Chapter 6

Beyond the Black-Scholes world

6.1 Volatility simile phenomena and defects in the Black-Scholes model

Before pointing out some of the flaws in the assumptions of the Black-Scholes world, we must emphasize how well the model has done in practice, how widespread its use is and how much impact it has had on financial markets. The model is used by everyone working in derivatives whether they are salesmen, traders or quants. The value of vanilla options are often not quoted in monetary terms, but in volatility terms with the understanding that the price of a contract is its Black-Scholes value using the quoted volatility. The idea of delta hedging and risk-neutral pricing have taken a formidable grip on the minds of academics and practitioners alike. In many ways, especially with regards to commercial success, the Black-Scholes model is remarkably robust.

Nevertheless, there is room for improvement.

6.1.1 Implied volatility and volatility similes

The one parameter in the Black-Scholes pricing formulas that cannot be directly observed is the volatility of the underlying asset price which is a measure of our uncertainty about the returns provided by the underlying asset. Typically values of the volatility of an underlying asset are in the range of 20% to 40% per annum.

The volatility can be estimated from a history of the underlying asset
price. However, it is more appropriate to mention an alternative approach that involves what is termed an implied volatility. This is the volatility implied by an option price observed in the market.

To illustrate the basic idea, suppose that the market price of a call on a non-dividend-paying underlying is $1.875$ when $S_0 = 21$, $X = 20$, $r = 0.1$ and $T = 0.25$. The implied volatility is the value of $\sigma$, that when substituted into the Black-Scholes formula

$$c = S_0 N \left( \frac{\ln S_0 + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right) + X e^{-rT} N \left( \frac{\ln S_0 + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right)$$

gives $c = 1.875$. In general, it is not possible to invert the formula so that $\sigma$ is expressed as a function of $S_0$, $X$, $r$, $T$, and $c$. But, it is not hard to use Matlab to find a numerical solution of $\sigma$ because

$$\frac{\partial c}{\partial \sigma} > 0.$$ 

In this example, the implied volatility is 23.5%.

Implied volatilities can be used to monitor the market’s opinion about the volatility of a particular stock. Analysts often calculate implied volatilities from actively traded options on a certain stock and use them to calculate the price of a less actively traded option on the same stock.

Black-Scholes assumes that volatility is a known constant. If it is true, then the implied volatility should keep invariant w.r.t. different strike prices. However, in reality, the shape of this implied volatility versus strike curve is often like ‘a smile’, instead of a flat line. This is the so-called ‘volatility simile’ phenomena. In some markets it shows considerable asymmetry, a skew, and sometimes it is upside down in a frown. The general shape tends to persist for a long time in each underlying.

The volatility simile phenomena implies that there are flaws in the Black-Scholes model.

### 6.1.2 Improved models

1. Local volatility model:

   Black-Scholes assumes that volatility is a known constant. If volatility is not a simple constant then perhaps it is a more complicated function of time and the underlying.

2. Stochastic volatility

   The Black-Scholes formulae require the volatility of the underlying to be a constant (or a known deterministic function of time). The Black-Scholes
6.2. **LOCAL VOLATILITY MODEL**

The Black-Scholes model assumes that the volatility is a constant function of time and asset value (i.e., the local volatility model). Neither of these is true. All volatility time series show volatility to be a highly unstable quantity. It is very variable and unpredictable. It is therefore natural to represent volatility itself as a random variable. Stochastic volatility models are currently popular for the pricing of contracts that are very sensitive to the behavior of volatility.

(3) Jump diffusion model.

Black-Scholes assumes that the underlying asset path is continuous. It is common experience that markets are discontinuous: from time to time they ‘jump’, usually downwards. This is not incorporated in the lognormal asset price model, for which all paths are continuous.

(4) Others:

Discrete hedging: Black-Scholes assumes the delta-hedging is continuous. When we derived the Black-Scholes equation we used the continuous-time Ito’s lemma. The delta hedging that was necessary for risk elimination also had to take place continuously. If there is a finite time between rehedges then there is risk that has not been eliminated.

Transaction costs: Black-Scholes assumes there are no costs in delta hedging. But not only must we worry about hedging discretely, we must also worry about how much it costs us to rehedge. The buying and selling of assets exposes us to bid-offer spreads. In some markets this is insignificant, then we rehedge as often as we can. In other markets, the cost can be so great that we cannot afford to hedge as often as we would like.

### 6.2 Local volatility model

Suppose the volatility of the underlying asset is a deterministic function of $S$ and $t$, i.e., $\sigma = \sigma(S, t)$. It is not hard to show that the option value still satisfies the Black-Scholes equations,

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S, t)S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \quad \text{for } S > 0, \ t \in [0, T).
$$

with the final condition

$$
V(S, T) = \begin{cases} 
(S - X)^+ & \text{for call} \\
(X - S) & \text{for put}
\end{cases}
$$

A natural question is how to calibrate the function $\sigma(S, t)$. In general, there are not closed form solutions to the pricing model provided that volatility is a function of $S$ and $t$. To identify the volatility function, we need to exploit more information.
Let $V(S, t; K, T)$ be the option price with strike $K$ and maturity $T$. It can be shown (see Wilmott (1998)) that as a function $K$ and $T$, the function $V(.,.; K, T)$ satisfies

$$-rac{\partial V}{\partial T} + \frac{1}{2} \sigma^2(K, T) K^2 \frac{\partial^2 V}{\partial K^2} - (r - q) K \frac{\partial V}{\partial K} - qV = 0, \text{ for } K > 0, \ T \geq t^*.$$ 

We can identify the function $\sigma(.,.)$ by

$$\sigma(K, T) = \sqrt{\frac{\frac{\partial V}{\partial T} + (r - q) K \frac{\partial V}{\partial K} + qV}{\frac{1}{2} K^2 \frac{\partial^2 V}{\partial K^2}}}.$$ 

In practice, one often calibrates the model from the market prices of vanilla options so as to price some exotic options of the OTC market.

### 6.3 Stochastic volatility model

Volatility doesn’t not behave how the Black-Scholes equation would like it to behave; it is not constant, it is not predictable, it’s not even directly observable. This make it a prime candidate for modelling as a random variable.

#### 6.3.1 Random volatility

We continue to assume that $S$ satisfies

$$dS = \mu S dt + \sigma S dW_1,$$

but we further assume that volatility satisfies

$$d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dW_2.$$ 

The two increments $dW_1$ and $dW_2$ have a correlation of $\rho$. The choice of functions $p(S, \sigma, t)$ and $q(S, \sigma, t)$ is crucial to the evolution of the volatility, and thus to the pricing of derivatives.

The value of an option with stochastic volatility is a function of three variables, $V(S, \sigma, t)$. 
6.3. STOCHASTIC VOLATILITY MODEL

6.3.2 The pricing equation

The new stochastic quantity that we are modeling, the volatility, is not a traded asset. Thus, when volatility is stochastic we are faced with the problem of having a source of randomness that cannot be easily hedged away. Because we have two sources of randomness we must hedge our option with two other contracts, one being the underlying asset as usual, but now we also need another option to hedge the volatility risk. We therefore must set up a portfolio containing one option, with value denoted by \( V(S, \sigma, t) \), a quantity \(-\Delta\) of the asset and a quantity \(-\Delta_1\) of another option with value \( V_1(S, \sigma, t) \). We have

\[ \Pi = V - \Delta S - \Delta_1 V_1. \]

The change in this portfolio in a time \( dt \) is given by

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt
\]

\[ -\Delta_1 \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} \right) dt \]

\[ + \left( \frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta \right) dS \]

\[ + \left( \frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} \right) d\sigma \]

where we have used Ito lemma on functions of \( S, \sigma \) and \( t \).

To eliminate all randomness from the portfolio we must choose

\[ \frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} = 0, \]

to eliminate \( dS \) terms, and

\[ \frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0, \]

to eliminate \( d\sigma \) terms. This leaves us with

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt
\]

\[ -\Delta_1 \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} \right) dt \]

\[ = r\Pi dt = r(V - \Delta S - \Delta_1 V_1) dt \]
where I have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate.

As it stands, this is one equation in two unknowns, $V$ and $V_1$. This contrasts with the earlier Black-Scholes case with one equation in the one unknowns.

Collecting all terms on the left-hand side and all $V_1$ terms on the right-hand side we find that

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} - r V$$

$$= \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} + r S \frac{\partial V_1}{\partial S} - r V_1.$$

We are lucky that the left-hand side is a function of $V$ but not $V_1$ and the right-hand side is a function of $V_1$ but not $V$. Since the two options will typically have different payoffs, strikes or expiries, the only way for this to be possible is for both sides to be independent of the contract type. Both sides can only be functions of the independent variables, $S$, $\sigma$ and $t$. Thus we have

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} - r V = -a(S, \sigma, t)$$

for some function $a(S, \sigma, t)$. Reordering this equation, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} + a(S, \sigma, t) \frac{\partial V}{\partial \sigma} - r V = 0.$$

The final condition is

$$V(S, \sigma, T) = \begin{cases} (S - X)^+, & \text{for call option} \\ (X - S)^+, & \text{for put option} \end{cases}.$$

The solution domain is $\{\sigma > 0, S > 0, t \in [0, T]\}$.

**Remark 16** In the risk-neutral world, the underlying asset $S$ follows the following process:

$$dS_t = r S_t dt + \sigma S_t dW_t.$$  

We can similarly get the risk-neutral process of $\sigma$ as follows

$$d\sigma = a(S, \sigma, t) dt + q(S, \sigma, t) dW_t.$$

Here $a(S, \sigma, t)$ is often rewritten as

$$a(S, \sigma, t) = p(S, \sigma, t) - \lambda(S, \sigma, t) q(S, \sigma, t),$$

where $\lambda(S, \sigma, t)$ is called the market price of risk.
6.3.3 Named models


\[ d \left( \sigma^2 \right) = k(b - \sigma^2)dt + c\sigma^2dW_2, \]

where \( k, b \) and \( c \) are constant.

Using the Ito lemma, we can get

\[ d\sigma = \left[ -\frac{1}{8}c^2\sigma + \frac{k}{2} \left( \frac{b}{\sigma} - \sigma \right) \right] dt + \frac{c}{2}\sigma dW_2. \]

2. Heston (1993)

\[ d\sigma = -\gamma\sigma dt + \delta dW_2. \]

Explicit price formulas are available for Heston model. For Hull-White model, explicit formulas exist when \( S \) and \( \sigma \) are uncorrelated.

6.4 Jump diffusion model

There is plenty of evidence that financial quantities do not follow the lognormal random walk that has been the foundation of the Black-Scholes model. One of the striking features of real financial markets is that every now and then there is a sudden unexpected fall or crash. These sudden movements occur far more frequently than would be expected from a Normally-distributed return with a reasonable volatility.

6.4.1 Jump-diffusion processes

The basic building block for the random walks we have considered so far is continuous Brownian motion based on the Normally-distributed increment. We can think of this as adding to the return from one day to the next a Normally-distributed random variable with variance proportional to timestep. The extra building block we need for the jump-diffusion model for an asset price is the Poisson process. A Poisson process \( dq \) is defined

\[ dq = \begin{cases} 
0, & \text{with probability } 1 - \lambda dt \\
1, & \text{with probability } \lambda dt 
\end{cases}. \]

There is therefore a probability \( \lambda dt \) of a jump in \( q \) in the timestep \( dt \). The parameter \( \lambda \) is called the intensity of the Poisson process.
This Poisson process can be incorporated into a model for an asset in the following way:

\[
\frac{dS}{S} = \mu dt + \sigma dW + (J - 1) dq.
\]

This is the jump-diffusion process. We assume that there is no correlation between the Brownian motion and the Poisson process. If there is a jump \((dq = 1)\) then \(S\) immediately goes to the value \(JS\). We can model a sudden 10% fall in the asset price by \(J = 0.9\). We can generalize further by allowing \(J\) to be a random quantity.

A jump-diffusion version of the Ito lemma is

\[
dV(S, t) = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW + (V(JS, t) - V(S, t)) dq.
\]

The random walk in \(\ln S\) follows

\[
d\ln S = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW + (\ln(JS) - \ln(S)) dq
\]

\[
= \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW + \ln J dq
\]

### 6.4.2 Hedging when there are jumps

Hold a portfolio of the option and \(-\Delta\) of the underlying:

\[
\Pi = V(S, t) - \Delta S.
\]

The change in the value of this portfolio is

\[
d\Pi = dV - \Delta dS
\]

\[
= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW + (V(JS, t) - V(S, t)) dq
\]

\[ -\Delta [\mu S dt + \sigma S dW + (J - 1) S dq].\]

### 6.4.3 Merton’s model (1976)

If we choose

\[
\Delta = \frac{\partial V}{\partial S},
\]
6.4. JUMP DIFFUSION MODEL

we are following a Black-Scholes type of strategy, hedging the diffusive movements. The change in the portfolio value is then

\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( V(JS, t) - V(S, t) - (J - 1)S \frac{\partial V}{\partial S} \right) dq. \]

The portfolio now evolves in a deterministic fashion, except that every so often there is a non-deterministic jump in its value. It can be argued (Merton 1976) if the jump component of the asset price process is uncorrelated with the market as a whole, then the risk in the discontinuity should not be priced in the option. Diversifiable risk should not be rewarded. In other words, we can take expectations of this expression and set that value equal to the riskfree return from the portfolio, namely

\[ E(d\Pi) = r\Pi dt. \]

This argument is not completely satisfactory, but is a common assumption whenever there is a risk that cannot be fully hedged.

Since there is no correlation between \( dW \) and \( dq \), and

\[ E(dq) = \lambda dt, \]

we arrive at

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda [V(JS, t) - V(S, t)] - \lambda [J - 1]S \frac{\partial V}{\partial S} = 0. \]

If the jump size \( J \) is a random quantity, we need to take the expectation over the \( J \). It follows

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda E^J [V(JS, t) - V(S, t)] - \lambda E^J [J - 1]S \frac{\partial V}{\partial S} = 0. \]

There is a simple solution of this equation in the special case that the logarithm of \( J \) is Normally distributed. If the logarithm of \( J \) is Normally distributed with standard deviation \( \sigma' \) and if we write

\[ k = E^J [J - 1] \]

then the price of a European non-path-dependent option can be written as

\[ \sum_{n=1}^{\infty} \frac{1}{n!} e^{-\lambda(T-t)} (\lambda'(T-t))^n V_{BS}(S, t; \sigma_n, r_n), \]
where

\[ \lambda' = \lambda(1 + k), \quad \sigma^2_n = (\sigma^2 + \frac{n\sigma^2}{T-t}) \text{ and } r_n = r - \lambda k + \frac{n\ln(1 + k)}{T-t}, \]

and \( V_{BS} \) is the Black-Scholes formula for the option value in the absence of jumps. This formula can be interpreted as the sum of individual Black-Scholes values, each of which assumes that there have been \( n \) jumps, and they are weighted according to the probability that there will have been \( n \) jumps before expiry.

### 6.4.4 Wilmott et al.’s model

In the above we hedged the diffusive element of the random walk for the underlying. Another possibility is to hedge both the diffusion and jumps as much as we can. For example, we could choose \( \Delta \) to minimize the variance of the hedged portfolio.

The changes in the value of the portfolio with an arbitrary \( \Delta \) is

\[
d\Pi = (...) dt + \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dW + (V(JS,t) - V(S,t) - \Delta(J-1)S) dq.
\]

The variance in this change, which is a measure of the risk in the portfolio, is

\[
\text{var}[d\Pi] = \sigma^2 S^2 \left( \frac{\partial V}{\partial S} - \Delta \right)^2 dt + \lambda (V(JS,t) - V(S,t) - \Delta(J-1)S)^2 dt + O(dt^2)
\]

If \( J \) is a random quantity, we take expectation over \( J \) to get

\[
\text{var}[d\Pi] = \sigma^2 S^2 \left( \frac{\partial V}{\partial S} - \Delta \right)^2 dt + \lambda E[J] \left[ (V(JS,t) - V(S,t) - \Delta(J-1)S)^2 \right] dt + O(dt^2)
\]

By neglecting \( O(dt^2) \), this is minimized by the choice

\[
\Delta = \frac{\lambda E[J] (V(JS,t) - V(S,t)) + \sigma^2 S^2 \frac{\partial V}{\partial S}}{\lambda SE[J] (J-1)^2 + \sigma^2 S}
\]

If we value the option as a pure discounted real expectation under this best-hedge strategy, then we have

\[
E[d\Pi] = r\Pi dt
\]
or
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left( \mu - \frac{\sigma^2}{d} (\mu + \lambda k - r) \right) \frac{\partial V}{\partial S} - rV
+ \lambda E^J \left[ (V(JS,t) - V(S,t)) \left( 1 - \frac{J-1}{d} (\mu + \lambda k - r) \right) \right] = 0
\]
where
\[
d = \lambda E^J \left[ (J-1)^2 \right] + \sigma^2 \text{ and } k = E^J [J - 1]
\]
When \( \lambda = 0 \) this recovers the Black-Scholes equation.

### 6.4.5 Summary
Jump diffusion models undoubtedly capture a real phenomenon that is missing from the Black-Scholes model. Yet they are rarely used in practice. There are three main reasons for this:

1. difficulty in parameter estimation. In order to use any pricing model one needs to be able to estimate parameters. In the lognormal model there is just the one parameter to estimate. This is just the right number. More than one parameter is too much work. The jump diffusion model in its simplest form needs an estimate of probability of a jump, measured by \( \lambda \) and its size \( J \). This can be made more complicated by having a distribution for \( J \).
2. difficulty in solution. The governing equation is no longer a diffusion equation (about the easiest problem to solve numerically), and is harder than the solution of the basic Black-Scholes equation.
3. impossibility of perfect hedging. Finally, perfect risk-free hedging is impossible when there are jumps in the underlying. The use of a jump-diffusion model acknowledges that one’s hedge is less than perfect.

In fact the above remarks also apply to the stochastic volatility model.

### 6.5 Appendix: Option pricing with no-short selling underlying

#### 6.5.1 A forward contract on a no-short selling asset
If the underlying is not held for investment purposes, we should be careful to use the no-arbitrage principle. For example, assume the underlying to be a consumption commodity: oil for which short selling is not allowed. In this case, we cannot short sell portfolio B, but we can still short sell portfolio A.
We claim that at time \( t \) the value of portfolio A is not greater than that of portfolio B, that is
\[
V(S_t, t) + Ke^{-r(T-t)} \leq S_t. \tag{6.1}
\]
Indeed, suppose that instead of equation (6.1), we have
\[
V(S_t, t) + Ke^{-r(T-t)} > S_t. \tag{6.2}
\]
Then one could short sell portfolio A and buy portfolio B at time \( t \). Then the strategy is certain to lead to a riskless positive profit of \( e^{r(T-t)}(V(S_t, t) + Ke^{-r(T-t)} - S_t) \) at expiry \( T \). Therefore, we conclude from the no-arbitrage principle that equation (6.2) cannot hold (for any significant length of time).

If short selling is allowed for the underlying (gold, for example), we are able to similarly deduce that \( V(S_t, t) + Ke^{-r(T-t)} < S_t \) cannot hold, and thus we are certain to have equation (1.6). However, all we can assert for the forward contract on a consumption commodity is only equation (6.1), or equivalently,
\[
V(S_t, t) \leq S_t - Ke^{-r(T-t)}. 
\]
Corresponding, the delivery price \( K \leq S_0e^{rT} \).

### 6.5.2 No Arbitrage Pricing: A General Framework

We consider the derivatives on a single underlying variable, \( \theta \), which follows
\[
\frac{d\theta}{\theta} = \mu(\theta, t)dt + \sigma(\theta, t)dW.
\]
Here the variable \( \theta \) need not be the price of an investment asset. For example, it might be the interest rate, and corresponding derivative products can be bonds or some interest rate derivatives. In this case the shorting selling for the underlying is not permitted and thus we cannot replicate the derivation process of the Black-Scholes equation where the underlying asset is used to hedge the derivative.

Suppose that \( f_1 \) and \( f_2 \) are the prices of two derivatives dependent only on \( \theta \) and \( t \). These could be options or other instruments that provide a payoff equal to some function of \( \theta \) at some future time. We assume that during the time period under consideration \( f_1 \) and \( f_2 \) provide no income.

Suppose that the processes followed by \( f_1 \) and \( f_2 \) are
\[
df_1 = a_1 dt + b_1 dW
\]
and
\[
df_2 = a_2 dt + b_2 dW,
\]
6.5. **APPENDIX: OPTION PRICING WITH NO-SHORT SELLING UNDERLYING**

where $a_1, a_2, b_1$ and $b_2$ are functions of $\theta$ and $t$. The $W$ is the same Brownian motion as in the process of $\theta$, because this is the only source of the uncertainty in their prices. To eliminate the uncertainty, we can form a portfolio consisting of $b_2$ of the first derivative and $-b_1$ of the second derivative. Let $\Pi$ be the value of the portfolio,

$$\Pi = b_2 f_1 - b_1 f_2.$$ 

Then

$$d\Pi = b_2 df_1 - b_1 df_2 = (a_1 b_2 - a_2 b_1) dt.$$ 

Because the portfolio is instantaneously riskless, it must earn the risk-free rate. Hence

$$d\Pi = r\Pi dt = r(b_2 f_1 - b_1 f_2) dt$$

Therefore,

$$a_1 b_2 - a_2 b_1 = r(b_2 f_1 - b_1 f_2)$$

or

$$\frac{a_1 - r f_1}{b_1} = \frac{a_2 - r f_2}{b_2} = \lambda.$$ 

Define $\lambda$ as the value of each side in the equation, so that

$$\frac{a_1 - r f_1}{b_1} = \frac{a_2 - r f_2}{b_2} = \lambda.$$ 

Dropping subscripts, we have shown that if $f$ is the price of a derivative dependent only on $\theta$ and $t$ with

$$df = adt + b dW$$

then

$$\frac{a - rf}{b} = \lambda. \quad (6.3)$$ 

The parameter $\lambda$ is known as the market price of risk of $\theta$. It may be dependent on both $\theta$ and $t$, but it is not dependent on the nature of any derivative $f$. At any given time, $(a - rf)/b$ must be the same for all derivatives that are dependent only on $\theta$ and $t$.

The market price of risk of $\theta$ measures the trade-offs between risk and return that are made for securities dependent on $\theta$. Eq. (6.3) can be written

$$a - rf = \lambda b.$$
For an intuitive understanding of this equation, we note that the variable \( \sigma \) can be loosely interpreted as the quantity of \( \theta \)-risk present in \( f \). On the right-hand side of the equation we are, therefore, multiplying the quantity of \( \theta \)-risk by the price of \( \theta \)-risk. The left-hand side is the expected return in excess of the risk-free interest rate that is required to compensate for this risk. This is analogous to the capital asset pricing model, which relates the expected excess return on a stock to its risk.

Because \( f \) is a function of \( \theta \) and \( t \), the process followed by \( f \) can be expressed in terms of the process followed by \( \theta \) using Ito’s lemma. The parameters \( \mu \) and \( \sigma \) are given by:

\[
\begin{align*}
a &= \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + \mu \theta \frac{\partial f}{\partial \theta} \\
b &= \sigma \theta \frac{\partial f}{\partial \theta}
\end{align*}
\]

Substituting these into equation (6.3), we obtain the following differential equation that must be satisfied by \( f \):

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + \left( \mu - \lambda \sigma \right) \theta \frac{\partial f}{\partial \theta} - rf = 0. \tag{6.4}
\]

This equation is structurally very similar to the Black-Scholes equation.

If the variable \( \theta \) is the price of a traded asset, then the asset itself can be regarded as a derivative on \( \theta \). Hence we can take \( f = \theta \) and substitute into Eq. (6.3) to get

\[\mu f - rf = \lambda \sigma f\]

or

\[\mu - r = \lambda \sigma.\]

Then the equation becomes precisely the Black-Scholes equation:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + r \theta \frac{\partial f}{\partial \theta} - rf = 0.
\]

**Remark 17** Eq (6.4) implies that the risk-neutral process of \( \theta \) is

\[d\theta = (\mu - \lambda \sigma) \theta dt + \sigma \theta dW.\]

**Remark 18** Applying Ito’s lemma gives

\[df = \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + \mu \theta \frac{\partial f}{\partial \theta} \right] dt + \sigma \theta \frac{\partial f}{\partial \theta} dW.\]
Substituting Eq (6.4) into the above expression, we have

\[
df = \left[ r f + \lambda \sigma \frac{\partial f}{\partial \theta} \right] dt + \sigma \theta \frac{\partial f}{\partial \theta} dW \\
= r f dt + \sigma \theta \frac{\partial f}{\partial \theta} [\lambda dt + dW].
\]

That is

\[
df - r f dt = \sigma \theta \frac{\partial f}{\partial \theta} [\lambda dt + dW].
\]

Observe that for every unit of risk, represented by \( dW \), there are \( \lambda \) units of extra return. That is why we call \( \lambda \) the market price of risk.