has been triggered. This value still satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - r V = 0.$$  \hfill (5.1)

The details of the barrier feature come in through the specification of the boundary conditions and solution domains.

‘Out’ Barriers

If the underlying asset reaches the barrier in an ‘out’ barrier option then the contract becomes worthless. This leads to the boundary condition

$$V(H, t) = 0 \text{ for } t \in [0, T].$$

The final condition is still

$$V(S, T) = \begin{cases} (S - X)^+ & \text{for call} \\ (X - S)^+ & \text{for put} \end{cases}.$$

The solution domain is $\{0 < S < H\} \times [0, T)$ for an up-out option, and $\{H < S < \infty\} \times [0, T)$ for a down-out option.

‘In’ Barriers

An ‘in’ option only has a payoff if the barrier is triggered. Remember that $V(S, t)$ stands for the option value before the barrier has been triggered. If the barrier is not triggered during the option’s life, the option expires worthless so that we have the final condition

$$V(S, T) = 0.$$

Once the barrier is triggered, the barrier-in option becomes a vanilla option then on the barrier the contract must have the same value as a vanilla contract:

$$V(H, t) = \begin{cases} C_E(H, t), & \text{for call} \\ P_E(H, t), & \text{for put} \end{cases}, \text{ for } t \in [0, T],$$
where $C_E(S,t)$ and $P_E(S,t)$ represent the values of the (European-style) vanilla call and put, respectively.

The solution domain is $\{0 < S < H\} \times [0,T)$ for an up-in option, and $\{H < S < \infty\} \times [0,T)$ for a down-in option.

*Explicit solutions*

Closed-form solutions for European-style barrier options are available. We refer interested readers to those references for these formulas. Here we only present the formula for the down-and-out call option with $H \leq X$:

$$C_{do}(S,t) = C_E(S,t) - \left(\frac{S}{H}\right)^{1-2(r-q)/\sigma^2} C_E\left(\frac{H^2}{S},t\right), \quad (5.2)$$

where $C_E(S,t)$ is the value of the European vanilla call.

Let us confirm that this is indeed the solution. Clearly, on the barrier $C_{do}(H,t) = C_E(H,t) - C_E(H,t) = 0$. At expiry

$$C_{do}(S,T) = (S - X)^+ - \left(\frac{S}{H}\right)^{1-2(r-q)/\sigma^2} \left(\frac{H^2}{S} - X\right)^+$$

$$= (S - X)^+, \text{ for } S > H.$$

So the remaining is to show $C_{do}(S,t)$ satisfies the Black-Scholes equation. The first term on the right-hand side of (5.2) does. The second term does also. Actually, if we have any solution $V_{BS}$, of the Black-Scholes equation it is easy to show that

$$S^{1-2(r-q)/\sigma^2} V_{BS}\left(\frac{A}{S},t\right)$$

is also a solution for any constant $A$.

**5.1.5 American early exercise**

For American knock-out options, the pricing model is a free boundary problem:

$$\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S \frac{\partial V}{\partial S} + rV, \phi \right\} = 0, \ (S,t) \in D$$

$$V(H,t) = 0$$

$$V(S,T) = \phi^+$$

where $D = (H,\infty) \times [0,T)$ for a down-out, and $D = (0,H) \times [0,T)$ for an up-out.
However, an American knock-in option price is governed still by the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \quad (S, t) \in D \quad (5.3)$$

The reason is $V(S, t)$ represents the option value before the barrier is triggered. Thus for $(S, t) \in D$, the option cannot be early exercised because it has not been activated. In fact, the option should be referred to as \textit{knock-into American option}. As in the European case, at expiry we have

$$V(S, T) = 0. \quad (5.4)$$

Once the barrier is triggered, the option becomes an American vanilla option then on the barrier. So,

$$V(H, t) = \begin{cases} C_A(H, t), \text{ for call} \\ P_A(H, t), \text{ for put} \end{cases}, \quad \text{for } t \in [0, T], \quad (5.5)$$

where $C_A(S, t)$ and $P_A(S, t)$ represent the values of the American vanilla call and put, respectively. (5.3-5.5) forms a complete model for American knock-in options.

### 5.1.6 BTM

The BTM can be readily extended to cope with barrier options. Recall the Black-Scholes still holds. So the BTM must hold too, that is

$$V(S, t) = \frac{1}{\rho} [pV(Su, t + \Delta t) + (1 - p)V(Sd, t + \Delta t)]$$

with appropriate final and boundary conditions. For instance, for a down-out call option,

$$\begin{aligned} V(H, t) &= 0 \\
V(S, T) &= (S - X)^+, \end{aligned}$$

and the backward procedure is conducted in the region \{S > H\} × [0, T).

It is easy to see that the BTM can handle early exercise involved in American barrier options.

### 5.1.7 Hedging

Barrier options may have discontinuous delta at the barrier. Such a discontinuity leads to difficulty in hedging a barrier option. We refer interested students to Wilmott (1998) and references therein.
5.1.8 Other features

The barrier option discussed above is the commonest one. Actually, the position of the barrier in the contract can be time dependent. The level may begin at one level and then rise, say. Usually the level is a piecewise-constant function of time.

Another style of barrier option is the double barrier. Here there is both an upper and a lower barrier, the first above and the second below the current asset price. In a double ‘out’ option the contract becomes worthless if either of the barriers is reached. In a double ‘in’ option one of the barriers must be reached before expiry, otherwise the option expires worthless. Other possibilities can be imagined: one barrier is an ‘in’ and the other ‘out’, at expiry the contract could have either and ‘in’ or an ‘out’ payoff.

Sometimes a rebate is paid if the barrier level is reached. This is often the case for ‘out’ barriers in which case the rebate can be thought of as cushioning the blow of losing the rest of the payoff. The rebate may be paid as soon as the barrier is triggered or not until expiry.

5.2 Asian options

Asian options give the holder a payoff that depends on the average price of the underlying during the options’ life.

5.2.1 Payoff types

The payoff from a fixed strike Asian call (or, average call) is \( (A_T - X)^+ \), and that from a fixed strike Asian put (or, average price put) is \( (X - A_T)^+ \), where \( A_T \) is the average value of the underlying asset calculated over a predetermined average period. Fixed strike Asian options are less expensive than vanilla options and are more appropriate than vanilla options for meeting some of the needs of corporate treasurers. Suppose that a U.S. corporate treasurer expects to receive a cash flow of 100 million Australian dollars spread evenly over the next year from the company’s Australian subsidiary. The treasurer is likely to be interested in an option that guarantees that the average exchange rate realized during the year is above some level. A fixed strike Asian put can achieve this more effectively than vanilla put options.

Another type of Asian options is a floating strike option (or, average strike option). A floating strike Asian call pays off \( (S_T - A_T)^+ \) and a floating strike Asian put pays off \( (A_T - S_T)^+ \). Floating strike options can guarantee that the average price paid for an asset in frequent trading over a period of
time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price.

5.2.2 Types of averaging

The two simplest and obvious types of average are the arithmetic average and the geometric average. The arithmetic average of the price is the sum of all the constituent prices, equally weighted, divided by the total number of prices used. The geometric average is the exponential of the sum of all the logarithms of the constituent prices, equally weighted, divided by the total number of prices used. Furthermore, the average may be based on discretely sampled prices or on continuously sampled prices. Then, we have

\[ A_T = \begin{cases} 
\frac{1}{n} \sum_{i=1}^{n} S_{t_i}, & \text{discretely sampled arithmetic} \\
\frac{1}{T} \int_0^T S_t \, dt, & \text{continuously sampled arithmetic} \\
\exp \left( \frac{1}{n} \sum_{i=1}^{n} \ln S_{t_i} \right) = (S_{t_1} S_{t_2} ... S_{t_n})^{1/n}, & \text{discretely sampled geometric} \\
\exp \left( \frac{1}{T} \int_0^T \ln S_t \, dt \right), & \text{continuously sampled geometric}
\end{cases} \]

5.2.3 Extending the Black-Scholes equation

Monte-Carlo simulation is easy to cope with the pricing of all kinds of European-style Asian options. However, for American-style Asian options, we have to fall back on PDE framework or BTM. In the following we only consider the continuously sampled Asian options. We refer interested reader to Wilmott et al. (1995), Wilmott (1998) for the PDE formulation of discretely sampled Asian options.

We start by assuming that the underlying asset follows the lognormal random walk

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \]

The value of an Asian option is not only a function of \( S \) and \( t \), but also a function of the historical average \( A \), that is \( V = V(S_t, A_t, t) \). Here \( A_t \) will be a new independent variable. Let us focus on the arithmetic average for which

\[ A_t = \frac{1}{t} \int_0^t S_{\tau} \, d\tau. \]

In anticipation of argument that will use Ito lemma, we need to know the stochastic differential equation satisfied by \( A \). It is not hard to check that

\[ dA_t = \frac{t S_t - \int_0^t S_{\tau} \, d\tau}{t^2} \, dt = \frac{S_t - A_t}{t} \, dt. \]
We can see that its stochastic differential equation contains no stochastic terms. So, the Ito lemma for $V = V(S, A, t)$ is
\[
dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial A} dA + \frac{\partial V}{\partial S} dS
\]
\[
= \left(\frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S} dS.
\]

To derive the pricing PDE, we set up a portfolio containing one of the path-dependent option and short a number $\Delta$ of the underlying asset:
\[
\Pi = V(S, A, t) - \Delta S
\]
The change in the value of this portfolio is given by
\[
d\Pi = dV - \Delta dS
\]
\[
= \left(\frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right) dS.
\]
Choosing $\Delta = \frac{\partial V}{\partial S}$ to hedge the risk, we find that
\[
d\Pi = \left(\frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.
\]
This change is risk free, thus earns the risk-free rate of interest $r$, leading to the pricing equation
\[
\frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (5.6)
\]
The solution domain is $\{S > 0, A > 0, t \in [0, T]\}$.

This is to be solved subject to
\[
V(S, A, T) = \begin{cases} 
(A - S)^+, & \text{floating put} \\
(S - A)^+, & \text{floating call} \\
(A - X)^+, & \text{fixed call} \\
(X - A)^+, & \text{fixed put}
\end{cases}
\]
The obvious changes can be made to accommodate dividends on the underlying. This completes the formulation of the valuation problem.
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For geometric average,

\[ dA = d \exp \left( \frac{1}{t} \int_0^t \ln S \, d\tau \right) = \exp \left( \frac{1}{t} \int_0^t \ln S \, d\tau \right) \frac{t \ln S - \int_0^t \ln S \, d\tau}{t^2} \, dt \]

\[ = \frac{A \ln S}{t}. \]

So the pde satisfied by geometric Asian options is

\[ \frac{\partial V}{\partial t} + \frac{A \ln S}{t} \frac{\partial V}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \]

The solution domain is \{ \( S > 0 \), \( A > 0 \), \( t \in [0, T) \) \}.

5.2.4 Early exercise

The only point to mention is that the details of the payoff on early exercise have to be well defined. The payoff at expiry depends on the value of the average up to expiry; this will, of course, not be known until expiry. Typically, on early exercise it is the average to date that is used. For example, in an American floating strike arithmetic put the early payoff would be

\[ \left( \frac{1}{t} \int_0^t S \, d\tau - S_t \right)^+. \]

In general, we denote the exercise payoff at time \( t \) by \( \Lambda(S_t, A_t) \), where

\[ \Lambda(S_t, A_t) = \begin{cases} 
(A_t - S_t)^+, & \text{floating put} \\
(S_t - A_t)^+, & \text{floating call} \\
(A_t - X)^+, & \text{fixed call} \\
(X - A_t)^+, & \text{fixed put} 
\end{cases} \]

Such representation is consistent with the terminal payoff. Then the pricing model is formulated by

\[
\min \left\{ -\frac{\partial V}{\partial t} - LV, V - \Lambda(S, A) \right\} = 0
\]

\[ V(S, A, T) = \Lambda(S, A) \]

where

\[
L = \begin{cases} 
\frac{S-A}{A \ln(S/A)} \frac{\partial}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-q)S \frac{\partial}{\partial S} - r, & \text{for arithmetic} \\
\frac{S-A}{A \ln(S/A)} \frac{\partial}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-q)S \frac{\partial}{\partial S} - r, & \text{for geometric} 
\end{cases}
\]

Remark 12 All European-style Asian geometric options have explicit price formulas. However, that is generally not true for Asian arithmetic options and all American-style Asian options which must be solved by numerical approaches.
5.2.5 Reductions in dimensionality

The Asian options value depends on three variables. For some cases, a reduction in dimensionality of the problem is permitted. We leave it as an exercise.

Remark 13 For any floating strike Asian option (arithmetic or geometric, European or American), one can find an appropriate transformation to reduce the model to a lower dimensional problem. As for fixed strike Asian options, it is true only for European-style.

5.2.6 Parity relation

We will give one example. Consider European-style floating strike arithmetic call options. The payoff at expiry for a portfolio of one call held long and one put held short is

\[(S - A)^+ - (A - S)^+,\]

where is simply

\[S - A.\]

The value of the portfolio then satisfies the equation (??) with the final condition

\[V(S, A, T) = S - A.\]

It can be verified that the solution is

\[V(S, A, t) = S - \frac{S}{rT} (1 - e^{-r(T-t)}) - \frac{t}{T} e^{-r(T-t)} A.\]

Therefore, for floating strike arithmetic Asian options, we have the put-call parity relation

\[V_{fc}(S, A, t) - V_{fp}(S, A, t) = S - \frac{S}{rT} (1 - e^{-r(T-t)}) - \frac{t}{T} e^{-r(T-t)} A.\]

We have similar results for other European Asian options.

5.2.7 Model-dependent and model-independent results

We need to point out that some results depends on assumptions of the model. For example, the above Asian put-call parity holds for the Black-Scholes model, but it might not be true in general provided that the geometric Brownian assumption is given up.

Recall some results are model-independent. For example:
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(1) the put-call parity for vanilla options
(2) the in-out parity for barrier options
(3) American call options should never be early exercised if there is no dividend payment.

To acquire these results, one only needs no-arbitrage principle.
But some are true only under Black-Scholes framework. For instance:
(1) the above Asian put-call parity
(2) the put-call symmetry relation

5.2.8 **Binomial tree method**

The binomial tree method for European-style Asian option is given as follows:

\[
V(S, A, t) = e^{-r\Delta t} \left[ pV(S_{u}, A_{u}, t + \Delta t) + (1 - p)V(S_{d}, A_{d}, t + \Delta t) \right]
\]

\[
V(S, A, T) = A(S, A)
\]

where

\[
p = \frac{e^{(r-q)\Delta t} - d}{u - d}
\]

and

\[
A_u = \frac{tA + \Delta tS_u}{t + \Delta t}, \quad A_d = \frac{tA + \Delta tS_d}{t + \Delta t}.
\]

Early exercise can be easily incorporated into the above algorithm as before to deal with American style options.

5.3 **Lookback options**

The dream contract has to be one that pays the difference between the highest and the lowest asset prices realized by an asset over some period. Any speculator is trying to achieve such a trade. The contract that pays this is an example of a lookback option, an option that pays off some function of the realized maximum and/or minimum of the underlying asset over some prescribed period. Since lookback options have such an extreme payoff they tend to be expensive.
5.3.1 Types of Payoff

For the basic lookback contracts, the payoff comes in two varieties, like the Asian option: the fixed strike and the floating strike, respectively. These have payoffs that are the same as vanilla options except that in the floating strike option the vanilla exercise price is replaced by the maximum or minimum. In the fixed strike option it is the asset value in the vanilla option that is replaced by the maximum or minimum. That is

$$\Lambda(S_T, M_T) = \begin{cases} 
M_T - S_T, & \text{floating put} \\
S_T - m_T, & \text{floating call} \\
(M_T - X)^+, & \text{fixed call} \\
(X - m_T)^+, & \text{fixed put}
\end{cases}$$

Here

$$M_T = \max_{0 \leq \tau \leq T} S_\tau \quad \text{and} \quad m_T = \min_{0 \leq \tau \leq T} S_\tau.$$ 

Note that for floating lookback options $^+$ can be removed.

5.3.2 Extending the Black-Scholes equations

Let $V = V(S_t, M_t, t)$ (or. $V = V(S_t, m_t, t)$) be the lookback option value, where

$$M_t = \max_{0 \leq \tau \leq t} S_\tau \quad \text{and} \quad m_t = \min_{0 \leq \tau \leq t} S_\tau.$$ 

We anticipate that Ito lemma will be used to derive the model. However, $\max_{0 \leq \tau \leq t} S_\tau$ (or. $\min_{0 \leq \tau \leq t} S_\tau$) is not differentiable. So we have to introduce another variable. (Let us consider the fixed call or floating put)

$$J_{nt} = \left( \frac{1}{t} \int_0^t S_\tau^nd\tau \right)^{1/n}.$$ 

It is easy to see

$$\lim_{n \to \infty} J_{nt} = \max_{0 \leq \tau \leq t} S_\tau = M_t$$

and

$$dJ_{nt} = \frac{J_{nt}^{1-n}}{nt} (S_t^n - J_{nt}^n) dt.$$ 

Next we consider the function $V(S_t, J_{nt}, t)$, which can be imagined as a product depends on the variable $J_{nt}$. Using the $\Delta$-hedging argument and Ito lemma, we find that $V(S_t, J_{nt}, t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{nt} \sum_{n=1}^{N_t} \frac{S^n - J_n^n}{J_n^n} \frac{\partial V}{\partial J_n} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$
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We now take the limit \( n \to \infty \). Note that \( S_t \leq \max_{0 \leq r \leq t} S_r = M_t \). When \( S < M \),

\[
\lim_{n \to \infty} \frac{1}{n} \left( S_n^n - J_n^n \right) = 0;
\]

Thus we obtain a governing equation for the floating strike lookback put option

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad \text{for } S < M.
\]

At expiry we have

\[
V(S, M, T) = \Lambda(S, M).
\]

The solution domain is \( \{ S \leq M \} \times [0, T] \). Here we require an auxiliary condition on \( S = M \),

\[
\frac{\partial V}{\partial M} \bigg|_{M=S} = 0.
\]

For a floating call or fixed call, we can establish its governing equation in a similar way. And the solution domain is \( \{ S \geq m \} \times [0, T] \).

### 5.3.3 BTM

Consider a floating put or fixed call. Similarly we have

\[
V(S, M, t) = e^{-r\Delta t} \left[ pV(S_u, M_u, t + \Delta t) + (1 - p)V(S_d, M_d, t + \Delta t) \right],
\]

where

\[
p = \frac{e^{(r-q)\Delta t} - d}{u - d}
\]

and

\[
M_u = \max(M, S_u), \quad M_d = \max(M, S_d).
\]

Due to \( M \geq S \) and \( d < 1 \), we have

\[
M_d = M.
\]

Then the binomial tree method is given by

\[
V(S, A, t) = e^{-r\Delta t} \left[ pV(S_u, M_u, t + \Delta t) + (1 - p)V(S_d, M, t + \Delta t) \right], \quad \text{for } S \leq M
\]

\[
V(S, A, T) = \Lambda(S, A)
\]
5.3.4 Consistency of the BTM and the continuous-time model:

Note that \( M \geq S \). We only need to consider two cases:

1. \( M \geq Su \) : In this case \( M_u = M_d = M \). Then the binomial tree scheme can be rewritten as

\[
V(S, M, t) = e^{-r\Delta t} [pV(Su, M, t + \Delta t) + (1 - p)V(Sd, M, t + \Delta t)].
\]

Using the Taylor expansion, we obtain

\[
-rV(S, M, t) + \frac{\partial V}{\partial t}(S, M, t) + (r - q)S \frac{\partial V}{\partial S}(S, M, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, M, t) = O(\Delta t).
\]

2. \( M = S \) : Then \( M_u = Su \) and \( M_d = S \), and the binomial tree scheme becomes

\[
V(S, S, t) = e^{-r\Delta t} [pV(Su, Su, t + \Delta t) + (1 - p)V(Sd, S, t + \Delta t)].
\]

By virtue of the Taylor expansion, we have

\[
V(S, S, t) - e^{-r\Delta t} [pV(Su, Su, t + \Delta t) + (1 - p)V(Sd, S, t + \Delta t)]
\]

\[
= -\left[\frac{\partial V}{\partial t}(S, S, t) + (r - q)S \frac{\partial V}{\partial S}(S, S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, S, t) - rV(S, S, t)\right] \Delta t
\]

\[
= -e^{-r\Delta t} p(Su - S) \frac{\partial V}{\partial M}(S, S, t) + O(\Delta t)
\]

\[
= \Delta t^{1/2} \frac{\partial V}{\partial M}(S, S, t) + O(\Delta t),
\]

which implies

\[
\frac{\partial V}{\partial M} = 0 \text{ at } M = S.
\]

**Remark 14** For Asian options, the Taylor expansion also gives the consistency.

5.3.5 Similarity reduction

The similarity reduction also applies to some lookback options. For example, for floating strike lookback put options, it follows from the transformations \( \frac{V(S, M, t)}{S} = W(x, t) \) and \( x = \frac{M}{S} \),

\[
\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} + (q - r)x \frac{\partial W}{\partial x} - qW = 0, \text{ } t \in [0, T), \text{ } x \in (1, \infty)
\]

\[
\frac{\partial W}{\partial x}\bigg|_{x=1} = 0
\]

\[
W(x, T) = x - 1.
\]
5.3. **LOOKBACK OPTIONS**

The reduction can be extended to the binomial tree model. Taking the same transformations, we have for the floating strike lookback call

\[
SW\left(\frac{M}{S}, t\right) = e^{-r\Delta t} \left[pSuW\left(\frac{\max(M, Su)}{Su}, t + \Delta t\right) + (1 - p)SdW\left(\frac{M}{Sd}, t + \Delta t\right)\right]
\]

or

\[
W(x, t) = e^{-r\Delta t} \left[puW\left(xd, t + \Delta t\right) + (1 - p)dW\left(xu, t + \Delta t\right)\right], \text{ for } x \geq 1.
\]

(5.7) can be rewritten as

\[
W(x, t) = e^{-r\Delta t} \left[puW\left(xd, t + \Delta t\right) + (1 - p)dW\left(xu, t + \Delta t\right)\right], \text{ for } x \geq u
\]

\[
W(1, t) = e^{-r\Delta t} \left[puW(1, t + \Delta t) + (1 - p)dW(u, t + \Delta t)\right],
\]

\[
W(x, T) = x - 1.
\]

5.3.6 **Russian options**

The similarity reduction can also be applied to the American-style lookback option. Here, we consider the Russian option, a special perpetual option, whose payoff is \(M\). Since it is a perpetual option, the option value is independent of time. Let \(V = V(S, M)\) denote the option value. The pricing model for the Russian option is given by

\[
\min \left\{ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - q)S \frac{\partial V}{\partial S} + rV, V - M \right\} = 0, \text{ for } S < M
\]

\[
\left. \frac{\partial V}{\partial M} \right|_{S=M} = 0.
\]

Using the transformations

\[
W(x) = \frac{V(S, M)}{S} \text{ and } x = \frac{M}{S},
\]

we have

\[
\min \left\{ \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (q - r)x \frac{\partial W}{\partial x} + qW, W - x \right\} = 0, \text{ for } x > 1
\]

\[
\left. \frac{\partial V}{\partial x} \right|_{x=1} = 0.
\]
For a fixed $S$, it becomes more attractive to exercise the Russian option if $M$ is sufficiently large. Then, the above model can be rewritten as

$$
-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (q - r)x \frac{\partial W}{\partial x} + qW = 0, \text{ for } 1 < x < x^*
$$

$$
W(x^*) = x^*
$$

$$
\frac{\partial W}{\partial x}(x^*) = 1
$$

$$
\frac{\partial W}{\partial x}(1) = 0.
$$

Solving the equations we can obtain the option value. A question: what about the option value if $q = 0$.

### 5.4 Miscellaneous exotics

#### 5.4.1 Forward start options

As its name suggests, a forward start option is an option that comes into being some time in the future. Let us consider an example: a forward start call option is bought now, at time $t = 0$, but with a strike price that is not known until time $T_1$, when the strike is set at the asset price on that date, say. The option expires later at time $T$.

The way to price this contract is to ask what happens at time $T_1$. At that time we get an at-the-money option with a time $T - T_1$ left to expiry. If the stock price at time $T_1$ is $S_1$ then the value of the contract is simply the Black-Scholes value with $S = S_1, X = S_1$ and with given values for $r$ and $\sigma$. For a call option this value, as a function of $S_1$, is $S_1 f(T_1)$, where

$$
f(t) = N(d_{1,t}) - e^{-r(T-t)} N(d_{2,t}),
$$

$$
d_{1,t} = \frac{r + \frac{1}{2}\sigma^2}{\sigma} \sqrt{T - t} \text{ and } d_{2,t} = \frac{r - \frac{1}{2}\sigma^2}{\sigma} \sqrt{T - t}.
$$

The value is proportional to $S$. Thus, at time $T_1$, we will hold an asset worth $S_1 f(T_1)$. Since this is a constant multiplied by the asset price at time $T_1$ the value today must be

$$
S f(T_1),
$$

where $S$ is today’s asset price.

We can also formulate the problem as a PDE model. Let $V = V(S, t)$ be the option value. Then $V(S, t)$ must satisfy the Black-Scholes equation

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
$$
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At time \( t = T_1 \), the option values are known as
\[
V(S, T_1) = S f(T_1).
\]
So we will solve the Black-Scholes equation in \((S, t) \in (0, \infty) \times [0, T_1]\) with
the above final condition. It is not hard to check that
\[
V(S, t) = S f(T_1)
\]
is just the solution.

5.4.2 Shout options

A shout call option is a vanilla call option but with the extra feature that
the holder can at any time reset the strike price of the option to the current
level of the asset. The action of resetting is called ‘shouting’.

There is clearly an element of optimization in the matter of shouting. One
would expect to see a free boundary problem occur quite naturally as
with American options. For a shout call option, if the shouting happens at
time \( t \), the holder essentially acquires an at-the-money option whose value
is
\[
S_t f(t).
\]
Here \( f(t) \) is given by 5.8). Then its pricing model is
\[
\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV - V_f(t) \right\} = 0, \quad S > 0, \quad t \in [0, T)
\]
\[
V(S, T) = (S - X)^+
\]
This problem has no closed-form solution and must be solved numerically.
It is not hard to extend to the binomial tree model and find a numerical
solution.

5.4.3 Compound options

A compound option is simply an option on an option. There are four main
types of compound options, namely, a call on a call, a call on a put, a put
on a call and a put on a put. A compound option has two strike prices and
two expiration dates. As an illustration, we consider a call on a call where
both calls are European-style. On the first expiration date \( T \), the holder of
the compound option has the right to buy the underlying call option for the
first strike price \( X \). The underlying call option again gives the right to the
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holder to buy the underlying asset for the second strike price \( X_1 \) on a later expiration date \( T_1 \).

Let \( S_t \) be the price of the underlying asset of the underlying option. Suppose \( S_t \) satisfies the geometric Brownian motion. Then the value of the underlying call option \( C(S_t; X, T) \) can be represented by the Black-Scholes formula. The value of the compound option is also a function of \( S \) and \( t \), denoted by \( V(S, t) \). Hence it satisfies the Black-Scholes equations for \( S > 0 \), \( t \in [0, T) \), that is,

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \quad \text{for } S > 0, \ t \in [0, T).
\]

At \( t = T \), we have the final condition

\[
V(S, T) = (C(S, T; X_1, T_1) - X)^+.
\]

The closed-form solution to the above model can be obtained. We refer interested readers to Hull (1998) or Kwok (1998).

**Remark 15** Recall the pricing model for a future call option. At the expiration date \( t = T \), the holder of a future call option has the right to get the cash \( (F_T - X) \) and a long position of a future contract that expires at later time \( T_1 \). Here \( F_t \) stands for the future price at time \( t \). Since it is free to enter a long position of a future contract, the payoff of the future option is \((F_T - X)^+ \). Let \( S_t \) denote the underlying asset price of the future contract. Due to \( F_t = S_te^{(r-q)(T_1-t)} \), the payoff of the future option can be rewritten as \((S_te^{(r-q)(T_1-T)} - X)^+ \). Then the option value, denoted by \( U(S, t) \), satisfies

\[
\frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (r - q)S \frac{\partial U}{\partial S} - rU = 0, \quad \text{for } S > 0, \ t \in [0, T)
\]

\[
U(S, T) = (S_te^{(r-q)(T_1-T)} - X)^+.
\]

If we consider the option value as a function of the future price, i.e. \( V = V(F, t) \), then we can make the transformations \( V(F, t) = U(S, t) \) and \( F = S_te^{(r-q)(T_1-t)} \) to get

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0, \quad \text{for } F > 0, \ t \in [0, T)
\]

\[
V(F, T) = (F - X)^+.
\]