Chapter 1

Basic Concepts of PDE

After thinking about the meaning of a partial differential equation, we will flex our mathematical muscles by solving a few of them. Then we will see how naturally they arise in the physical sciences. The physics will motivate the formulation of boundary conditions and initial conditions.

1.1 What is a partial Differential Equation

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable $x, y, \cdots$. There is a dependent variable that is an unknown function of these variables $u(x, y, \cdots)$. We will often denote its derivatives by subscripts; thus $\partial u / \partial x = u_x$, and so on. A PDE is an identity that relates the independent variables, the dependent variable $u$, and the partial derivatives of $u$. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0.$$  \hfill (1.1.1)

This is the most general PDE in two independent variables of first order. The order of an equation is the highest derivative that appears. The most general second order PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$  \hfill (1.1.2)

A solution of a PDE is a function $u(x, y, \cdots)$ that satisfies the equation identically, at least in some region of the $x, y, \cdots$ variables. When solving an ordinary differential equation (ODE), one sometimes reverses the roles of the independent and dependent variables—for instance, for the separable ODE $\partial u / \partial x = u^5$. For PDEs, the distinction between the independent variables and the dependent variable (the unknown) is always maintained.

Some examples of PDEs are:

1. $u_x + u_y = 0$ (transport)
2. \( u_x + y u_y = 0 \) (transport)
3. \( u_x + u u_y = 0 \) (shock wave)
4. \( u_{xx} + u_{yy} = 0 \) (Laplace’s equation)
5. \( u_{tt} - u_{xx} + u^3 = 0 \) (wave with interaction)
6. \( u_t + uu_x + u_{xxx} = 0 \) (dispersive wave)
7. \( u_{tt} + u_{xxxx} = 0 \) (vibrating bar)
8. \( u_t - iu_{xx} = 0 (i = \sqrt{-1}) \) (quantum mechanics)
9. \( \Delta u + u^{n+2/(n-2)} = 0 \) (Scalar Curvature equation)
10. \( u_t - u_{xx} = 0 \) (heat equation)
11. \( u_{tt} - u_{xx} = 0 \) (wave equation)
12. \( u_t + xu_x + x^2u_{xx} - u = 0 \) (Black-Scholes equation)

Each of these has two independent variables, written either as \( x \) and \( y \) or as \( x \) and \( t \).

**Linearity** means the following. Write the equation in the form \( Lu = 0 \), where \( L \) is an operator. That is, if \( v \) is any function, \( Lv \) is a new function. For example, \( L = \partial / \partial x \) is the operator that takes \( v \) into its partial derivative \( v_x \). In example 12, the operator \( L \) is \( L = \partial / \partial t + x \partial / \partial x + x^2 \partial^2 / \partial x^2 - 1 \). The definition we want for linearity is

\[
L(u + v) = Lu + Lv, \quad L(cu) = cLu \quad \text{(1.1.3)}
\]

for any functions \( u, v \) and any constant \( c \). Whenever (1.1.3) holds (for all choices of \( u, v \) and \( c \)), \( L \) is called a linear operator. The equation

\[
Lu = 0, \quad \text{(1.1.4)}
\]

is called linear if \( L \) is a linear operator. Equation (1.1.4) is called a homogeneous linear equation. The equation

\[
Lu = g, \quad \text{(1.1.5)}
\]

where \( g \neq 0 \) is a given function of the independent variables, is called an inhomogeneous linear equation. For instance, the equation

\[
(cos xy^3)u_x - y^4u_y = \ln(x^2 + y^2), \quad \text{(1.1.6)}
\]

is an inhomogeneous linear equation.

As you can easily verify, eight of the twelve equations above are linear as well as homogeneous. Example 5, on other hand, is not linear because although \( (u + v)_{xx} = u_{xx} + v_{xx} \) and \( (u + v)_{tt} = u_{tt} + v_{tt} \) satisfy the property (1.1.3), the cubic term does not

\[
(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \neq u^3 + v^3.
\]
1.1. WHAT IS A PARTIAL DIFFERENTIAL EQUATION

The advantage of linearity for the equation $Lu = 0$ is that if $u$ and $v$ are both solutions, so is $u + v$. If $u_1, \cdots, u_n$ are all solutions, so is any linear combination:

$$c_1 u_1 + \cdots + c_n u_n = \sum_{k=1}^{n} c_k u_k$$

where $c_k$ are constants. This is sometimes called the superposition principle.

Let us look at some PDEs:

Example 1.1.1 Find all $u(x, y)$ such that $u_{xx} = 0$.

Example 1.1.2 Solve the PDE $u_{xx} + u = 0$.

Example 1.1.3 Solve the equation $u_{xy} = 0$. 
**Moral:** A PDE has arbitrary functions in its solution. In these examples the arbitrary functions are functions of one variable that combine to produce a function $u(x, y)$ of two variables which is only partly arbitrary.

A function of two independent variables contains more information than a function of only one variable. Hence solving PDE is much harder than solving ODE just as the graph of a function of two variable is more complicated than a graph of a function of one variable.

Here are some basic facts you need to know in order to follow this course:

1. Derivatives are local.
2. Mixed derivatives are equal: $u_{xy} = u_{yx}$. We will assume this throughout this course. Unless stated otherwise, all derivatives are assumed to exist and be continuous.
3. The chain rule is used frequently in PDEs.
4. Green’s theorem and the divergence theorem are needed.
5. Derivatives of integrals like $I(t) = \int_{a(t)}^{b(t)} f(x, t)\,dx$.
7. Directional derivatives.
8. We will often reduce PDEs to ODEs, so we must know how to solve simple ODEs.
1.2  First Order Linear Equations

1.2.1 Constant Coefficient Equations
Let us solve the transport equation

\[ au_x + bu_y = 0, \quad (1.2.1) \]

where \( a \) and \( b \) are constants not both zero.

Geometric method The equation can be written as

\[ (a, b) \cdot \nabla u = 0, \quad \text{i.e.} \quad \frac{\partial u}{\partial l} = 0, \]

where \( l = (a, b) = a\vec{i} + b\vec{j} \). Thus the solution is constant on each line parallel to \( l \) (characteristic line). Such lines have the equations

\[ bx - ay = c, \]

where \( c \) is an arbitrary constant. On the line \( bx - ay = c \), the solution has a constant value. Call this value \( f(c) \). Then \( u(x, y) = f(c) = f(bx - ay) \). Since \( c \) is arbitrary, we have

\[ u(x, y) = f(bx - ay), \quad \text{for all} \ x \text{ and} \ y. \quad (1.2.2) \]

where \( f \) is any differentiable function of one variable.
Coordinate Method Change variables (or “make a change of coordinates”;) to
\[ x' = ax + by; \quad y' = bx - ay. \] (1.2.3)
and let \( u(x, y) = \pi(x', y') \).
Replace all \( x \) and \( y \) derivatives by \( x' \) and \( y' \) derivatives. By the chain rule,
\[ u_x = \frac{\partial u}{\partial x} = \frac{\partial \pi}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial \pi}{\partial y'} \frac{\partial y'}{\partial x} = a\pi_{x'} + b\pi_{y'} \]
and
\[ u_y = \frac{\partial u}{\partial y} = \frac{\partial \pi}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial \pi}{\partial y'} \frac{\partial y'}{\partial y} = b\pi_{x'} - a\pi_{y'} \].
Hence \( au_x + bu_y = a(a\pi_{x'} + b\pi_{y'}) + b(b\pi_{x'} - a\pi_{y'}) = (a^2 + b^2)\pi_{x'} \). So, since \( a^2 + b^2 \neq 0 \),
the equation takes the form \( \pi_{x'} = 0 \) in the new (primed) variables. Thus the solution is
\( u(x, y) = \pi(x', y') = f(y') = f(bx - ay) \) with \( f \) an arbitrary function of one variable. This is exactly the same as before!
1.2. Variable Coefficient Equations

Example 1: \( u_x + yu_y = 0 \).

The equation can be written as \((1, y) \cdot \nabla u = 0\), i.e. at any point \((x, y)\), the directional derivative of \(u\) in the direction \((1, y)\) is zero. Thus \(u\) is constant on each of the curves which have \((1, y)\) as tangential vector at the point \((x, y)\). Such curves are called the characteristic curves of the equation.

Step 1: Let us find the characteristic curves:

\[
\frac{dx}{1} = \frac{dy}{y}, \text{ i.e. } \frac{dy}{dx} = \frac{y}{1}.
\]

We have \(y = ce^x\), where \(c\) is an arbitrary constant. When \(c\) changes, the family of curves fill out the \(xy\)-plane without intersecting.

Step 2: Along the characteristic curves: \(y = ce^x\), we have

\[
\frac{d}{dx} [u(x, ce^x)] = \frac{\partial u}{\partial x} + ce^x \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.
\]

Then along the characteristic curves

\[
u(x, y) = u(x, ce^x) = \text{constant} = u(0, c) = f(c) = f(e^{-x}y).
\]

Therefore the value of \(u\) depends only on \(e^{-x}y\). The solution is

\[
u = f \left( e^{-x}y \right),
\]

where \(f\) is an arbitrary function which has first derivative.
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Example 2: $u_x + 2xy^2u_y = 0$.

The equation for the characteristic curves is

$$\frac{dx}{1} = \frac{dy}{2xy^2}, \text{ i.e. } \frac{dy}{dx} = 2xy^2.$$

The solution of the ODE is $x^2 + \frac{1}{y} = c$. The value of $u$ depends only on the value of $x^2 + \frac{1}{y}$. So the solution is

$$u = f(x^2 + \frac{1}{y}),$$

where $f$ is an arbitrary function which has first derivative.

Example 3: $(1 + x^2)u_x + u_y = 0$

Example 4: $\sqrt{1 - x^2}u_x + u_y = 0$

1.3 Flows, Vibrations, and Diffusions

The subject of PDEs was practically a branch of physics until the twentieth century. In this section we present a series of examples of PDEs as they occur in physics. They provide the basic motivation for all the PDE problems we study in the rest of course. We shall see that most often in physical problems the independent variables are those of space $x, y, z$ and time $t$. 
Example 1.3.1 Simple Transport

Consider a fluid, water, say, flowing at a constant rate $c$ along a horizontal pipe of fixed cross section in the positive $x$ direction. A substance, say a pollutant, is suspended in the water. Let $u(x,t)$ be its concentration in grams/centimeter at time $t$. Then

$$u_t + cu_x = 0 \quad (1.3.1)$$

(That is, the rate of change $u_t$ of concentration is proportional to the gradient $u_x$. Diffusion is assumed to be negligible.) Solving this equation as in previous section, we find that the concentration is a function of $x - ct$ only. This means that the substance is transported to the right at a fixed speed $c$. Each individual particle moves to the right at speed $c$; that is, in the $xt$-plane, it moves precisely along a characteristic line.

**Derivation of Equation (1.3.1):** The amount of pollutant in the interval $[0, b]$ at the time $t$ is $M = \int_0^b u(x, t)dx$, in grams, say. At the later time, $t + h$, the same molecules of pollutant have moved to the right by $c \cdot h$ centimeters. Hence,

$$M = \int_0^b u(x, t)dx \quad \text{and} \quad M = \int_{b+ch}^{b+ch} u(x, t+h)dx.$$  

Differentiating with respect to $b$, we get

$$u(b, t) = u(b + ch, t + h).$$

Differentiating with respect to $h$ and putting $h = 0$, we get

$$0 = cu_x(b, t) + u_t(b, t),$$

which is equation (1.3.1).

Example 1.3.2 Vibrating String

A flexible, elastic homogenous string of length $l$ undergoes relatively small transverse vibrations. Assume that it remains in a plane. Let $u(x, t)$ be its displacement from equilibrium position in the vertical direction at time $t$ and position $x$. Because the string is perfectly flexible, the tension (force) is directed tangentially along the string. Let $\rho$ be the density (mass per unit length) of the string. It is a constant because the string is homogenous. Let $T(x)$ be the magnitude of this tension vector and $\alpha(x, t)$ be the angle between the tensor and the horizontal direction. The slope of the tangential line of the string $u_x(x, t)$. Thus

$$\cos \alpha(x, t) = \frac{1}{\sqrt{1 + u_x^2}} \quad \text{and} \quad \sin \alpha(x, t) = \frac{u_x}{\sqrt{1 + u_x^2}}.$$  

Consider a part of the string between any two points at $x_0$ and $x_1$. Applying the Newton’s second law $\vec{F} = m\vec{a}$ to the horizontal ($x$) and vertical ($u$) components

$$T(x_1) \cos \alpha(x_1, t) - T(x_0) \cos \alpha(x_0, t) = 0 \quad (\text{horizontal})$$
\[ T(x_1) \sin \alpha (x_1, t) - T(x_0) \sin \alpha(x_0, t) = \int_{x_0}^{x_1} \rho u_{tt} \, dx \quad \text{(vertical)}. \]

Now we also assume that the motion is small, which means that \(|u_x|\) is quite small. Then \(\sqrt{1 + u_x^2}\) can be replaced by 1. If we do so, then the first equation says that \(T\) is a constant along the string. The second equation, differentiated, says that

\[ (Tu_x)_x = \rho u_{tt}. \]

That is,

\[ u_{tt} = c^2 u_{xx} \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}}. \quad \text{(1.3.2)} \]

This is the wave equation. At this point it is not clear why \(c\) is defined in this manner, but shortly we will see that \(c\) is the wave speed.

There are many variations of this equation:

(i) If significant air resistance \(r\) is present, we have an extra term proportional to the speed \(u_t\), thus

\[ u_{tt} - c^2 u_{xx} + ru_t = 0 \]

(ii) If there is a vertical elastic force, we have an extra term proportional to the displacement \(u\), as in a coiled spring, thus

\[ u_{tt} - c^2 u_{xx} + ku = 0, \quad \text{where} \quad k > 0 \]

(iii) If there is an externally applied force, it appears as an extra term, thus

\[ u_{tt} - c^2 u_{xx} = f(x, t), \]

which makes the equation inhomogeneous.

**Example 1.3.3 Vibrating Drumhead**

The two dimensional version of a string is an elastic, flexible, homogenous drumhead, that is, a membrane stretched over a frame. Say the frame lies in the \(xy\)-plane, \(u(x, y, t)\) is the vertical displacement and there is no horizontal motion. The horizontal components of Newton’s law again give constant tension \(T\). Let \(D\) be any domain in the \(xy\)-plane, say a circle or a rectangle. Let \(\partial D\) denote the boundary curve of \(D\). We use reasoning similar to the one dimensional case. The vertical component gives (approximately)

\[ F = \int_{\partial D} T \frac{\partial u}{\partial \eta} \, ds = \int_D \int \rho u_{tt} \, dx \, dy = ma, \]

where the left side is the total force acting on the piece \(D\) of the membrane, and where \(\partial u / \partial \eta = \eta \cdot \nabla u\) is the directional derivative in the outward normal direction, \(\eta\) being the unit outward normal vector on \(\partial D\). By Green’s theorem, this can be written as
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\[ \int_D \nabla \cdot (T \nabla u) dxdy = \int_D \rho u_{tt} dxdy. \]

Since \( D \) is arbitrary, we deduce from the second vanishing theorem that \( \rho u_{tt} = \nabla \cdot (T \nabla u) \). Since \( T \) is constant, we get

\[ u_{tt} = c^2 \nabla \cdot (\nabla u) = c^2 (u_{xx} + u_{yy}), \tag{1.3.3} \]

where \( c = \sqrt{T/\rho} \) as before and \( \nabla \cdot \nabla u = \text{divgradu} = u_{xx} + u_{yy} \) is known as the two dimensional Laplacian. Equation (1.3.3) is the two dimensional wave equation.

Now the pattern is clear. Three dimensional vibrations obey the equation

\[ u_{tt} = c^2 (u_{xx} + u_{yy} + u_{zz}). \tag{1.3.4} \]

The operator on \( u \) on the right of equation (1.3.4) is called three dimensional Laplacian.

Example 1.3.4 **Diffusion**

Let us imagine a motionless liquid filling a straight tube or pipe and a chemical substance, say a dye, which is diffusing through the liquid. Simple diffusion is characterized by the following law. [It is not to be confused with convection (transport), which refers to currents in the liquid.] The dye moves from the region of higher concentration to region of lower concentration. The rate of motion is proportional to the concentration gradient. (This is known as Fick’s law of diffusion in physics.) Let \( u(x, t) \) be the concentration (mass per unit length) of the dye at position \( x \) of the pipe at time \( t \).

In the section of pipe from \( x_0 \) to \( x_1 \), the mass of dye is

\[ M(t) = \int_{x_0}^{x_1} u(x, t) dx, \quad \text{so } \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x, t) dx. \]

The mass in this section of pipe cannot change except by flowing in or out of its ends. By Fick’s law,

\[ \frac{dM}{dt} = \text{flow in} - \text{flow out} = ku_x(x_1, t) - ku_x(x_0, t), \]

where \( k \) is proportionally constant. Therefore, those two expressions are equal:

\[ \int_{x_0}^{x_1} u_t(x, t) dx = ku_x(x_1, t) - ku_x(x_0, t). \]

Differentiating with respect to \( x_1 \), we get

\[ u_t = ku_{xx}. \tag{1.3.5} \]
This is the diffusion equation.
In three dimensions we have
\[\int \int \int_D u_t dxdydz = \int \int_{\partial D} k(\eta \cdot \nabla u) ds,\]
where \(D\) is any solid domain and \(\partial D\) is its bounding surface. By the divergence theorem (using the arbitrariness of \(D\) as in previous example), we get the three dimensional diffusion equation
\[u_t = k(u_{xx} + u_{yy} + u_{zz}) = k\Delta u. \quad (1.3.6)\]
If there is an external source (or a “sink”) of the dye, and if the rate \(k\) of diffusion is a variable, we get the more general inhomogeneous equation
\[u_t = \nabla (k\nabla u) + f(x,t).\]

Example 1.3.5 Heat Flow
We let \(u(x,y,z,t)\) be the temperature and let \(H(t)\) be the amount of heat (in calories, say) contained in a region \(D\). Then
\[H(t) = \int \int \int_D c\rho u dxdydz,\]
where \(c\) is the “specific heat” of the material and \(\rho\) is its density (mass per unit volume). The change in heat is
\[\frac{dH}{dt} = \int \int \int_D c\rho u_t dxdydz.\]

Fourier’s law says that heat flows from hot to cold regions proportionately to the temperature gradient. But heat cannot be lost from \(D\) except by leaving it through the boundary. This is the law of conservation of energy. Therefore, the change of heat energy in \(D\) also equals the heat flux across the boundary,
\[\frac{dH}{dt} = \int \int_{\partial D} \kappa(\eta \cdot \nabla u) ds,\]
where \(\kappa\) is a proportionality factor (the “heat conductivity”). By the divergence theorem,
\[\int \int \int_D c\rho u_t dxdydz = \int \int \int_D \nabla \cdot (\kappa \nabla u) dxdydz\]
and we get the heat equation
\[cp \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u). \quad (1.3.7)\]
If \(c, \rho\) and \(\kappa\) are constants, it is exactly the same as the diffusion equation!
Example 1.3.6 Stationary Waves and Diffusions

Consider any of the four preceding examples in a situation where the physical
state does not change with time. Then \( u_t = u_{tt} = 0 \). So both the wave and
the diffusion equations reduce to

\[
\Delta u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{1.3.8}
\]

This is called the Laplace equation. Its solutions are called harmonic functions.

There are many other examples such as the equations coming from geometry, financial
market, biology, material science. Due to time concern, I will not try to detail
them.

Complementary Materials: Black-Scholes Equations

Let \( V = V(S_t, t) \) be the value of a financial derivative, where the underlying asset
price \( S_t \) follows a geometric Brownian motion

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
\]

with constant expected return \( \mu \), constant volatility \( \sigma \) \((W_t \text{ is a Brownian motion})\).
To derive the model for \( V = V(S_t, t) \), we construct a portfolio of one long option
position and a short position in some quantity \( \Delta \), of the underlying.

\[
\Pi = V(S_t, t) - \Delta S_t.
\]

The increment of the value of the portfolio in one time-step is

\[
d\Pi = dV(S_t, t) - \Delta dS_t \\
= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS_t - \Delta dS_t.
\]

At this point the Ito’s Lemma has been used. To eliminate the risk, we take

\[
\Delta = \frac{\partial V}{\partial S}
\]

and then

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt.
\]

Note that there is no random term, the portfolio is riskless. So we must have

\[
d\Pi = r\Pi dt = r(V - S_t \frac{\partial V}{\partial S}) dt
\]

From the above two equalities, we obtain an equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.
\]

This is the well-known Black-Scholes equation. The solution domain is
\( D = \{(S, t) : S > 0, t \in [0, T]\} \), \( T \) is the maturity of the financial derivative.
1.4 Initial and boundary conditions

Because PDEs typically have so many solutions, we attempt to formulate the conditions so as to specify a unique solution. These conditions are motivated by the physics and they come in two varieties, initial conditions and boundary conditions. An initial condition specifies the physical state at a particular time $t_0$. For the diffusion equation the initial condition is

$$ u(x, y, z, t_0) = \phi(x, y, z), \quad (1.4.1) $$

where $\phi(x, y, z)$ is a given function. For a diffusing substance, $\phi(x, y, z)$ is the initial concentration. For heat flow, $\phi(x, y, z)$ is the initial temperature.

For the wave equation there is a pair of initial conditions

$$ u(x, y, z, t_0) = \phi(x, y, z) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, z, t_0) = \psi(x, y, z), \quad (1.4.2) $$

where $\phi(x, y, z)$ is the initial position and $\psi(x, y, z)$ is the initial velocity. It is clear on physical grounds that both of them must be specified in order to determine the position $u(x, y, z, t)$ at later time.

In each physical problem we have seen that there is a domain $D$ in which the PDE is valid. For the vibrating string, $D$ is the interval $0 < x < l$, so the boundary of $D$ consists only of the two points $x = 0$ and $x = l$. For the drumhead, the domain is a plane region and its boundary is a closed curve. For the diffusing chemical substance, $D$ is the container holding the liquid, so its boundary is a surface $S = \partial D$. For the hydrogen atom, the domain is all of space, so it has no boundary, or, in other words, its boundary is empty.

It is clear, again from our physical intuition, that it is necessary to specify some boundary condition if the solution is to be determined. The three most important kinds of boundary conditions are:

1. **Dirichlet condition** (D)
   - $u$ is specified

2. **Neumann condition** (N)
   - the normal derivative $\frac{\partial u}{\partial n}$ is specified

3. **Robin condition** (R)
   - $\frac{\partial u}{\partial n} + au$ is specified

where $a$ is a given function of $x, y, z, t$. Each is to hold for all $t$ and for $(x, y, z)$ belonging to $\partial D$. Usually, we write (D), (N), and (R) as equations. For instance,

$$ u|_{\partial D} = g(x, y, z, t) $$

$$ \frac{\partial u}{\partial \eta}|_{\partial D} = g(x, y, z, t) \quad (1.4.3) $$

where $g$ is a given function which could be called the boundary datum. Any of these boundary condition is called homogeneous if the specified function $g(x, y, z, t)$ vanishes (equals zero). Otherwise, it is called inhomogeneous. As usual, $n = (n_1, n_2, n_3)$
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denotes the unit normal vector on \( \partial D \), which points outward from \( D \). Also, \( \partial u / \partial \eta = n \cdot \nabla u \) denotes the directional derivative of \( u \) in the outward normal direction.

In one-dimensional problems, where \( D \) is just an interval \( 0 < x < l \), the boundary consists of just the two endpoints, and these boundary conditions take the simple form:

\[
\begin{align*}
(D) \quad & u(0, t) = g(t) \quad \text{and} \quad u(l, t) = h(t) \\
(N) \quad & \frac{\partial u}{\partial x}(0, t) = g(t) \quad \text{and} \quad \frac{\partial u}{\partial x}(l, t) = h(t)
\end{align*}
\]

and similarly for the Robin condition.

Following are some illustrations of physical problems corresponding to these boundary conditions.

**The vibrating string**

If the string is held fixed at both ends, as for a violin string, we have the homogeneous Dirichlet conditions \( u(0, t) = u(l, t) = 0 \).

Imagine, on the other hand, that one end of the string is free to move transversely without any resistance (say, along a frictionless track); then there is no tension \( T \) at the end, so \( u_x = 0 \). This is a Neumann condition.

If an end of the string were simply moved in a specified way, we would have an inhomogeneous Dirichlet condition at that end.

A Robin condition is also possible (See P 21, Strauss (1992))
Diffusion
If the diffusing substance is enclosed in a container $D$ so that none can escape or enter, then the concentration gradient in the normal direction must vanish, by Fick’s law. Thus $\partial u/\partial n = 0$ on $\partial D$, which is the Neumann condition.
If, on the other hand, the container is permeable and is so constructed that any substance which escapes to the boundary of container is immediately washed away, then we have $u = 0$ on $\partial D$.

Heat
Heat conduction is described by the diffusion equation with $u(x, t) =$temperature. If the object $D$ through which the heat is flowing is perfectly insulated, the no heat flows across the boundary and we have the Neumann condition $\partial u/\partial n = 0$.
On the other hand, if the object were immersed in a large reservoir of specified temperature $g(t)$ and there were perfect thermal conduction, then we’d have the Dirichlet condition $u = g(t)$ on $\partial D$.

Conditions at infinity
In case the domain $D$ is unbounded, sometimes we need to give conditions at infinity, and sometimes not. These can be tricky.

Jump conditions
An example is heat conduction, where the domain $D$ has two parts, $D = D_1 \cup D_2$ with different physical properties. A jump condition is to guarantee the continuity of the temperature $u$ and the heat flux at the weld $\partial D_1 \cap \partial D_2$.
Another example is caused in option pricing when a discrete dividend is paid.
1.5 First Order Equations with Initial / Boundary Value Conditions

Example 1: Consider the initial value problem (IVP):

\[ u_t(x, t) + au_x(x, t) = 0, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = f(x), \quad -\infty < x < \infty, \]

where \( a = \text{const} \), and \( f(x) \) is a specified function.

Step 1: The characteristic line is

\[ \frac{dt}{1} = \frac{dx}{a} \]

or

\[ x(t) = at + x(0). \]

Step 2: Along the characteristic line \( x(t) = at + x(0) \),

\[ \frac{du(x(t), t)}{dt} = u_x(x(t)) u_t + u_t = au_x + u_t = 0. \]

Then

\[ u(x(t), t) = \text{const}. \]

Step 3: Along the characteristics

\[ u(x(t), t) = u(x(0), 0) = f(x(0)) = f(x(t) - at). \]

Then in general

\[ u(x, t) = f(x - at) \text{ for all } x \in R, \quad t > 0. \]

Example 2: In the IVP,

\[ u_t(x, t) + \frac{1}{2} u_x(x, t) = 0, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = \begin{cases} \sin x, & 0 \leq x \leq \pi, \\ 0, & \text{otherwise} \end{cases} \]
Example 3:

\[ u_t(x, t) + 3t u_x(x, t) = u, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = \cos x, \quad -\infty < x < \infty \]

Step 1: The characteristic:
\[ \frac{dt}{1} = \frac{dx}{3t} \]
or
\[ x(t) = \frac{3}{2} t^2 + x(0). \]

Step 2: Along the characteristic:
\[ \frac{d}{dt} u(x(t), t) = u_t + x'(t) u_x = u_t + 3tu_x = u. \]

Then
\[ u(x(t), t) = u(x(0), 0) e^t. \]

Step 3: Using the initial condition, we obtain
\[ u(x(t), t) = \cos (x(0)) e^t = \cos \left( x(t) - \frac{3}{2} t^2 \right) e^t. \]

Then
\[ u(x, t) = \cos \left( x - \frac{3}{2} t^2 \right) e^t. \]

Example 4. In the IVP

\[ u_t(x, t) + xu_x(x, t) = 1, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = x^2, \quad -\infty < x < \infty \]

Example 5. The initial boundary value problem (IBVP) (semi-unbounded problem)

\[ u_t(x, t) + u_x(x, t) = 0, \quad x > 0, \quad t > 0, \]
\[ u(0, t) = t, \quad t > 0, \]
\[ u(x, 0) = \sin x, \quad x > 0 \]
Example 6:

\[ u_t(x, t) - u_x(x, t) = 0, \quad x > 0, \quad t > 0, \]
\[ u(x, 0) = \sin x, \quad x > 0 \]
Two exercises:
(a)\[ u_t(x, t) + u_x(x, t) = x, \quad x > 0, \ t > 0, \]
\[ u(0, t) = t, \quad t > 0, \]
\[ u(x, 0) = \sin x, \quad x > 0 \]

(b)\[ u_t(x, t) - u_x(x, t) = x, \quad x > 0, \ t > 0, \]
\[ u(x, 0) = \sin x, \quad x > 0 \]

Example 7. First Order Quasilinear Equations:
A PDE of the form
\[ u_t(x, t) + c(u)u_x(x, t) = q(x, t, u) \]
is called quasilinear. It is a nonlinear equation, but it is linear in the first order derivatives of \( u \). Such equations arise in the modeling of a variety of phenomena (for example, traffic flow) and can be solved by the method of characteristics.

\[ u_t(x, t) + u^3(x, t)u_x(x, t) = 0, \quad -\infty < x < \infty, \ t > 0, \]
\[ u(x, 0) = x^{1/3}, \quad -\infty < x < \infty \]

If \( x = x(t) \) satisfies \( x'(t) = u^3(x(t), t) \), then on the characteristic through \( (x, t) \) the PDE becomes
\[ \frac{d}{dt} u(x(t), t) = u_t(x(t), t) + u_x(x(t), t)x'(t) = 0, \]
which yields \( u(x(t), t) = C = \text{const.} \). Hence from the IC we see have
\[ u(x(t), t) = u(x(0), 0) = x(0)^{1/3} \]

Recall the characteristic line satisfies
\[ x'(t) = u^3(x(t), t) = x(0). \]

So,
\[ x(t) - x(0) = x(0)t \]
or
\[ x(t) = (t + 1)x(0). \]

This gives
\[ u(x(t), t) = \left( \frac{x(t)}{t + 1} \right)^{1/3}. \]

Then the solution of the given IVP is
\[ u(x, t) = \left( \frac{x}{t + 1} \right)^{1/3}, \text{ for all } x \in \mathbb{R} \text{ and } t > 0. \]

Exercise:
\[ u_t(x, t) + u(x, t)u_x(x, t) = 2t, \quad -\infty < x < \infty, \ t > 0, \]
\[ u(x, 0) = x, \quad -\infty < x < \infty \]
1.6 Well-Posed Problems

One PDE with suitable initial or boundary conditions or both is said to be well-posed if the problem satisfies the following three conditions:

Existence: There exists at least one solution satisfying all these conditions;

Uniqueness: There is at most one such solution;

Stability: The unique solution depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little.

For a physical problem modeled by a PDE, the scientist normally tries to formulate physically realistic auxiliary conditions which all together make a well-posed problem. The mathematician tries to prove that a given problem is or is not well-posed. If too few auxiliary are imposed, then there may be more than one solution (nonuniqueness) and the problem is called underdetermined. If, on the other hand, there are too many auxiliary conditions, there may be no solution at all (nonexistence) and the problem is called overdetermined.

The stability property is normally required in models of physical problems. This is because you could never measure the data with mathematical precision but only up to some number of decimal places. You cannot distinguish a set of data from a tiny perturbation of it. The solution ought not be significantly affected by such tiny perturbations, so it should change very little.
Chapter 2

Waves

2.1 Wave on the Line: Cauchy Problem.

\[ u_{tt} = c^2 u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \ t > 0 \tag{2.1.1} \]

where \( c > 0 \) is a constant. This equation has a very nice operator factors:

\[ u_{tt} - c^2 u_{xx} = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u. \tag{2.1.2} \]

2.1.1 General Solution

**Conclusion** The general solution of (2.1.1) is

\[ u = f(x + ct) + g(x - ct) \tag{2.1.3} \]

where \( f \) and \( g \) are two arbitrary \( C^2 \) functions of a single variable.

A function is called \( C^k \) if it has continuous \( k \)th order derivatives (or partial derivatives).

**Proof:** We introduce the characteristic coordinates

\[ \xi = x + ct, \ \eta = x - ct. \]

By the chain rule, we have easily reached the new equation

\[ u_{\xi\eta} = 0. \]

The solution is

\[ u = f(\xi) + g(\eta). \]

Returning to the variables \( x \) and \( t \), we get (2.1.3).

**Remark 2.1** The wave equation has a nice simple geometry. There are two families of characteristic lines, \( x \pm ct = \text{constant} \). The most general solution is the sum of two functions. One, \( g(x - ct) \), is a wave of arbitrary shape traveling to the right at the speed \( c \). The other, \( f(x + ct) \), is another shape traveling to the left at the speed.
2.1.2 Initial Value Problem

\[
\begin{align*}
    u_{tt} - c^2 u_{xx} &= 0, \ x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) &= \phi(x), \ x \in \mathbb{R} \\
    u_t(x, 0) &= \psi(x), \ x \in \mathbb{R}
\end{align*}
\] (2.1.4)

where \( \phi \) and \( \psi \) are given functions of \( x \).

**Conclusion.** If \( \phi \in C^2 \) and \( \psi \in C^1 \), (2.1.4) has a unique solution \( u \in C^2 \) which is given by

\[
    u(x, t) = \frac{1}{2} \left[ \phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)ds.
\]

Proof: From general solution \( u(x, t) = f(x + ct) + g(x - ct) \) and applying initial conditions, we get

\[
    u(x, 0) = f(x) + g(x) = \phi(x).
\] (2.1.5)

Then using the chain rule, we differentiate (2.1.3) with respect to \( t \) and put \( t = 0 \) to get

\[
    u_t(x, 0) = cf'(x) - cg'(x) = \psi(x).
\] (2.1.6)

Now differentiate (2.1.5) with respect to \( y \) and divide (2.1.6) by \( c \) to get

\[
    \frac{\phi'}{c} = f' + g' \quad \text{and} \quad \frac{1}{c} \psi = f' - g'.
\]

Adding and subtracting the last pair of equations gives us

\[
    f' = \frac{1}{2} (\phi' + \frac{\psi}{c}) \quad \text{and} \quad g' = \frac{1}{2} (\phi' - \frac{\psi}{c}).
\]

Integrating, we have

\[
    f(y) = \frac{1}{2} \phi(y) + \frac{1}{2c} \int_{0}^{y} \psi(s)ds + A
\]

and

\[
    g(y) = \frac{1}{2} \phi(y) - \frac{1}{2c} \int_{0}^{y} \psi(s)ds + B,
\]

where \( A \) and \( B \) are constants. Because of (2.1.5), we have \( A + B = 0 \). This tells us what \( f \) and \( g \) are in the general formula (2.1.3). Substituting \( y = x + ct \) into the formula for \( f \) and \( y = x - ct \) into that of \( g \), we get

\[
    u(x, t) = \frac{1}{2} \phi(x + ct) + \frac{1}{2c} \int_{0}^{x+ct} \psi(s)ds + \frac{1}{2} \phi(x - ct) - \frac{1}{2c} \int_{0}^{x-ct} \psi(s)ds.
\]

This simplifies to

\[
    u(x, t) = \frac{1}{2} \left[ \phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)ds. \quad (2.1.7)
\]

This is the solution formula for the initial value problem, due to D’Alembert in 1746.
2.1. WAVE ON THE LINE: CAUCHY PROBLEM.

Example (1) Solve the initial value problem of (2.1.1) with $\phi = 0$ and $\psi = \cos x$.

Example (2) Solve the initial value problem of (2.1.1) with $\phi = 2 \sin x$ and $\psi = x$.

Example (3) The Plucked String
Consider an infinitely long string with the speed is $c = \sqrt{T/\rho}$, initial position

$$
\phi(x) = \begin{cases} 
  b - \frac{b|x|}{a} & \text{for } |x| < a, \\
  0 & \text{for } |x| > a.
\end{cases}
$$

and initial velocity $\psi(x) = 0$ for all $x$. 

\( (2.1.8) \)
### 2.1.3 Waves with a Source

Consider the problem
\[
\begin{align*}
    u_{tt} - c^2 u_{xx} & = f(x, t) \quad \text{for} \quad x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) & = \phi(x), \ x \in \mathbb{R} \\
    u_t(x, 0) & = \psi(x), \ x \in \mathbb{R}
\end{align*}
\]
where \( f(x, t) \) is a given function. For instance, \( f(x, t) \) could be interpreted as an external force acting on a vibrating string.

**Conclusion** The unique solution of the above problem is
\[
u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(\xi) d\xi \\
+ \frac{1}{2c} \int_0^t d\tau \int_{x - c(t-\tau)}^{x + c(t-\tau)} f(\xi, \tau) d\xi
\]
(2.1.9)

Proof: It suffices to show that the equation with a source and homogeneous initial conditions
\[
\begin{align*}
    u_{tt} - c^2 u_{xx} & = f(x, t) \quad \text{for} \quad x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) & = 0, \ u_t(x, 0) = 0, \ x \in \mathbb{R}
\end{align*}
\]
has a unique solution
\[
u(x, t) = \frac{1}{2c} \int_0^t d\tau \int_{x - c(t-\tau)}^{x + c(t-\tau)} f(\xi, \tau) d\xi
\]
Let us use the method of characteristics. Since
\[
u_{tt} - c^2 u_{xx} = \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u,
\]
we can let
\[
\begin{align*}
    \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u & = v \\
    \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) v & = f
\end{align*}
\]
(2.1.10)  (2.1.11)
where \( v(x, 0) = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u \bigg|_{t=0} = 0 \). The characteristic line of (2.1.11) is \( x(t) = ct + x(0) \), along which
\[
\frac{d}{dt} v(x(t), t) = f(x(t), t).
\]
Then
\[ v(x(t), t) = v(x(0), 0) + \int_0^t f(x(\tau), \tau) d\tau \]
\[ = \int_0^t f(x(t) - c(t-\tau), \tau) d\tau. \]
So
\[ v(x, t) = \int_0^t f(x - c(t-\tau), \tau) d\tau. \]
The characteristic line of (2.1.10) is \( x(t) = -ct + x(0), \) along which
\[ \frac{d}{dt} u(x(t), t) = v(x(t), t). \]
Then
\[ u(x(t), t) = u(x(0), 0) + \int_0^t v(x(s), s) ds \]
\[ = \int_0^t ds \int_0^s f(x(s) - c(s-\tau), \tau) d\tau \]
\[ = \int_0^t ds \int_0^s f(x(t) + c(t-s) - c(s-\tau), \tau) d\tau \]
\[ = \int_0^t ds \int_0^s f(x(t) + c(t+\tau) - 2cs, \tau) d\tau \]
So
\[ u(x, t) = \int_0^t ds \int_0^s f(x + c(t+\tau) - 2cs, \tau) d\tau \]
\[ = \int_0^t d\tau \int_0^t f(x + c(t+\tau) - 2cs, \tau) ds \]
\[ = \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi, \tau) d\xi \]

2.1.4 Well-Posedness of Cauchy Problem of Wave Equations

We discuss well-posedness of the Cauchy problem of wave equation with a source
\[ \begin{align*}
    u_{tt} - c^2 u_{xx} &= f(x,t), \quad x \in \mathbb{R}, \quad t > 0 \\
    u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}.
\end{align*} \]
We have got the existence and uniqueness. Next we consider the stability, which means that if the data \((\phi, \psi, f)\) change a little, then \(u\) also changes only a little. To make this precise, we need a way to measure the “nearness” of functions, that is, a metric or norm on function spaces. We will illustrate this concept using the uniform norms:
\[ ||w|| = \max_{x \in \mathbb{R}} |w(x)|, \quad ||w||_T = \max_{x \in \mathbb{R}, t \in [0,T]} |w(x, t)|. \]
Here $T$ is fixed. Suppose that $u_1(x, t)$ is the solution with data $(\phi_1(x), \psi_1(x), f_1(x, t))$ and $u_2(x, t)$ is the solution with data $(\phi_2(x), \psi_2(x), f_2(x, t))$. We have the same formula (2.1.9) satisfied by $u_1$ and $u_2$ except for the different data. We subtract the two formulas to get

$$|u_1(x, t) - u_2(x, t)| \leq \max |\phi| + \frac{1}{2c} \max |\psi| \cdot 2ct + \frac{1}{2c} \max |f| \cdot ct^2$$

Therefore,

$$||u_1 - u_2||_T \leq ||\phi_1 - \phi_2|| + T||\psi_1 - \psi_2|| + \frac{T^2}{2} ||f_1 - f_2||_T.$$ 

So if $||\phi_1 - \phi_2|| < \delta$, $||\psi_1 - \psi_2|| < \delta$, and $||f_1 - f_2||_T < \delta$, where $\delta$ is small, then

$$||u_1 - u_2||_T \leq \left(1 + T + T^2\right) \delta.$$

### 2.2 Physics of Waves: Causality and Energy

#### 2.2.1 Causality

**Principle of causality.** We have just learnt that the effect of an initial position $\phi(x)$ is a pair of waves travelling in either direction at speed $c$ and at half the original amplitude. The effect of an initial velocity $\psi$ is wave spreading out at speed $\leq c$ in both direction. So part of the wave may lag behind (if there is an initial velocity), but no part goes faster than speed $c$. The last assertion is called the principle of causality. It can be visualized in the $xt$ plane.

**Domain of influence.** An initial condition (position or velocity or both) at the point $(x_0, 0)$ can effect the solution for $t > 0$ only in a sector in $xt$ plane bounded by two ray

$$x - x_0 = \pm ct, \ t > 0,$$

which is called the domain of influence of the point $(x_0, 0)$. As a consequence, if $\phi$ and $\psi$ vanish for $|x| > R$, then $u(x, t) = 0$ for $|x| > R + ct$. In other words, the domain of influence of an interval ($|x| \leq R$) is a sector ($|x| \leq R + ct$).

**Interval of dependence.** An “inverse” way to express causality is the following. Fix a point $(x, t)$ for $t > 0$, the value $u(x, t)$ depends only on the values of $\phi$ at the two points $x \pm ct$, and on the values of $\psi$ within the interval $[x - ct, x + ct]$. We therefore say that the interval $(x - ct, x + ct)$ is the interval of dependence of the point $(x, t)$ on $t = 0$. Sometimes we call the triangle bounded by the pair of characteristic lines that pass through $(x, t)$ the domain of dependence or the past history of the point $(x, t)$. 
2.2.2 Energy

Imagine an infinite string with constant $\rho$ and $T$. Then $\rho u_t = Tu_{xx}$ for $-\infty < x < \infty$. From physics we know that the kinetic energy is $\frac{1}{2}mv^2$, which in our case takes the form

$$KE = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2 dx.$$  

To be sure that the integral converges, we assume that $\phi$ and $\psi$ vanish outside an interval $\{|x| \leq R\}$. As mentioned above, $u(x,t)$ [and therefore $u_t(x,t)$] vanish for $|x| > R + ct$. Differentiating the kinetic energy, we can pass the derivative under the integral sign to get

$$\frac{dKE}{dt} = \rho \int_{-\infty}^{\infty} u_t u_{tt} dx.$$  

Then we substitute the PDE $\rho u_{tt} = Tu_{xx}$ and integrate by parts to get

$$\frac{dKE}{dt} = T \int_{-\infty}^{\infty} u_t u_{xx} dx = T u_t u_x |_{-\infty}^{\infty} - T \int_{-\infty}^{\infty} u_t u_x dx.$$  

The term $T u_t u_x$ is evaluated at $x = \pm \infty$ and so it vanishes. But the final term is a pure derivative since $u_t u_x = \left(\frac{1}{2}u_x^2\right)_t$. Therefore,

$$\frac{dKE}{dt} = -\frac{d}{dt} \int \frac{1}{2} Tu_x^2 dx.$$  

Let $PE = \frac{1}{2}T \int u_x^2 dx$ and let $E = KE + PE$. Then $dKE/dt = -dPE/dt$, or $dE/dt = 0$. Thus,

$$E = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2) dx$$  

is a constant independent of $t$. This is the law of conservation of energy.

In physics courses, we learn that $PE$ has the interpretation of the potential energy. The only thing we need mathematically is the total energy $E$. The conservation of energy is one of the most basic facts about the wave equation. Sometimes the definition of $E$ is modified by a constant factor, but that does not affect its conservation. Notice that the energy is necessarily positive. The energy can also be used to derive causality as we will see in multidimensional waves.

**Example** An infinitely long string with the speed $c = \sqrt{T/\rho}$, initial position

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a, \\ 0 & \text{for } |x| > a. \end{cases}$$

and initial velocity $\psi(x) = 0$, has the energy

$$E = \frac{T}{2} \int_{-\infty}^{\infty} \phi_x^2 dx = \frac{T}{2} \int_{-a}^{a} \left(\frac{b}{a}\right)^2 dx = \frac{Tb^2}{a}.$$
Remark 2.2 In electromagnetic theory the equations are Maxwell’s. Each component of the electric and magnetic fields satisfies the (three-dimensional) wave equation, where \( c \) is the speed of light. The principle of causality, discussed above, is the cornerstone of the theory of relativity. It means that a signal located at the position \( x_0 \) at the instant \( t_0 \) cannot move faster than the speed of light. The domain of influence of \((x_0, t_0)\) consists of all the points that can be reached by a signal of speed \( c \) starting from the point \( x_0 \) at the time \( t_0 \). It turns out that the solutions of the three-dimensional wave equation always travel at the speeds exactly equal to \( c \) and never slower. Therefore, the causality principle is sharper in three dimensions than in one. This sharp form is called Huygens’ principle.

Remark 2.3 Flatland is an imaginary two-dimensional world. You can think of yourself as a waterbug confined to the surface of a pond. You would not want to live there because Huygens’ principle is not valid in two dimensions. Each sound you make would automatically mix with the “echoes” of your previous sounds. And each view would be mixed fuzzily with the previous views. Three is the best of all possible dimensions you can live comfortably.

### 2.3 Waves on the Half-Line: Reflections.

#### 2.3.1 Homogeneous boundary conditions

Consider the following problem:

\[
\begin{align*}
\text{DE: } & \quad v_{tt} - c^2 v_{xx} = 0 \quad \text{for } 0 < x < \infty \quad \text{and } 0 < t < \infty \\
\text{IC: } & \quad v(x, 0) = \phi(x) \quad \text{for } t = 0 \\
& \quad v_t(x, 0) = \psi(x) \quad \text{and } 0 < x < \infty \\
\text{BC: } & \quad v(0, t) = 0 \quad \text{for } x = 0 \quad \text{and } 0 < t < \infty.
\end{align*}
\]

(2.3.1)

**Conclusion.** Assume that \( \phi \) is a \( C^2 \) function and \( \psi \) is a \( C^1 \) function, and

\[
\phi(0) = \phi''(0) = 0, \quad \psi(0) = 0,
\]

(2.3.2)

then (2.3.1) has a unique solution, which is given by

\[
v(x, t) = \begin{cases}
\frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{ct-x}^{ct+x} \psi(y)dy & \text{for } x > ct \\
\frac{1}{2} [\phi(ct + x) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{ct+x} \psi(y)dy & \text{for } 0 < x < ct.
\end{cases}
\]

(2.3.3)

**Proof:** We use reflection method to solve (2.3.1), i.e. extend the initial condition to the entire line and solve the Cauchy problem on the entire line. Let \( \phi_{\text{odd}}(x) \) and \( \psi_{\text{odd}}(x) \) be the odd extension of \( \phi \) and \( \psi \) respectively. From the condition (2.3.2), \( \phi_{\text{odd}} \) is \( C^2 \) and \( \psi_{\text{odd}}(x) \) is \( C^1 \) on \( R \). By our previous formula, we know that the solution on the whole line with initial conditions \( \phi_{\text{odd}} \) and \( \psi_{\text{odd}} \) is given by the formula:

\[
u(x, t) = \frac{1}{2} [\phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy.
\]

(2.3.4)
It can be shown that \( u(x, t) \) itself is also an odd function. In particular, \( u(0, t) = 0 \) for all \( t \). Thus if we set \( v(x, t) = u(x, t) \) for \( x > 0 \), it will be clear that \( v \) satisfies the differential equation as well as the boundary condition. In formula (2.3.4), set \( x > 0 \) and \( t = 0 \), we can see that \( v(x, 0) = \frac{1}{2} [\phi'(x) + \phi(x)] = \phi(x) \) and \( v_t(x, 0) = \frac{1}{2} [c\phi'(x) - c\phi(x)] + \psi(x) = \psi(x) \). Hence it satisfies the initial condition as well. Thus, we conclude that \( v(x, t) \) is a unique solution of (2.3.1).

To get (2.3.3), observe that if \( x > ct > 0 \), then \( x + ct > 0 \) and \( x - ct > 0 \), thus \( \phi_{\text{odd}}(x \pm ct) = \phi(x \pm ct) \) and for \( 0 < x - ct < y < x + ct \), \( \psi_{\text{odd}}(y) = \psi(y) \). Thus the first part of formula (2.3.3) follows.

Now if \( 0 < x < ct \), then \( \phi_{\text{odd}}(x - ct) = -\phi(ct - x) \), thus (2.3.4) implies

\[
v(x, t) = \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(y) dy + \frac{1}{2c} \int_0^0 \psi(-y) dy.
\]

Change variable in last term to get the second equation of (2.3.3).

Geometrically, the result can be interpreted as follows. Draw the backward characteristics from the point \((x, t)\). In case \((x, t)\) is in the region \( x < ct \), one of the characteristics hits the \( t\)-axis before it hits the \( x\)-axis. The second formula in (2.3.3) shows that the reflection induces a change of sign. The value of \( v(x, t) \) now depends on the values of \( \phi \) at the pair of points \( ct \pm x \) and on the values of \( \psi \) in the short interval between these points.

**Remark 2.4** If the condition (2.3.2) is not satisfied, the function \( u \) given in (2.3.3) may have singularity.

**Example:** Solve (2.3.1) for \( \phi(x) = 1 \) and \( \psi(x) = 0 \) by using the reflection method.

\[
v(x, t) = \begin{cases} 
1 & \text{if } x > c|t| \\
0 & \text{if } 0 < x < c|t| 
\end{cases}
\]

Note that \( v \) has singularity at the characteristic lines \( x = \pm ct \).
**Exercise:** Solve the initial-Neumann boundary value problem

\[
\begin{align*}
DE: \quad & v_{tt} - c^2 v_{xx} = 0 \quad \text{for } 0 < x < \infty \\
& \quad \text{and } 0 < t < \infty \\
IC: \quad & v(x, 0) = \phi(x) \quad \text{for } t = 0 \\
& \quad v_t(x, 0) = \psi(x) \quad \text{and } 0 < x < \infty \\
BC: \quad & v_x(0, t) = 0 \quad \text{for } x = 0 \\
& \quad \text{and } 0 < t < \infty.
\end{align*}
\]

(2.3.5)

**Remark 2.5** The reflection method can also be used to solve the Initial-Dirichlet boundary value problem on a finite interval:

\[
\begin{align*}
DE: \quad & v_{tt} - c^2 v_{xx} = 0 \quad \text{for } 0 < x < l \\
& \quad \text{and } 0 < t < \infty \\
IC: \quad & v(x, 0) = \phi(x) \\
& \quad v_t(x, 0) = \psi(x) \quad \text{for } 0 < x < l \\
BC: \quad & v(0, t) = 0 \\
& \quad v(l, t) = 0 \quad \text{for } 0 < t < \infty.
\end{align*}
\]

(2.3.6)

The way to solve this equation is again to try to extend the functions \(\phi\) and \(\psi\) to be odd functions with respect to 0 as well as \(l\):

\[
\phi_{\text{ext}}(x) = \begin{cases} 
\phi(x) & \text{for } 0 < x < l \\
-\phi(-x) & \text{for } -l < x < 0
\end{cases}
\]

(2.3.7)

Similarly we have \(\psi_{\text{ext}}(x)\). Thus the solution will be

\[
v(x, t) = \frac{1}{2} [\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)dy.
\]

(2.3.8)

However it is not easy to find a formula containing only \(\phi\) and \(\psi\) instead of \(\phi_{\text{ext}}\) and \(\psi_{\text{ext}}\).
2.3. WAVES ON THE HALF-LINE: REFLECTIONS.

2.3.2 Inhomogeneous boundary conditions

Inhomogeneous Dirichlet conditions

\[
\begin{align*}
DE &: \quad v_{tt} - c^2 v_{xx} = 0 \quad \text{for } 0 < x < \infty \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{and } 0 < t < \infty \\
IC &: \quad v(x, 0) = \phi(x) \quad \text{for } t = 0 \\
&\quad \quad v_t(x, 0) = \psi(x) \quad \text{and } 0 < x < \infty \\
BC &: \quad v(0, t) = g(t) \quad \text{for } x = 0 \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{and } 0 < t < \infty.
\end{align*}
\]

(2.3.9)

We can make use of the transformation \( u(x, t) = v(x, t) - g(t) \) to reduce it to the problem with homogeneous Dirichlet boundary condition.

Inhomogeneous Neumann conditions

\[
\begin{align*}
DE &: \quad v_{tt} - c^2 v_{xx} = 0 \quad \text{for } 0 < x < \infty \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{and } 0 < t < \infty \\
IC &: \quad v(x, 0) = \phi(x) \quad \text{for } t = 0 \\
&\quad \quad v_t(x, 0) = \psi(x) \quad \text{and } 0 < x < \infty \\
BC &: \quad v_x(0, t) = g(t) \quad \text{for } x = 0 \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{and } 0 < t < \infty.
\end{align*}
\]

(2.3.10)

The transformation \( u(x, t) = v(x, t) - xg(t) \) can be used.

Remark 2.6 The reflection (extension) method fails to apply to the problems with Robin boundary conditions.

Remark 2.7 For the finite interval problem, similarly we can reduce the problem with inhomogeneous Dirichlet or Neumann boundary conditions to the one with homogeneous boundary conditions. However, the above method cannot handle the mixed boundary conditions, that is, Dirichlet condition at one endpoint and Neumann condition at the other endpoint.
Chapter 3

The Diffusion Equation

In this chapter we begin a study of the one-dimensional diffusion equation

\[ u_t - ku_{xx} = f(x,t). \]  \hspace{1cm} (3.0.1)

3.1 Maximum Principle

Diffusions are very different from the waves, and this is reflected in the mathematical properties of the equations. Because (3.0.1) is harder to solve than the wave equation, we are going to focus on a general discussion of some of the properties of diffusions. We begin with the maximum principle, from which we will deduce the uniqueness of an initial-boundary value problem. We postpone until the next section the derivation of the solution formula for (3.0.1).

**Weak Maximum Principle** Let \( u(x,t) \in C^2 \) satisfies the diffusion equation in a rectangle (say, \( 0 \leq x \leq l, 0 \leq t \leq T \)) in space-time. If \( f(x,t) \leq 0 \), then the maximum value of \( u(x,t) \) is attained either initially (\( t = 0 \)) or on the lateral sides (\( x = 0 \) or \( x = l \)).

**Remark 3.8** In fact, there is a stronger version of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but only on the bottom or the lateral sides (unless \( u \) is a constant).

The minimum value has the same property; it can also be attained only on the bottom or lateral sides if \( f(x,t) \geq 0 \).

These principles have a natural interpretation in terms of diffusion or heat flow. If you have a rod with no internal heat source and sink, the hottest spot and the coldest spot can occur only initially or on one of the two ends of the rod. Thus a hot spot at time zero will cool off (unless heat is fled into the rod at the end). You can burn one of its ends but the maximum temperature will always be at the hot end, so that it will be cooler away from that end. Similarly, if you have a substance diffusing along a tube, its highest concentration can occur only initially or at one of the ends of the tube.
If we draw a “movie” of the solution, the maximum drops while the minimum comes upon (provided $f = 0$). So the differential equation tends to smooth the solution out. (This is very different from the behaviour of the wave equation!)

**Proof of the Weak Maximum Principle:**
3.1. MAXIMUM PRINCIPLE

Uniqueness
The maximum principle can be used to give a proof of uniqueness for the Dirichlet problem for the diffusion equation. That is, there is at most one solution of

\[
\begin{align*}
    u_t - ku_{xx} &= f(x, t) \quad \text{for } 0 < x < l \text{ and } t > 0 \\
    u(x, 0) &= \phi(x) \\
    u(0, t) &= g(t) \\
    u(l, t) &= h(t)
\end{align*}
\]  

(3.1.1)

for four given functions \(f, \phi, g,\) and \(h.\) Uniqueness means that any solution is determined completely by its initial and boundary conditions. Indeed, let \(u_1(x, t)\) and \(u_2(x, t)\) be two solutions of (3.1.1). Let \(w = u_1 - u_2\) be their difference. Then, \(w_t - kw_{xx} = 0, w(x, 0) = 0, w(0, t) = 0, w(l, t) = 0.\) Let \(T > 0.\) By the maximum principle, \(w(x, t)\) has its maximum for the rectangle on its bottom or sides–exactly where it vanishes. So \(w(x, t) \leq 0.\) The same for the minimum shows that \(w(x, t) \geq 0.\) Therefore, \(w(x, t) \equiv 0,\) so that \(u_1(x, t) \equiv u_2(x, t)\) for all \(t \geq 0.\)

Here is a second proof of uniqueness for the problem (3.1.1), by a very different technique, the energy method. Multiplying the equation for \(w = u_1 - u_2\) by \(w\) itself, we can write

\[
0 = 0 \cdot w = (w_t - kw_{xx})(w) = \frac{1}{2} w^2_t + (-kww_x)_x + kw^2_x.
\]

(Verify this by carrying out the derivatives on the right side.) Upon integrating over the interval \(0 < x < l,\) we get

\[
0 = \int_0^l \left( \frac{1}{2} w^2 \right)_t dx - kw_xw|_{x=0} + k \int_0^l w^2_x dx.
\]

Because of the boundary conditions \((w = 0 \text{ at } x = 0, l),\)

\[
\frac{d}{dt} \int_0^l \frac{1}{2} [w(x, t)]^2 dx = -k \int_0^l [w_x(x, t)]^2 dx \leq 0,
\]

where the derivative has been pulled out of the \(x\) integral. Therefore, \(\int_0^l w^2 dx\) is decreasing in \(t,\) so

\[
\int_0^l [w(x, t)]^2 dx \leq \int_0^l [w(x, 0)]^2 dx
\]

(3.1.2)

for \(t \geq 0.\) The right side of (3.1.2) vanishes because of the initial conditions of \(u_1\) and \(u_2\) are the same, so that \(\int_0^l [w(x, t)]^2 dx = 0\) for all \(t > 0.\) So \(w \equiv 0\) and \(u_1 \equiv u_2\) for all \(t \geq 0.\)

Stability
This is the third ingredient of well-posedness. It means that the initial and boundary conditions are correctly formulated. The energy method leads to the following form of stability of the problem (3.1.1), in case \(h = g = f = 0.\) Let \(u_1(x, 0) = \phi_1(x)\) and
\[ u_2(x, 0) = \phi_2(x). \] Then \( w = u_1 - u_2 \) is the solution with the initial datum \( \phi_1 - \phi_2 \). So from (3.1.2) we have
\[
\int_0^l [u_1(x, t) - u_2(x, t)]^2 dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2 dx. \tag{3.1.3}
\]
On the hand right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at \( t = 0 \)), we stay nearby. This is exactly the meaning of stability in the “square integral” sense.

The maximum principle also proves the stability, but with a different way to measure the nearness. Consider two solutions of (3.1.1) in a rectangle. We then have \( w = u_1 - u_2 = 0 \) on the lateral sides of the rectangle and \( w = \phi_1 - \phi_2 \) on the bottom. The maximum principle asserts that throughout the rectangle
\[
u_1(x, t) - u_2(x, t) \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|.
\]
The minimum principle says that
\[
u_1(x, t) - u_2(x, t) \geq -\max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|.
\]
Therefore,
\[
\max_{0 \leq x \leq l} |u_1(x, t) - u_2(x, t)| \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|, \tag{3.1.4}
\]
valid for all \( t > 0 \). Equation (3.1.4) is in the same spirit as (3.1.3), but with a quite different method of measuring the nearness of functions. It is called stability in “uniform” sense.

### 3.2 Diffusion on the whole line: Cauchy problem

Consider the Cauchy problem
\[
\begin{cases}
u_t - ku_{xx} = f(x, t), & x \in R, \ t > 0 \\
u(x, 0) = \phi(x), & x \in R
\end{cases} \tag{3.2.1}
\]
We shall use the Fourier transform method to find the solution to the above problem. First, let us introduce the Fourier transform.

#### 3.2.1 Fourier Transform

Assume that \( f(x) \) is an absolutely integrable function on \( R \), namely,
\[
\int_{-\infty}^{\infty} |f(x)| < \infty.
\]
The Fourier transform of \( f(x) \) is defined by \( g = F[f] \), where
\[
g(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx, \tag{3.2.2}
\]
3.2. DIFFUSION ON THE WHOLE LINE: CAUCHY PROBLEM

where $i = \sqrt{-1}$.

The Fourier inverse transform of $f(\xi)$ is defined by

$$F^{-1}[f](x) = \lim_{M \to \infty} \frac{1}{2\pi} \int_{-M}^{M} f(\xi) e^{i\xi x} d\xi.$$  \hspace{1cm} (3.2.3)

We may simply write (3.2.3) as

$$F^{-1}[f](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{i\xi x} d\xi,$$

understanding that the integral in the right is defined by the limit in (3.2.3).

**Theorem 3.1** (Convergence Theorem of Fourier Transform) Assume that $f$ is a $C^1$ function and is absolutely integrable on $\mathbb{R}$, and $g(\xi) = F[f](\xi)$ is the Fourier transform of $f(x)$. Then

$$\lim_{M \to \infty} \frac{1}{2\pi} \int_{-M}^{M} g(\xi) e^{i\xi x} d\xi = f(x),$$

Hence

$$F^{-1} [F[f]] = f$$

More general, if $f$ is absolutely integrable on $\mathbb{R}$, and $f$ and $f'$ are piecewise continuous on $\mathbb{R}$, then we have

$$\lim_{M \to \infty} \frac{1}{2\pi} \int_{-M}^{M} g(\xi) e^{i\xi x} d\xi = \frac{f(x - 0) + f(x + 0)}{2}.$$

**Example.** Let

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq a, \\ 0, & \text{if } |x| > a, \end{cases}$$

where $a > 0$. Then

$$F[f](\xi) = \frac{2 \sin (a\xi)}{\xi}.$$  Using the Convergence Theorem we get

$$F^{-1} \left[ \frac{2 \sin (a\xi)}{\xi} \right](x) = \begin{cases} 1, & \text{if } |x| < a, \\ \frac{1}{2}, & \text{if } |x| = a, \\ 0, & \text{if } |x| > a \end{cases}$$

or

$$F^{-1} \left[ \frac{\sin (a\xi)}{\xi} \right](x) = \begin{cases} \frac{1}{2}, & \text{if } |x| < a, \\ \frac{1}{4}, & \text{if } |x| = a, \\ 0, & \text{if } |x| > a \end{cases}$$

Let $a = ct$ for $c, t > 0$, we find

$$F^{-1} \left[ \frac{\sin (ct\xi)}{\xi} \right](x) = \begin{cases} \frac{1}{2}, & \text{if } |x| < ct, \\ \frac{1}{4}, & \text{if } |x| = ct, \\ 0, & \text{if } |x| > ct. \end{cases}$$
Example. \( f(x) = e^{-\alpha x^2} \), where \( \alpha > 0 \).

\[
F[f](\xi) = \int_{-\infty}^{\infty} e^{-i\xi x - \alpha x^2} \, dx = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-y^2 - \frac{i\xi y}{\sqrt{\alpha}}} \, dy
\]
\[
= \frac{e^{-\frac{\xi^2}{4\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-\left(y + \frac{i\xi}{2\sqrt{\alpha}}\right)^2} \, dy = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\xi^2}{4\alpha}}.
\]

Here we have used the formula
\[
\int_{-\infty}^{\infty} e^{-(y+ib)^2} \, dy = \int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi}
\]
for all real number \( b \).

By using the convergence theorem, we have

\[
F^{-1}\left[e^{-\frac{\xi^2}{4\alpha}}\right] = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}.
\]

Let \( \alpha = \frac{1}{4kt} \) for \( k, t > 0 \),

\[
F^{-1}\left[e^{-\frac{k\xi^2}{4t}}\right] = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.
\]

It turns out that the Fourier transform has the following properties.

1. Linearity:
   \[
   F\left[c_1 f_1 + c_2 f_2\right] = c_1 F\left[f_1\right] + c_2 F\left[f_2\right],
   \]
   \[
   F^{-1}\left[c_1 f_1 + c_2 f_2\right] = c_1 F^{-1}\left[f_1\right] + c_2 F^{-1}\left[f_2\right],
   \]
   for any constants \( c_1 \) and \( c_2 \).

2. Differentiation: Assume \( f(x) \) has Fourier transform and \( f(x) \to 0 \) as \( |x| \to \infty \), then
   \[
   F\left[f'(x)\right](\xi) = i\xi F\left[f\right](\xi).
   \]

3. Multiplication: Assume \( f(x) \) and \( x f(x) \) have Fourier transform, then
   \[
   \frac{d}{d\xi} F\left[f\right](\xi) = F\left[-ix f(x)\right](\xi)
   \]
   namely,
   \[
   F\left[x f(x)\right] = i \frac{d}{d\xi} F\left[f\right].
   \]

Corollary:

\[
F\left[\frac{d^m f}{dx^m}\right](\xi) = (i\xi)^m F\left[f\right](\xi), \text{ for } m \geq 1
\]
\[
F\left[x^m f(x)\right](\xi) = i^m \frac{d^m}{d\xi^m} F\left[f\right](\xi).
\]
Applying the inverse Fourier transform, we have
\[
Solving this ODE, we get
\]
We now apply the Fourier transform to derive the solution of problem (3.2.1). To
Proof of (4): We only prove the first equality.
\[
where \( f_1 \ast f_2 \) is the convolution of \( f_1 \) and \( f_2 \) defined by
\[
(f_1 \ast f_2) (x) = \int_{-\infty}^{\infty} f_1(x - y) f_2(y) dy
\]
3.2.2 Poisson Formula
We now apply the Fourier transform to derive the solution of problem (3.2.1). To simplify notation, let \( \hat{u}(\xi, t) \), \( f(\xi, t) \) and \( \phi(\xi, t) \) be the Fourier transform of \( u(x, t) \), \( f(x, t) \) and \( \phi(x) \) in \( x \) respectively. Applying Fourier transform to (3.2.1), we have, for fixed \( \xi \),
\[
\frac{d \hat{u}}{dt} + k\xi^2 \hat{u} = \hat{f}(\xi, t)
\]
\[
\hat{u}(\xi, 0) = \hat{\phi}(\xi).
\]
Solving this ODE, we get
\[
\hat{u}(\xi, t) = e^{-k\xi^2 t} \hat{\phi}(\xi) + \int_{0}^{t} e^{-k\xi^2 (t-s)} \hat{f}(\xi, s) ds.
\]
Applying the inverse Fourier transform, we have
\[
u(x, t) = F^{-1}
\]
\[
= F^{-1} \left[ e^{-k\xi^2 t} \hat{\phi}(\xi) \right] + F^{-1} \left[ \int_{0}^{t} e^{-k\xi^2 (t-s)} \hat{f}(\xi, s) ds \right]
\]
\[
= F^{-1} \left[ e^{-k\xi^2 t} \right] \ast F^{-1} \left[ \hat{\phi}(\xi) \right] + \int_{0}^{t} F^{-1} \left[ e^{-k\xi^2 (t-s)} \hat{f}(\xi, s) \right] ds
\]
\[
= F^{-1} \left[ e^{-k\xi^2 t} \right] \ast \phi + \int_{0}^{t} F^{-1} \left[ e^{-k\xi^2 (t-s)} \right] \ast f(x, s) ds
\]
3.2. DIFFUSION ON THE WHOLE LINE: CAUCHY PROBLEM

(4) Multiplication and convolution:
\[
F [f_1 \ast f_2] = F [f_1] \cdot F [f_2],
\]
\[
F^{-1} [g_1 g_2] = F^{-1} [g_1] \ast F^{-1} [g_2],
\]
and
\[
F^{-1} [g_1 \ast g_2] = 2\pi F^{-1} [g_1] \cdot F^{-1} [g_2],
\]
\[
F [f_1 f_2] = \frac{1}{2\pi} F [f_1] \ast F [f_2],
\]
where \( f_1 \ast f_2 \) is the convolution of \( f_1 \) and \( f_2 \) defined by
\[
(f_1 \ast f_2) (x) = \int_{-\infty}^{\infty} f_1(x - y) f_2(y) dy
\]
CHAPTER 3. THE DIFFUSION EQUATION

Since
\[
F^{-1} \left[ e^{-k\xi^2 t} \right] = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}
\]
and
\[
F^{-1} \left[ e^{-k\xi^2 (t-s)} \right] = \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{x^2}{4k(t-s)}},
\]
we obtain
\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} * \phi + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{x^2}{4k(t-s)}} * f(x, s)ds
\]
\[
= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} f(y, s)dyds.
\]

This is called Poisson formula.

We can also use Fourier transform method to find solutions of other type equations. We can also define Fourier transform in several variables, and use it to find solutions of equations in higher dimensions. It is important to remember that, this method formally gives us solutions (as we do not know in advance whether the solutions satisfy the conditions of convergence theorem of Fourier transform), and we must verify they satisfy the equations and initial/boundary conditions.

Theorem 3.2 If \( f(x, t) \) and \( \phi(x) \) are continuous and bounded functions, then (3.2.1) has a solution given by
\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} f(y, s)dyds.
\]

Example: **Black-Scholes formula for European put options**: The option pricing problem
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad S > 0, \quad t \in [0, T)
\]
\[
V(S, T) = \max(K - S, 0)
\]
has a solution
\[
V(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1),
\]
where
\[
d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln \frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}.
\]
3.2.3 Diffusion on the half-line: Reflection

\[
\begin{aligned}
&\begin{cases}
  u_t - ku_{xx} = 0, \ x > 0, \ t > 0 \\
u(x, 0) = \phi(x), \ x > 0 \\
u(0, t) = 0, \ t > 0
\end{cases}
\end{aligned}
\] (3.2.5)

Physical interpretation: \(u\) is the temperature in a very long rod with one end immersed in a reservoir of temperature zero.

Conclusion: If \(\phi\) is a continuous function and bounded on \([0, \infty)\), then (3.2.5) has a solution given by

\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[ e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y) dy
\]

Proof. We use the idea of reflection to solve (3.2.5). Define the odd extension of \(\phi(x)\) by

\[
\phi_{\text{odd}}(x) = \begin{cases}
  \phi(x) & \text{for } x > 0 \\
  -\phi(-x) & \text{for } x < 0 \\
  0 & \text{for } x = 0
\end{cases}
\]

Consider the following problem

\[
\begin{aligned}
&\begin{cases}
  u_t - ku_{xx} = 0, \ x \in \mathbb{R}, \ t > 0 \\
u(x, 0) = \phi_{\text{odd}}(x), \ x \in \mathbb{R}
\end{cases}
\end{aligned}
\]

whose solution is

\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_{\text{odd}}(y) dy
\]

\[
= \frac{1}{\sqrt{4\pi kt}} \left[ \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_{\text{odd}}(y) dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} \phi_{\text{odd}}(y) dy \right]
= \frac{1}{\sqrt{4\pi kt}} \left[ \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy + \int_{0}^{\infty} e^{-\frac{(x+y)^2}{4kt}} (-\phi(y)) dy \right]
= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[ e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y) dy.
\]

It is not hard to check that \(u(x, t)\) is an odd function and is the solution to (3.2.5).

Exercise: Barrier options:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \ S > B, \ t \in [0, T) \\
V(S, T) = \max(K - S, 0) \\
V(B, t) = 0, \ t \in [0, T)
\]
Exercise: Derive the solution for the Neumann boundary on the half line. That is, solve
\[
\begin{align*}
& \quad \quad u_t - ku_{xx} = 0, \ x > 0, \ t > 0 \\
& u(x, 0) = \phi(x), \ x > 0 \\
& u_x(0, t) = 0, \ t > 0
\end{align*}
\] (3.2.6)
3.3 Comparison of waves and diffusions

The basic property of waves is that information gets transported in both directions at a finite speed. The basic property of diffusions is that the initial disturbance gets spread out in a smooth fashion and gradually disappears. The fundamental properties of these two equations can be summarized in the following table.

<table>
<thead>
<tr>
<th>Property</th>
<th>Waves</th>
<th>Diffusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Speed of propagation</td>
<td>Finite</td>
<td>Infinite</td>
</tr>
<tr>
<td>(ii) Singularities for ( t &gt; 0 )?</td>
<td>Transported along Characteristics</td>
<td>Lost immediately</td>
</tr>
<tr>
<td>(iii) Well-posed for ( t &gt; 0 )?</td>
<td>Yes</td>
<td>Yes (at least for bounded solution)</td>
</tr>
<tr>
<td>(iv) Well-posed for ( t &lt; 0 )?</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>(v) Maximum principle</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>(vi) Behavior as ( t \to +\infty )</td>
<td>Energy is constant so does not decay</td>
<td>Decays to zero</td>
</tr>
</tbody>
</table>

For the wave equation we have seen most of these properties already. That there is no maximum principle is easy to see. Generally speaking, the wave equation just moves information along the characteristic lines.

For the diffusion equation we discuss property (ii), that singularities are immediately lost. The solution is differentiable to all orders even if the initial data are not. Properties (iii), (v), and (vi) have been shown already in section 3.1.

As for property (i) for the diffusion equation, it is not hard to check from the formula that the solution at any point \((x, t)\) depends on the values of initial datum \(\phi(y)\) for all \(y \in \mathbb{R}\). Conversely, the value of \(\phi\) at a point \(x_0\) has an immediate effect everywhere (for \(t > 0\)), even though most of its effect is only for a short time near \(x_0\). Therefore, the speed of propagation is infinite. This is in stark contrast to the wave equation.

As for (iv), there are several ways to see that the diffusion equation is not well posed for \(t < 0\) ("backward in time"). We only introduce one way as follows. Let

\[
  u_n(x, t) = \frac{1}{n} \sin (nx) e^{-n^2kt}.
\]

You can check that this satisfies the diffusion equation for all \(x, t\). Also, \(u_n(x, 0) = n^{-1} \sin (nx) \to 0\) uniformly as \(n \to \infty\). But consider any \(t < 0\), say \(t = -1\). Then

\[
  u_n(x, -1) = n^{-1} \sin (nx) e^{kn^2} \to \pm\infty \text{ uniformly as } n \to \infty \text{ except for a few } x.
\]

Thus \(u_n\) is close to the zero solution at time \(t = 0\) but not at time \(t = -1\). This violates the stability, in the uniform sense at least.
Chapter 4

Boundary Value Problems

In this chapter we finally come to the physically realistic case of a finite interval $0 < x < l$. We will employ the method of separation of variables.

4.1 Fourier Series

Recall that for positive integers $m, n$,

$$\int_{-l}^{l} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{m \pi x}{l} \right) dx = l \delta_{mn},$$
$$\int_{-l}^{l} \cos \left( \frac{n \pi x}{l} \right) \cos \left( \frac{m \pi x}{l} \right) dx = l \delta_{mn},$$
$$\int_{-l}^{l} \sin \left( \frac{n \pi x}{l} \right) \cos \left( \frac{m \pi x}{l} \right) dx = 0,$$
$$\int_{-l}^{l} 1 \cdot \sin \left( \frac{n \pi x}{l} \right) = \int_{-l}^{l} 1 \cdot \cos \left( \frac{n \pi x}{l} \right) dx = 0,$$
$$\int_{0}^{l} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{m \pi x}{l} \right) dx = \frac{l}{2} \delta_{mn},$$
$$\int_{0}^{l} \cos \left( \frac{n \pi x}{l} \right) \cos \left( \frac{m \pi x}{l} \right) dx = \frac{l}{2} \delta_{mn},$$

where

$$\delta_{mn} = \begin{cases} 
1, & \text{if } m = n, \\
0, & \text{if } m \neq n. 
\end{cases}$$

The Fourier series of a piecewise continuous function $\phi(x)$ on $(-l, l)$ is

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n \pi x}{l} \right) + B_n \sin \left( \frac{n \pi x}{l} \right) \right], \quad (4.1.1)$$

where

$$\begin{cases} 
A_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \cos \left( \frac{n \pi x}{l} \right) dx, \\
B_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \sin \left( \frac{n \pi x}{l} \right) dx. 
\end{cases} \quad (4.1.2)$$
If \( \phi(x) \) is odd, then \( A_n = 0 \) for all \( n \geq 0 \) and (4.1.1) becomes
\[
\phi(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{l} \right),
\] (4.1.3)
where, for \( n > 0 \),
\[
B_n = \frac{2}{l} \int_0^l \phi(x) \sin \left( \frac{n\pi x}{l} \right) dx.
\] (4.1.4)
If \( \phi(x) \) is even, then \( B_n = 0 \) for all \( n > 0 \) and (4.1.1) becomes
\[
\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{l} \right),
\] (4.1.5)
where, for \( n \geq 0 \),
\[
A_n = \frac{2}{l} \int_0^l \phi(x) \cos \left( \frac{n\pi x}{l} \right) dx.
\] (4.1.6)

Now, given a function \( \phi(x) \) on the interval \((0, l)\), we can find an expansion by the following methods.

(i) Extend \( \phi \) to an odd function on \((-l, l)\). The Fourier series of the extended function is given by (4.1.3), (4.1.4). We call the expansion given in (4.1.3), where \( B_n \)'s are determined by (4.1.4), the Fourier sine series of \( \phi \) on \((0, l)\).

(ii) Extend \( \phi \) to an even function on \((-l, l)\). The Fourier series of the extended function is given by (4.1.5), (4.1.6). We call the expansion given in (4.1.5), where \( A \)'s are determined by (4.1.6), the Fourier cosine series of \( \phi \) on \((0, l)\).

**Example 4.1** Find the Fourier series of \( \phi(x) = x^3 \) over \((-l, l)\).

**Example 4.2** Find the Fourier sine series for
1. \( \phi(x) = 1 \), over \((0, l)\);
2. \( \phi(x) = x \), over \((0, l)\);
3. \( \phi(x) = x^3 \), over \((0, \pi)\).

**Example 4.3** Find the Fourier cosine series for
1. \( \phi(x) = 1 \), over \((0, l)\);
2. \( \phi(x) = x \), over \((0, 1)\);
3. \( \phi(x) = x^4 \), over \([0, 1]\). Then find the sum of \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \).
4. \( \phi(x) = |\sin x| \), over \((-\pi, \pi)\). Then show that
\[
\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4}.
\]

**Definition 4.1** A series \( \sum_{n=1}^{\infty} f_n(x) \) is said to converge to \( f(x) \) in \([a, b]\)
(i) pointwise if it converges to \( f(x) \) for each \( x \in [a, b] \), i.e. for each \( x \in [a, b] \),
\[
\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \text{ as } N \to \infty;
\]
(ii) uniformly if
\[ \max_{0 \leq x \leq b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \text{ as } N \to \infty; \]
(iii) in the $L^2$ sense if
\[ \int_{a}^{b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 \, dx \to 0 \text{ as } N \to \infty. \]

**Exercise 4.1** Show that if a series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function $f(x)$ on $[a, b]$, then it converges to $f(x)$ pointwise and in the sense of $L^2$ in $(a, b)$.

**Theorem 4.3** Suppose that
\[ \int_{a}^{b} |f(x)|^2 \, dx < \infty. \]
Then its Fourier series converges to $f(x)$ in the sense of $L^2$.

**Theorem 4.4** (i) If $f(x)$ is continuous and $f'(x)$ is piecewise continuous on $[-l, l]$, then its Fourier series converges to $f(x)$ pointwise on $(-l, l)$, and converges to $\frac{1}{2} [f(-l) + f(l)]$ at $x = \pm l$.

(ii) More generally, if both $f(x)$ and $f'(x)$ are piecewise continuous on $[-l, l]$, then its Fourier series pointwise converges to $\frac{1}{2} [f(x^+) + f(x^-)]$ on $(-l, l)$, and converges to $\frac{1}{2} [f(-l) + f(l)]$ at $x = \pm l$.

### 4.2 Review: Eigenvalue Problems of ODE

Consider Sturm-Liouville problem
\[
\begin{align*}
\frac{d}{dx} [p(x)y'] + q(x)y + \lambda r(x)y &= 0, \quad a < x < b, \\
a_1 y(a) + a_2 y'(a) &= 0, \\
b_1 y(b) + b_2 y'(b) &= 0,
\end{align*}
\] (4.2.1)

where $\lambda$ is a parameter and $a_j$’s and $b_j$’s are real numbers. We shall always assume that

\[
\begin{align*}
p(x), \ p'(x), \ q(x) \text{ and } r(x) \text{ are continuous on } [a, b], \\
p(x) > 0 \text{ and } r(x) > 0 \text{ on } [a, b], \ a_1^2 + a_2^2 > 0, \ b_1^2 + b_2^2 > 0.
\end{align*}
\] (4.2.2)

There are special types of boundary conditions which occur frequently in applications:

- **Dirichlet:** $y(a) = c_1, \ y(b) = c_2$.
- **Neumann:** $y'(a) = c_1, \ y'(b) = c_2$.
- **Robin:** $y'(a) - \alpha y(a) = c_1, \ y'(b) + \beta y(b) = c_2$.
- **Periodic:** $y(0) = y(T), \ y'(0) = y'(T)$.

For (4.2.1), $y(x) \equiv 0$ is always a solution. In some case there may exist non-trivial solutions. If (4.2.1) has a non-trivial solution $y_1(x)$, then it has a one-parameter family of solutions $y = cy_1(x)$. 

Definition 4.2 \(\lambda_0\) is called an eigenvalue of (4.2.1) if (4.2.1) has a non-trivial solution when \(\lambda = \lambda_0\). In this case, the non-trivial solutions are called eigenfunctions associated with the eigenvalue \(\lambda_0\).

An eigenvalue \(\lambda_0\) of (4.2.1) is called simple if all the eigenfunctions associated with \(\lambda_0\) are just scalar multiplications of each other.

Example 4.4 The Dirichlet eigenvalue problem

\[ y'' + \lambda y = 0, \quad y(0) = y(L) = 0 \]

has eigenvalues \(\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots\), and the associated eigenfunctions are \(c_n \sin \left(\frac{n\pi x}{L}\right)\), where \(c_n\)'s are arbitrary non-zero constants.

The Neumann eigenvalue problem

\[ y'' + \lambda y = 0, \quad y'(0) = y'(L) = 0 \]

has eigenvalues \(\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots\), and the associated eigenfunctions are \(c_n \cos \left(\frac{n\pi x}{L}\right)\), where \(c_n\)'s are arbitrary non-zero constants.

The periodic eigenvalue problem

\[ y'' + \lambda y = 0, \quad y(0) = y(L), \quad y'(0) = y'(L) \]

has eigenvalues \(\lambda_n = \left(\frac{2n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots\), and the associated eigenfunctions are \(\phi_0(x) = A_0, \quad \phi_n(x) = A_n \cos \left(\frac{2n\pi x}{L}\right) + B_n \sin \left(\frac{2n\pi x}{L}\right), \quad n \geq 1\), where \(A_n\)'s, \(B_n\)'s are arbitrary non-zero constants.

Let \(W(a, b)\) denote the set of all (real-valued) functions \(y(x)\) that have continuous second derivative on \([a, b]\) and satisfy the boundary conditions in (4.2.1). For \(y(x), z(x) \in W(a, b)\) we define the inner product

\[(y, z) = \int_a^b y(x)z(x)dx.\]

Two functions \(y(x)\) and \(z(x)\) are called orthogonal with respective to a weight function \(r(x)\) on \([a, b]\), if

\[\int_a^b r(x)y(x)z(x)dx = 0.\]

If \(r(x) \equiv 1\), then we say \(y(x)\) and \(z(x)\) are orthogonal.

Define a linear operator \(L\) by

\[Ly(x) = \frac{d}{dx} \left[p(x)y'(x)\right] + q(x)y(x).\]

We call \(L\) the Sturm-Liouville operator. The equation in (4.2.1) can be written in the form

\[Ly + \lambda r(x)y = 0.\]
Lemma 4.5 \( L \) is self-adjoint in \( W(a, b) \), namely, for any \( u, v \in W(a, b) \) we have
\[
(u, Lv) = (Lu, v).
\]

Proof. For any \( C^2 \) function \( u, v \) we have
\[
uLv - vLu = \frac{d}{dx} \left[p \left( uv' - vv' \right) \right].
\]
Integrating yields
\[
\int_a^b \{uLv - vLu\} \, dx = p \left( uv' - vv' \right) |^b_a.
\]
Keep in mind that \( a_1^2 + a_2^2 > 0, \ b_1^2 + b_2^2 > 0 \). If \( u \) and \( v \) satisfy the boundary conditions in (4.2.1), then the linear system
\[
u(a)x + u'(a)y = 0
\]
\[
v(a)x + v'(a)y = 0
\]
has a non-trivial solution \( x = a_1, y = a_2 \). Hence \( \int_a^b \{uLv - vLu\} \, dx = 0 \).

Theorem 4.6 Assume the condition (4.2.2).
(1) The eigenvalues of (4.2.1) are real, and have real-valued eigenfunctions.
(2) All the eigenvalues are simple.
(3) Eigenfunctions associated with distinct eigenvalues are orthogonal with respect to the weight function \( r(x) \) on \([a, b]\).

Proof. (1) Assume that \( \lambda \) is an eigenvalue of (4.2.1) and \( \phi(x) \) is an associated eigenfunction. \( \lambda \) may be complex and \( \phi \) may be complex-valued. Multiplying (4.2.1) by \( \overline{\phi} \) and integrating we get
\[
\int_a^b \overline{\phi}(x)(L\phi)(x) \, dx = -\lambda \int_a^b r(x) |\phi(x)|^2 \, dx.
\]
We can verify that \( \overline{\lambda} \) is also an eigenvalue and \( \overline{\phi} \) is an eigenfunction associated with \( \overline{\lambda} \). So we also have
\[
\int_a^b \phi(x)(L\overline{\phi})(x) \, dx = -\overline{\lambda} \int_a^b r(x) |\phi(x)|^2 \, dx.
\]
Sine \( L \) self-adjoint,
\[
\int_a^b \phi(x)(L\overline{\phi})(x) \, dx = \int_a^b \overline{\phi}(x)(L\phi)(x) \, dx.
\]
Thus
\[
\lambda \int_a^b r(x) |\phi(x)|^2 \, dx = \overline{\lambda} \int_a^b r(x) |\phi(x)|^2 \, dx.
\]
Since
\[
\int_a^b r(x) |\phi(x)|^2 \, dx > 0,
\]
we have $\lambda = \bar{\lambda}$. Thus $\lambda$ is real.

Now we show that $\lambda$ has a real-valued eigenfunction. If $\phi(x)$ is real-valued, we are done. Otherwise, we take the real part or the imaginary part of $\phi$ as an eigenfunction (they are not both zero).

(2) Assume that $u(x)$ and $v(x)$ are eigenfunctions associated with an eigenvalue $\lambda$ (thus they are solutions of the same equation). As in the proof of Lemma 4.1 we can show $u$ and $v$ are linearly dependent.

(3) Let $\lambda \neq \mu$ be eigenvalues of (4.2.1), with associated eigenfunctions $u$ and $v$ respectively.

\[
0 = \int_a^b \{ v(Lu) - u(Lv) \} dx = \int_a^b \{ v(x) [-\lambda r(x)u(x)] - u(x) [-\mu r(x)v(x)] \} dx = (\lambda - \mu) \int_a^b r(x)u(x)v(x) dx.
\]

Since $\lambda \neq \mu$ we get
\[
\int_a^b r(x)u(x)v(x) dx = 0.
\]

**Theorem 4.7** Under the condition (4.2.2), the eigenvalue problem (4.2.1) has an increasing sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lim_{n \to \infty} \lambda_n = \infty$.

**Remark 4.9** We can show that, the eigenfunctions associated with $\lambda_n$ have exactly $n - 1$ zeros in the open interval $(a, b)$.

Now we consider
\[
\begin{cases}
y'' + \lambda y = 0, & a < x < b, \\
a_1 y(a) + a_2 y'(a) = 0, \\
b_1 y(b) + b_2 y'(b) = 0,
\end{cases}
\] (4.2.3)
where $a_1 + a_2 > 0$, $b_1 + b_2 > 0$. From Theorem 4.6 and 4.7, (4.2.3) has a sequence of eigenvalues $\{\lambda_n\}$, and for each $n$ we can choose an eigenfunction $\phi_n(x)$ associated with $\lambda_n$, such that
\[
\int_a^b \phi_n(x)\phi_m(x) dx = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}
\] (4.2.4)

We call the sequence $\{\phi_n\}_{n=1}^\infty$ an orthonormal sequence of eigenfunctions. If $f(x)$ is a continuous function on $[a, b]$, we can write a series
\[
f(x) = \sum_{n=1}^\infty c_n \phi_n(x),
\] (4.2.5)
where
\[
c_n = \int_a^b r(x)\phi_n(x)f(x) dx.
\] (4.2.6)
(4.2.5) is called the eigenfunction expansion, or generalized Fourier series of $f$. 
4.3. SEPARATION OF VARIABLES

Theorem 4.8 (i) If
\[ \int_a^b |f(x)|^2 \, dx < \infty, \]
then the eigenfunction expansion (4.2.5) converges to \( f \) in the \( L^2 \) sense on \([a,b]\).

(ii) If \( f \), \( f' \) and \( f'' \) are continuous on \([a,b]\) and if \( f \) satisfies the boundary condition in (4.2.3), then the eigenfunction expansion (4.2.5) converges to \( f \) uniformly on \([a,b]\).

Example 4.5 Consider the eigenvalue problem
\[ \begin{cases} y'' + \lambda y = 0, & 0 < x < \pi, \\ y(0) = 0, \\ y'(\pi) = 0. \end{cases} \] (4.2.7)
The eigenvalues are \( \lambda_n = \frac{(2n-1)^2}{4}, \ n = 1, 2, 3, \ldots \), with the associated eigenfunction \( \phi_n(x) = a_n \sin \left( \frac{2n-1}{2} x \right) \). The eigenfunctions are orthogonal. We compute
\[ \int_0^\pi \pi \sin^2 \left( \frac{2n-1}{2} x \right) \, dt = \frac{\pi}{2}. \]
Choose \( a_n = \sqrt{\frac{2}{\pi}} \). We obtain an orthonormal sequence of eigenfunctions of (4.2.7):
\[ \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin \left( \frac{2n-1}{2} x \right), \ n = 1, 2, 3, \ldots \]
Consider a function
\[ f(x) = \begin{cases} \frac{2x}{\pi} & \text{if } 0 \leq x \leq \frac{\pi}{2}, \\ 1 & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases} \]
We have
\[ c_n = \int_0^\pi \sqrt{\frac{2}{\pi}} \sin \left( \frac{2n-1}{2} x \right) f(x) \, dx = \frac{2^{7/2}}{\pi^{3/2}(2n-1)^2} \sin \left( \frac{n\pi}{2} - \frac{\pi}{4} \right). \]
Thus we get an eigenfunction expansion for \( f(x) \):
\[ f(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{\pi}} \sin \left( \frac{2n-1}{2} x \right) = \sum_{n=1}^{\infty} \frac{2^{7/2}}{\pi^{3/2}(2n-1)^2} \sin \left( \frac{n\pi}{2} - \frac{\pi}{4} \right) \sin \left( \frac{2n-1}{2} x \right), \]
which converges uniformly to \( f(x) \) on \([0, \pi]\).

4.3 Separation of variables

4.3.1 Dirichlet condition

Let us consider the initial-Dirichlet boundary value problem for the wave equation
\[ \begin{cases} u_{tt} = c^2 u_{xx} & \text{for } 0 < x < l, \ t > 0 \\ u(0,t) = 0 = u(l,t) \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x). \end{cases} \] (4.3.1)
At moment let us ignore the initial condition and look for a solution of the wave equation with the boundary condition
\[
\begin{align*}
  u_{tt} &= c^2 u_{xx} \quad \text{for } 0 < x < l, \ t > 0, \\
  u(0, t) &= 0 = u(l, t)
\end{align*}
\]  \hfill (4.3.2)

Due to the nature form of the equation, we would like to consider the solution in the special form:
\[
  u(x, t) = X(x)T(t).
\]

Notice that here we have used the capital letter for dependent variable and lowercase letter for the independent variables. Our aim is to find the particular solution \(X\) and \(T\). It follows from the wave equation that
\[
  X'' + \lambda X = 0,
\]
\[
  T'' + \lambda c^2 T = 0.
\]

From the boundary condition we get \(X(0) = 0, X(l) = 0\). Thus \(X\) satisfies
\[
\begin{align*}
  X'' + \lambda X &= 0, \ 0 < x < l, \\
  X(0) &= X(l) = 0.
\end{align*}
\]  \hfill (4.3.3)

This is an eigenvalue problem (Keep in mind that the trivial solution has no help to us). The eigenvalues and eigenfunctions are
\[
  \lambda_n = \left(\frac{n\pi}{\lambda}\right)^2, \ X_n = \sin\left(\frac{n\pi x}{l}\right), \ n = 1, 2, \cdots.
\]

For each \(n\), we solve
\[
  T'' + \lambda_n c^2 T = 0
\]
to get
\[
  T = T_n = A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right).
\]

Hence for every \(n\), (4.3.2) has a solution
\[
  u_n(x, t) = X_nT_n = \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right).
\]

Any finite sum of \(u_n\)'s is also a solution. The following series, if convergent, will be again a solution to (4.3.2)
\[
  u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right).
\]  \hfill (4.3.4)
Formally we have
\[ \phi(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{l} \right) \tag{4.3.5} \]
and
\[ \psi(x) = u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \left( \frac{n\pi x}{l} \right). \tag{4.3.6} \]

It turns out that \( A_n \)'s and \( B_n \)'s are determined through the Fourier expansions of \( \phi \) and \( \psi \), namely
\[ A_n = \frac{2}{l} \int_0^l \phi(x) \sin \left( \frac{n\pi x}{l} \right) dx, \]
\[ B_n = \frac{2}{an\pi} \int_0^l \psi(x) \sin \left( \frac{n\pi x}{l} \right) dx. \]

**Example 4.6** Solve the diffusion problem
\[
\begin{cases}
  u_t = ku_{xx} \quad \text{for } 0 < x < l, \ t > 0 \\
  u(0, t) = 0 = u(l, t) \\
  u(x, 0) = \phi(x)
\end{cases}
\]

### 4.3.2 The Neumann Condition

Consider the initial-Neumann boundary value problem for the wave equation
\[
\begin{cases}
  u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l, \ t > 0 \\
  u(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x), \ 0 < x < l \\
  u_x(0, t) = 0, \ u_x(l, t) = 0, \ t > 0
\end{cases} \tag{4.3.7}
\]

We can use the same method as before. However, the boundary condition will enforce us to consider the eigenfunctions of the problem:
\[ X'' + \lambda X = 0, \ 0 < x < l, \]
\[ X'(0) = X'(l) = 0. \tag{4.3.8} \]

and
\[ T'' + c^2 \lambda T = 0. \]

Of course, we will exclude the trivial solution. The reasoning as before should show that the eigenvalues \( \lambda \) should be real and non-negative.

The eigenvalues and eigenfunctions of (4.3.8) are
\[ \lambda_0 = 0, \ \lambda_n = \left( \frac{n\pi}{l} \right)^2, \ X_0 = 1, \ X_n = \cos \left( \frac{n\pi x}{l} \right), \ n = 1, 2, \ldots. \]

For each \( n \) we solve \( T'' + c^2 \lambda_n T = 0 \) to get
\[ T_0 = \frac{A_0}{2} + \frac{B_0 t}{2}, \ T_n = A_n \cos \left( \frac{n\pi c t}{l} \right) + B_n \sin \left( \frac{n\pi c t}{l} \right), \ n = 1, 2, \ldots. \]
For later convenience we include the fact $1/2$ in $T_0$. Thus we can conclude that the solution for wave equation with Neumann conditions is given by
\[ u(x, t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}) \cos \frac{n\pi x}{l}. \] (4.3.9)

Applying the initial data and Fourier series, we get
\[ \phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \] (4.3.10)
and
\[ \psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos \frac{n\pi x}{l}, \] (4.3.11)
where $A_n$’s and $B_n$’s determined through the Fourier expansions of $\phi$ and $\psi$.

**Example 4.7** Do the same for the diffusion equation.

**Example 4.8** Solve the initial-mixed boundary value problem of diffusion equation
\[
\begin{cases}
  u_t = ku_{xx} & \text{for } 0 < x < l, \ t > 0 \\
  u(x, 0) = \phi(x), & 0 < x < l \\
  u(0, t) = 0, \ u_x(l, t) = 0, & t > 0.
\end{cases}
\] (4.3.12)

4.3.3 The Robin Condition

A general form of homogeneous Robin condition on the boundary $\partial \Omega$ is
\[ \frac{\partial u}{\partial n} + a(x, t)u = 0, \]
where $n$ is the unit outer-normal vector of $\partial \Omega$. When $\Omega = (0, l)$ and $a = a(x)$, it takes the form
\[ -u_x + a_0 u|_{x=0} = 0, \ \text{and} \ -u_x + a_l u|_{x=l} = 0. \]

Consider the initial-Robin boundary value problem for the wave equation
\[
\begin{cases}
  u_{tt} = c^2 u_{xx} & \text{for } 0 < x < l, \ t > 0 \\
  u(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x), & 0 < x < l \\
  u_x(0, t) - a_0 u(0, t) = 0, \ u_x(l, t) - a_l u(l, t) = 0, & t > 0.
\end{cases}
\] (4.3.12)

For simplicity we only consider the case where the constants $a_0$ and $a_l$ satisfy the condition
\[ a_0 > 0, \ a_l > 0. \] (4.3.13)

Using the method of separation of variables, we are led to the eigenvalue problem
\[
\begin{align*}
  X'' + \lambda X &= 0, \ 0 < x < l \\
  X'(0) - a_0 X(0) &= 0, \ X'(l) + a_l X(l) &= 0
\end{align*}
\] (4.3.14)
and

\[ T'' + c^2 \lambda T = 0. \]

Under the condition (4.3.13), we can show that (4.3.14) has no zero or negative eigenvalues. Here we only show \( \lambda = 0 \) is not an eigenvalue. Note that, when \( \lambda = 0 \), the general solution of the first equation of (4.3.14) is

\[ X = A + Bx. \]

Using the boundary condition,

\[ a_0 A - B = 0, \quad a_l A + (1 + l a_l) B = 0. \]

Therefore, \( \lambda = 0 \) is an eigenvalue of (4.3.14) iff

\[ a_0 + a_l + l a_0 a_l = 0. \]

This equality cannot hold under the condition (4.3.13).

When \( \lambda > 0 \), the general solution of the first equation of (4.3.14) is

\[ X = A \cos (\sqrt{\lambda} x) + B \sin (\sqrt{\lambda} x). \]

Using the boundary condition, we have

\[ a_0 A - \sqrt{\lambda} B = 0, \]
\[ p A + q B = 0, \]

where

\[ p = -\sqrt{\lambda} \sin (\sqrt{\lambda} l) + a_l \cos (\sqrt{\lambda} l), \]
\[ q = \sqrt{\lambda} \cos (\sqrt{\lambda} l) + a_l \sin (\sqrt{\lambda} l). \]

Since \( A, B \) are not all zero, we have

\[ 0 = \begin{vmatrix} a_0 & -\sqrt{\lambda} \\ p & q \end{vmatrix} = a_q q + \sqrt{\lambda} p. \]

So we have

\[ \tan (\sqrt{\lambda} l) = \frac{(a_0 + a_l) \sqrt{\lambda}}{\lambda - a_0 a_l}. \quad (4.3.15) \]

We can show that (4.3.15) has a sequence of positive solutions \( \lambda_n, n = 1, 2, \ldots \),

\[ \left( \frac{n \pi}{l} \right)^2 < \lambda_n < \left( \frac{(n+1) \pi}{l} \right)^2, \]

and

\[ \lim_{n \to \infty} \left( \lambda_n - \left( \frac{n \pi}{l} \right)^2 \right) = 0. \]
Thus (4.3.14) has a sequence of positive eigenvalues \( \{ \lambda_n \} \), and the eigenfunction associated with \( \lambda_n \) is

\[
X_n = \sqrt{\lambda_n} \cos \left( \sqrt{\lambda_n} x \right) + a_0 \sin \left( \sqrt{\lambda_n} x \right).
\]

For each \( n \) we solve \( T'' + c^2 \lambda_n T = 0 \) to find

\[
T_n = A_n \cos \left( \sqrt{\lambda_n} ct \right) + B_n \sin \left( \sqrt{\lambda_n} ct \right).
\]

Then (4.3.12) has a solution

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \sqrt{\lambda_n} ct \right) + B_n \sin \left( \sqrt{\lambda_n} ct \right) \right] \left[ \sqrt{\lambda_n} \cos \left( \sqrt{\lambda_n} x \right) + a_0 \sin \left( \sqrt{\lambda_n} x \right) \right],
\]

where \( A_n \)'s and \( B_n \)'s are determined through the Fourier expansions of \( \phi \) and \( \psi \):

\[
\phi(x) = \sum_{n=1}^{\infty} A_n \left[ \sqrt{\lambda_n} \cos \left( \sqrt{\lambda_n} x \right) + a_0 \sin \left( \sqrt{\lambda_n} x \right) \right]
\]

\[
\psi(x) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n} c \left[ \sqrt{\lambda_n} \cos \left( \sqrt{\lambda_n} x \right) + a_0 \sin \left( \sqrt{\lambda_n} x \right) \right].
\]

### 4.3.4 Inhomogeneous equation and inhomogeneous boundary conditions

We have used the method of separation of variables to solve homogeneous PDEs with homogeneous boundary conditions. The following problem has inhomogeneous boundary conditions

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = f(x, t) & \text{for } 0 < x < l, \ t > 0 \\
  u(x, 0) = \phi(x), & 0 < x < l \\
  u_t(x, 0) = \psi(x), & 0 < x < l \\
  u(0, t) = h(t), u(l, t) = g(t), & t > 0.
\end{cases}
\]

**Method (I):** We can shift the inhomogeneous boundary condition to the source and initial condition by transform. Set

\[
w(x, t) = (1 - \frac{x}{l})h(t) + \frac{x}{l}g(t).
\]

Then \( w(0, t) = h(t), w(l, t) = g(t) \). Set

\[
v(x, t) = u(x, t) - w(x, t),
\]

then \( v \) satisfies an initial-boundary value problem of wave equation with homogeneous boundary conditions:

\[
\begin{cases}
  v_{tt} - c^2 v_{xx} = f(x, t) - \left(1 - \frac{x}{l}\right) h''(t) - \frac{2}{l} g''(t), & \text{for } 0 < x < l, \ t > 0 \\
  v(x, 0) = \phi(x) - \left(1 - \frac{x}{l}\right) h(0) - \frac{2}{l} g(0), & 0 < x < l \\
  v_t(x, 0) = \psi(x) - \left(1 - \frac{x}{l}\right) h'(0) - \frac{2}{l} g'(0), & 0 < x < l \\
  v(0, t) = 0, \ v(l, t) = 0, & t > 0.
\end{cases}
\]
4.3. **SEPARATION OF VARIABLES**

So, essentially we need to solve the inhomogeneous equation with homogeneous boundary conditions.

**Method (II):** In the following we employ another method to solve problem (4.3.17). Recall that the homogeneous wave equation with homogeneous Dirichlet boundary conditions has a solution

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \left( \frac{n\pi x}{l} \right), \quad (4.3.18) \]

where

\[ u_n(t) = \frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx, \quad n \geq 1. \]

Now we look for a solution of (4.3.17) in the form of series (4.3.18). We avoid differentiating the series term by term. Instead, we expand

\[
\begin{align*}
u_{tt}(x, t) &= \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{l}, \quad A_n(t) = \frac{2}{l} \int_0^l u_{tt}(x, t) \sin \frac{n\pi x}{l} dx, \\
u_{xx}(x, t) &= \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{l}, \quad B_n(t) = \frac{2}{l} \int_0^l u_{xx}(x, t) \sin \frac{n\pi x}{l} dx, \\
u_t(x, t) &= \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi x}{l}, \quad C_n(t) = \frac{2}{l} \int_0^l u_t(x, t) \sin \frac{n\pi x}{l} dx, \\
f(x, t) &= \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}, \quad f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx.
\end{align*}
\]

Using the boundary conditions, we compute

\[
\begin{align*}
A_n(t) &= \frac{d^2}{dt^2} \frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx = u_n''(t), \\
C_n(t) &= u_n'(t), \\
B_n(t) &= \frac{2}{l} u_x(x, t) \sin \frac{n\pi x}{l} \bigg|_0^l - \frac{2n\pi}{l^2} \int_0^l u_x(x, t) \cos \frac{n\pi x}{l} dx \\
&= -\frac{2n\pi}{l^2} u(x, t) \cos \frac{n\pi x}{l} \bigg|_0^l - \frac{2n^2\pi^2}{l^3} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx \\
&= -\frac{2n\pi}{l^2} \left[ (-1)^n u(t, t) - u(0, t) \right] - \frac{n^2\pi^2}{l^2} u_n(t) \\
&= -\frac{2n\pi}{l^2} \left[ (-1)^n g(t) - h(t) \right] - \frac{n^2\pi^2}{l^2} u_n(t).
\end{align*}
\]

From the equation we have

\[
\begin{align*}
0 &= u_{tt} - c^2 u_{xx} - f(x, t) \\
&= \sum_{n=1}^{\infty} \left[ A_n(t) - c^2 B_n(t) - f_n \right] \sin \frac{n\pi x}{l}.
\end{align*}
\]
So
\[
0 = A_n(t) - c^2 B_n(t) - f_n(t)
= u_n'' + c^2 \left\{ \frac{2n\pi}{l^2} \left[ \left(-1\right)^n g(t) - h(t) \right] + \frac{n^2\pi^2}{l^2} u_n(t) \right\} - f_n.
\]

From the initial conditions,
\[
\phi(x) = u(x, 0) = \sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi x}{l} \Rightarrow u_n(0) = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} \, dx \equiv \phi_n,
\]
\[
\psi(x) = u_t(x, 0) = \sum_{n=1}^{\infty} C_n(0) \sin \frac{n\pi x}{l} \Rightarrow u_n'(0) = C_n(0) = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} \, dx \equiv \psi_n.
\]

Set
\[
F_n(t) = f_n(t) + \frac{2n\pi c^2}{l^2} \left[ h(t) - \left(-1\right)^n g(t) \right].
\]

\(u_n\) satisfies
\[
u_n''(t) + \frac{n^2\pi^2 c^2}{l^2} u_n(t) = F_n(t), \ t > 0, \quad u_n(0) = \phi_n, \ u_n'(0) = \psi_n.
\]

We solve the ODE to get
\[
u_n(t) = \phi_n \cos \frac{n\pi ct}{l} + \frac{l}{n\pi c} \sin \frac{n\pi ct}{l} + \frac{l}{n\pi c} \int_0^t F_n(s) \sin \left( \frac{n\pi c}{l} (t - s) \right) \, ds
= \phi_n \cos \frac{n\pi ct}{l} + \frac{l}{n\pi c} \sin \frac{n\pi ct}{l} + \frac{l}{n\pi c} \int_0^t \left\{ f_n(s) + \frac{2n\pi c^2}{l^2} \left[ h(s) - \left(-1\right)^n g(s) \right] \right\} \sin \left( \frac{n\pi c}{l} (t - s) \right) \, ds.
\]

Example 4.9
\[
\begin{cases}
  u_t - c^2 u_{xx} = f(x, t), \quad 0 < x < l, \ t > 0 \\
  u(x, 0) = \phi(x), \ 0 < x < l \\
  u(0, t) = h(t), \ u(l, t) = g(t), \ t > 0.
\end{cases}
\]
Chapter 5

Elliptic Equations

In this chapter we discuss the stationary equations like

\[-\Delta u = 0, \text{ Laplace equation}\]
\[-\Delta u = f, \text{ Poisson’s equation}\]
\[-\Delta u + \sum b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f.\]

where \(x = (x_1, x_2, ..., x_n) \in \Omega \subset \mathbb{R}^n; \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\). Let \(\partial \Omega\) denote the boundary and \(\bar{\Omega} = \Omega \cup \partial \Omega\). We are going to consider these equations with one of the following boundary conditions: (1) Dirichlet conditions: \(u = h\) on \(\partial \Omega\); (2) Neumann condition: \(\frac{\partial u}{\partial n} = h\) on \(\partial \Omega\); (3) Robin condition: \(\frac{\partial u}{\partial n} + \alpha u = h\) on \(\partial \Omega\).

5.1 Maximum Principle

We take into account a more general equation

\[Lu \equiv -\Delta u + c(x)u = f(x), \ x \in \Omega \tag{5.1.1}\]

Here we always assume

\[c(x) \geq 0 \text{ for } x \in \Omega. \tag{5.1.2}\]

It turns out that the condition plays a critical role in the maximum principle of elliptic equations.

Lemma 5.1 Let \(\Omega\) be a bounded domain. Assume \(u \in C^2(\Omega) \cap C(\bar{\Omega})\) satisfies \(Lu = f < 0\) for \(x \in \Omega\). Then \(u\) cannot take its non-negative maximum value inside \(\Omega\).

Proof. Suppose not, that is, \(u\) attains its maximum at \(\tilde{x}\). Then

\[\left. \frac{\partial^2 u}{\partial x_i^2} \right|_{x=\tilde{x}} \leq 0, \text{ for } i = 1, 2, ..., n.\]

Then

\[Lu|_{x=\tilde{x}} \geq 0.\]

A contradiction!
Theorem 5.2 Let $\Omega$ be a bounded domain. Assume $c(x) \geq 0$ is bounded in $\Omega$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu = f \leq 0$ for $x \in \Omega$. Then
\[
\max_{x \in \Omega} u(x) = \max_{x \in \partial \Omega} u^+(x).
\]
Here $u^+(x) = \max\{u(x), 0\}$.

Proof. For any $\varepsilon > 0$ given, let
\[
w(x) = u(x) + \varepsilon e^{\alpha x_1},
\]
where $\alpha$ is to be chosen. Then
\[
Lw = Lu + \varepsilon e^{\alpha x_1} (-a^2 + c(x)).
\]
Since $c(x)$ is bounded in $\Omega$, we can choose a sufficiently large $\alpha$ such that
\[-a^2 + c(x) < 0.\]
With the $\alpha$, we have
\[Lw < 0.\]
Due to Lemma 5.1,
\[
\max_{x \in \Omega} w(x) = \max_{x \in \partial \Omega} w^+(x).
\]
So,
\[
\max_{x \in \Omega} u(x) \leq \max_{x \in \Omega} w(x) = \max_{x \in \partial \Omega} w^+(x) \\
\leq \max_{x \in \partial \Omega} u(x) + \varepsilon \max_{x \in \partial \Omega} e^{\alpha x_1}.
\]
We then obtain the desired result by taking $\varepsilon \to 0$.

Remark 5.1 If $c(x) \equiv 0$, then “non-negative maximum” in the above lemma and theorem can be replaced by “maximum”.

Remark 5.2 If $Lu = f \geq 0$, then $\min_{x \in \Omega} u(x) = \min_{x \in \partial \Omega} u^-(x)$, where $u^-(x) = \min\{u(x), 0\}$.

Corollary 5.3 (Comparison principle) Assume the same conditions as in Theorem 5.2. If $Lu_1 \leq Lu_2$ and $u_1|_{\partial \Omega} \leq u_2|_{\partial \Omega}$, then
\[u_1 \leq u_2 \text{ in } \Omega.\]

Remark 5.3 We can deduce the uniqueness of solution directly from comparison. Recall that we have comparison principle also for parabolic equation, where condition (5.1.2) can be relaxed.

Remark 5.4 Concerning why condition (5.1.2) cannot be relaxed in elliptic equations, we refer to eigenvalue problems:
\[-\Delta u + \lambda u = 0.\]
5.2 Prior Estimate

5.2.1 Estimate in maximum norm

Consider the Poisson equation with boundary value condition:

\[-\Delta u = f, \ x \in \Omega \]
\[u|_{\partial \Omega} = \phi(x).\]  

**Theorem 5.4** Let \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) be the solution to problem (5.2.3). Then

\[\max_{\Omega} |u(x,t)| \leq \Phi + CF,\]

where \( \Phi = \max_{\partial \Omega} |\phi(x)|, \ F = \sup |f(x)|, \ C \) is a constant depending only on the bounded domain \( \Omega. \)

**Remark 5.5** From the above estimate, we immediately obtain the uniqueness and stability of problem (5.2.3).

Proof of Theorem 5.4: Since the domain is bounded, we can assume

\[|x|^2 \leq d^2, \text{ for all } x \in \Omega.\]

Construct a function

\[z(x) = \Phi + F \frac{d^2 - |x|^2}{2n},.\]

It is easy to check that

\[-\Delta z = F\]
\[z|_{\partial \Omega} \geq \Phi\]

By comparison principle, we have

\[z \geq \pm u, \text{ in } \Omega.\]

So

\[\max_{\Omega} |u(x)| \leq \max_{\Omega} z(x) \leq \Phi + \frac{d^2}{2n} F,\]

which is desired.

5.2.2 Estimate in energy norm

Let us consider the following Neumann problem

\[-\Delta u + c(x)u = f(x), \ x \in \partial \Omega\]
\[\frac{\partial u}{\partial n}|_{\partial \Omega} = 0.\]
Theorem 5.5 Suppose
\[ c(x) \geq c_0 > 0 \] (5.2.4)
and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be the solution to the above problem. Then we have
\[ \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{c_0}{2} \int_{\Omega} |u(x)|^2 \, dx \leq M \int_{\Omega} |f(x)|^2 \, dx, \]
where \( M \) depends only on \( c_0 \).

Proof: We multiply both sides of the equation by \( u \) and integrate to get
\[ -\int_{\Omega} u \Delta u \, dx + \int_{\Omega} c(x) u^2 \, dx = \int_{\Omega} f u \, dx. \]
Applying Green formula to the first term of the left hand side, and Cauchy inequality to the right hand side, we have
\[ \int_{\Omega} |\nabla u(x)|^2 \, dx + c_0 \int_{\Omega} |u(x)|^2 \, dx \leq c_0 \int_{\Omega} u^2 \, dx + \frac{1}{2c_0} \int_{\Omega} f^2 \, dx, \]
which yields (5.2.5) immediately.

Remark 5.6 For Dirichlet condition or Robin condition, we have similar conclusions.

Remark 5.7 To illustrate the importance of the condition (5.2.4), we consider the case of \( c(x) \equiv 0 \).

Remark 5.8 All results in this section can be generalized to the heat equation.

5.3 Laplace equation in cubes

Laplace equation on rectangles or cubes with the usual boundary conditions can be solved using the method of separation of variables. Consider the Laplace equation in a cube
\[ \Omega = \{(x, y, z) : 0 < x < \pi, \ 0 < y < \pi, \ 0 < z < \pi\} \]
with boundary conditions on the six sides:
\[ \begin{cases} 
\Delta u = 0 \text{ in } \Omega \\
 u|_{x=0} = 0, \ u|_{x=\pi} = g(y, z) \\
 u|_{y=0} = 0, \ u|_{y=\pi} = 0 \\
 u|_{z=0} = 0, \ u|_{z=\pi} = 0 
\end{cases} \]

We first look for a solution in the form
\[ u = X(x)Y(y)Z(z). \]
Then
\[
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0
\]
\[X(0) = Y(0) = Y(\pi) = Z(0) = Z(\pi).
\]
So \(\frac{X''}{X}, \frac{Y''}{Y}\) and \(\frac{Z''}{Z}\) are constants, denoted by \(\gamma, -\lambda\) and \(-\mu\), respectively. Then \(\gamma = \lambda + \mu\),

\[
X'' - (\lambda + \mu) X = 0, \quad X(0) = 0,
\]
\[
Y'' + \lambda Y = 0, \quad Y(0) = Y(\pi) = 0,
\]
\[
Z'' + \lambda Z = 0, \quad Z(0) = Z(\pi) = 0.
\]

We find
\[
\lambda = m^2, \quad Y(y) = \sin(my), \quad m = 1, 2, ...
\]
\[
\mu = n^2, \quad Z(z) = \sin(nz), \quad n = 1, 2, ...
\]
\[
X(x) = A \sinh(\sqrt{m^2 + n^2}x).
\]

Then we sum them up to get
\[
u = \sum_{m,n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2}x) \sin(my) \sin(nz).
\]

When \(x = \pi\),
\[
g(y, z) = u(\pi, y, z) = \sum_{m,n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2}\pi) \sin(my) \sin(nz).
\]

Multiplying the equality by \(\sin(iy)\sin(jz)\) and integrating, we get
\[
A_{ij} = \frac{4}{\pi^2 \sinh(\sqrt{i^2 + j^2} \pi)} \int_0^\pi \int_0^\pi g(y, z) \sin(iy) \sin(jz) dy dz.
\]

### 5.4 Poisson’s Formula and Mean Value Property (in 2-dimension)

#### 5.4.1 Poisson’s formula

Consider Laplace equation in a disc \(\Omega = \{(x, y) : x^2 + y^2 < a^2\}\),
\[
\Delta u = 0 \quad \text{in} \; \Omega
\]
\[
u = h(\theta) \quad \text{on} \; \partial \Omega,
\]
where \(\theta\) is the polar angle.

First we compute the Laplace operator in polar coordinates \((r, \theta)\), where
\[
x = r \cos \theta,
\]
\[
y = r \sin \theta.
\]
It is not hard to check
\[ \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \]

Then the problem becomes
\[ u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < a, \quad 0 \leq \theta < 2\pi \]
\[ u(a, \theta) = h(\theta), \quad 0 \leq \theta < 2\pi \]
with auxiliary conditions: (1) \( u \) is periodic in \( \theta \) of \( 2\pi \); (2) \( u \) is bounded as \( r \to 0 \).

**Theorem 5.6 (Poisson formula)** Let \( h(\theta) \) be a continuous function. The solution \( u(r, \theta) \) of the above problem is given by
\[ u = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\alpha)}{a^2 - 2ar \cos(\theta - \alpha) + r^2} d\alpha. \]
\( u \) is continuous on \( \overline{\Omega} \), and is infinitely differentiable inside \( \Omega \).

**Proof:** We first look for a solution of separated variable
\[ u = R(r)\Theta(\theta), \quad \Theta(\theta + 2\pi) = \Theta(\theta). \]

Then
\[ R''\Theta + \frac{R'}{r}\Theta + \frac{R}{r^2}\Theta'' = 0. \]

Thus
\[ \Theta'' + \lambda \Theta = 0, \]
\[ r^2R'' + rR' - \lambda R = 0. \]

We solve these equations to find
\[ \lambda_0 = 0, \quad \Theta_0 = \frac{A_0}{2}, \quad R_0 = C_0 + D_0 \log r, \]
\[ \lambda_n = n^2, \quad \Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \]
\[ R_n = C_nr^n + D_nr^{-n}, \quad n = 1, 2, \ldots \]

We choose \( D_n = 0 \) for \( n \geq 0 \) as \( u \) is bounded when \( r \to 0 \). Take the sum
\[ u = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)). \]

Using the boundary condition,
\[ h(\theta) = u|_{r=a} = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos(n\theta) + B_n \sin(n\theta)). \]
So
\[ A_n = \frac{1}{\pi a^n} \int_{0}^{2\pi} h(\alpha) \cos(n\alpha) \, d\alpha, \quad B_n = \frac{1}{\pi a^n} \int_{0}^{2\pi} h(\alpha) \sin(n\alpha) \, d\alpha. \]

Then
\[ u = \frac{1}{2\pi} \int_{0}^{2\pi} h(\alpha) \, d\alpha + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_{0}^{2\pi} h(\alpha) \left[ \cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta) \right] \, d\alpha \]
\[ = \frac{1}{2\pi} \int_{0}^{2\pi} h(\alpha) \left[ 1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \cos(n(\theta - \alpha)) \right] \, d\alpha. \]

We compute
\[
1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \cos(n(\theta - \alpha)) = 2 \text{Re} \left\{ \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n e^{in(\theta - \alpha)} \right\} - 1 = 2 \text{Re} \left\{ \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n e^{i(\theta - \alpha)} \right\} - 1 = 2 \text{Re} \left\{ \frac{r}{a} e^{i(\theta - \alpha)} \left[ \frac{a}{a - r \cos(\theta - \alpha) - ir \sin(\theta - \alpha)} \right] - 1 \right\} - 1 = 2 \text{Re} \left\{ \frac{a(r - r \cos(\theta - \alpha) + ir \sin(\theta - \alpha))}{(a - r \cos(\theta - \alpha))^2 + r^2 \sin(\theta - \alpha)} \right\} - 1 \right\} - 1 = \frac{2a}{a^2 - r^2} \left[ a - r \cos(\theta - \alpha) \right] - 1 \right\} - 1 = a^2 - 2ar \cos(\theta - \alpha) + r^2. \]

This is desired.

Let us come back to the orthogonal coordinates. Let \((x, y) = (r \cos \theta, r \sin \theta), (x', y') = (a \cos \alpha, a \sin \alpha)\). Then
\[
(x, y) \cdot (x', y') = ra \cos \theta \cos \alpha + \sin \theta \sin \alpha = ra \cos(\theta - \alpha),
\]
\[
|(x, y) - (x', y')|^2 = |(x, y)|^2 + |(x', y')|^2 - 2(x, y) \cdot (x', y') = r^2 + a^2 - 2ar \cos(\theta - \alpha)
\]

So Poisson’s formula takes the alternative form:
\[
u(x, y) = \frac{a^2 - |(x, y)|^2}{2\pi a} \int_{x'^2 + y'^2 = a} \frac{u(x', y')}{|(x, y) - (x', y')|^2} \, dx \, dy.
\]

5.4.2 Mean value property and strong maximum principle (in 2-dimension)

The following theorem is virtually a corollary of the Poisson formula.

**Theorem 5.7** Let \(u\) be a harmonic function in a planar domain \(\Omega\) and is continuous on \(\overline{\Omega}\). Then the value of \(u\) at any point \((x_0, y_0) \in \Omega\) is equal to the average of \(u\) on
the circumference of any disc in the interior of $\Omega$ with center at $(x_0, y_0)$, namely, if $B_a((x_0, y_0)) \subset \Omega$, where $B_a((x_0, y_0))$ is the disc with radius $a$ and center $(x_0, y_0)$, then

$$u(x_0, y_0) = \frac{1}{2\pi a} \int \frac{u(x', y')}{(x' - x_0)^2 + (y' - y_0)^2 = a^2} ds.$$  

Using the mean value property, we can show the strong maximum principle of Laplace equation.

**Theorem 5.8** (2-dimension case) Let $\Omega$ be a bounded domain. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\Delta u = f \leq 0$ for $x \in \Omega$. Then $u$ cannot take its non-negative maximum value inside $\Omega$ unless $u$ is a constant.

Proof: Let $M = \max_{x \in \Omega} u(x)$ and $D = \{x \in \Omega : u(x) = M\}$. We show $D$ is an open subset of $\Omega$. Write the disc of radius $a$ centered at $x$ by $B_a(x)$. For any $x_0 \in D$ given, we have $u(x_0) = M$. Choose $a > 0$ such that $B_a(x_0) \subset \Omega$. Using the average value property we obtain for $0 < r \leq a$,

$$M = u(x_0) = \frac{1}{2\pi r} \int_{|x-x_0|=r} u(x) ds.$$  

So

$$\int_{|x-x_0|=r} M - u(x) ds = 0.$$  

Since $M \geq u(x)$ for all $x \in \Omega$, we deduce $u(x) \equiv M$ on the circle $\{|x-x_0|=r\}$. Then $B_a(x_0) \subset D$. So $D$ is open. It is clear that $D$ is a closed subset of $\Omega$. Because $\Omega$ is connected, the only open and closed subsets are $\Omega$ and empty set $\emptyset$. Therefore either $u(x) \not\equiv M$ for all $x \in \Omega$, or $u(x) \equiv M$ on $\Omega$. 