Optimal Stock Selling/Buying Strategy with reference to the Ultimate Average*

Min Dai† Yifei Zhong
Dept of Math, National University of Singapore, Singapore

First version: July 2008
This version: Dec 2009

Abstract

We are concerned with the optimal decision to sell or buy a stock in a given period with reference to the ultimate average of the stock price. More precisely, we aim to determine an optimal selling (buying) time to maximize (minimize) the expectation of the ratio of the selling (buying) price to the ultimate average price over the period. This is an optimal stopping time problem which can be formulated as a variational inequality problem. The problem gives rise to a free boundary that corresponds to the optimal selling (buying) strategy. We provide a partial differential equation approach to characterize the free boundary (or equivalently, the optimal selling (buying) region). It turns out that the optimal selling strategy is bang-bang, which is the same as that obtained by Shiryaev, Xu and Zhou (2008b) taking the ultimate maximum of the stock price as benchmark, whereas the optimal buying strategy can be a feedback one subject to the type of averaging and parameter values. Moreover, by a thorough characterization of free boundary, we reveal that the bang-bang optimal selling strategy heavily depends on the assumption that no time-vesting restrictions are imposed. If a time-vested stock is considered, then the optimal selling strategy can also be a feedback one. In terms of a similar analysis developed by the present paper, the same phenomenon can be proved when taking the ultimate maximum as benchmark.

Key words: Optimal selling/buying strategy, ultimate average, time-vesting

1 Introduction

Assume that the discounted stock price evolves according to

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where constants $\mu \in (-\infty, +\infty)$ and $\sigma > 0$ are the discounted expected rate of return and volatility, respectively, and $\{B_t; t > 0\}$ is a standard 1-dimension Brownian motion on a filtered probability space $(\mathbb{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $B_0 = 0$ almost surely. We are interested

---

*Dai is also an affiliated member of Risk Management Institute (RMI), National University of Singapore (NUS). Zhong is now at Mathematical Institute, University of Oxford. This project is supported by Singapore Tier 1 AcR grant (No. R-146-000-096-112) and NUS RMI research grant (No. R-146-000-117-720/646). The authors thank seminar participants at Oxford University and Tongji University for helpful discussion and comments. All errors are our own.

†Email: matdm@nus.edu.sg. Fax: (65) 67795452.
in the following optimal decision to sell or buy a stock in a given period \([0, T]\) with reference to the ultimate average:

\[
\text{Buy case: } \min_{0 \leq \nu \leq T} \mathbb{E} \left( \frac{S_{\nu}}{A_T} \right),
\]

\[\tag{1.2}\]

\[
\text{Sell case: } \max_{0 \leq \nu \leq T} \mathbb{E} \left( \frac{S_{\nu}}{A_T} \right),
\]

\[\tag{1.3}\]

where \(\mathbb{E}\) stands for the expectation, \(\nu\) is a stopping time, and the benchmark value \(A_T\) is taken as either geometric or arithmetic average price over the period \([0, T]\), namely,

\[
A_T = \begin{cases} 
\exp \left( \frac{1}{T} \int_0^T \log S_{\nu} d\nu \right), & \text{geometric average}, \\
\frac{1}{T} \int_0^T S_{\nu} d\nu, & \text{arithmetic average}.
\end{cases}
\]

\[\tag{1.4}\]

Problem (1.2) and (1.3) are motivated by Shiryaev, Xu and Zhou (2008b) that addressed the optimal stock selling strategy with reference to the ultimate maximum, that is, the benchmark \(A_T\) is taken as \(\max_{0 \leq \nu \leq T} S_{\nu}\). They derived a surprising optimal selling strategy (henceforth called “bang-bang strategy”): one either sells the stock immediately or holds it until expiry. More precisely, if \(\mu > \sigma^2/2\), it is optimal to hold the stock until expiry; if \(\mu \leq \sigma^2/2\), it is optimal to sell the stock immediately at time zero. Hence, \(\mu/\sigma^2\) can be defined as the “goodness index” of a stock.

We are concerned with the robustness of the above bang-bang selling strategy. A natural question is whether we still have such a simple strategy if we use alternative benchmarks, say, the average price\(^2\). Taking the average price as benchmark is not only less aggressive than taking the maximum price, but is also realistic. For example, some shares are often granted to a newly hired executive as a long-term equity-based compensation. It is reasonable for the executive to consider the average price as benchmark. Another example is about a strategic investor who purchases a large stake of a company, usually at a discounted price, before or in the runup to its planned IPO (initial public offering). After the IPO, the strategic investor can profit a lot through selling his stake even with reference to the average price in the secondary market.\(^3\)

To analyze optimal strategy, Shiryaev, Xu and Zhou (2008b) adopted a stochastic analysis approach of the underlying optimal stopping problem related to Graversen, Peskir and Shiryaev (2001) which is the first paper to predict the maximum of a Brownian motion. Other papers along this direction include Pedersen (2003) and Du Toit and Peskir (2007, 2009). In the present paper, the arithmetic average involved makes the problems intractable. We will instead make use of a partial differential equation (PDE) approach.

The PDE formulation, described by a variational inequality, seemingly resembles the pricing model of American-style Asian options which has been extensively studied by numerous researchers (e.g., Geman and Yor (1993), Roger and Shi (1995), Wu, Kwok and

---

\(^1\)A rigorous proof for the case \(\mu > \sigma^2/2\) was provided by Shiryaev, Xu and Zhou (2008b), and the proof for the case \(\mu \leq \sigma^2/2\) was later provided by Dai et al. (2008) using the PDE approach adopted in the present paper, and by Du Toit and Peskir (2009) using a probabilistic approach, respectively. Majumdar and Bouchaud (2008) considered a similar problem in which the optimal time is deterministic (see also Shiryaev, Xu and Zhou (2008a) for comments). In addition, Shiryaev, Xu and Zhou (2008) also consider power utility, while we will restrict attention to linear utility.

\(^2\)The arithmetic average price during a period can be attained through continuous trading, which, however, may incur significant amount of fixed transaction costs. Moreover, continuous trading is prohibited if the stock is time-vested.

\(^3\)It is also true for institutional investors who gain IPOs that are usually underpriced.
Yu (1999), Vecer (2001), Ben-Ameur, Breton and Ecuyer (2002), Wu and Fu (2003), Halluin, Forsyth and Labahn (2005), Dai and Kwok (2006), and reference therein). However, the present problem involves a different obstacle function, and theoretical analysis of the resulting optimal strategy is distinct from the previous ones.

To gain some insights about optimal strategy in the average case, we will first consider the geometric averaging in which case analytical solutions are available so that optimal strategy can be readily figured out. It turns out that the optimal buying strategy depends on the parameter values and time to expiry: one should buy the stock immediately if $\mu \geq \sigma^2$, and never buy the stock before expiry if $\mu \leq 0$; when $0 < \mu < \sigma^2$, one should not buy the stock until time to expiry $\frac{\mu}{\sigma} T$. In contrast, the optimal selling strategy is still bang-bang, the same as that obtained by Shiryaev, Xu and Zhou (2008b).

Our focus will be on the case of arithmetic average, for the arithmetic average price is well received by technical analysis investors and is more realistic than the geometric average counterpart. We will show that similar strategies can be obtained. However, the problem is challenging due to lack of analytical solutions. We will resort to qualitative analysis of the resulting free boundary. It is worthwhile pointing out that the buy case is relatively easy because the corresponding free boundary proves to be monotone. Unfortunately, it is difficult to show the monotonicity of free boundary w.r.t. time in the sell case. To attack the sell case, the key point is to show that the free boundary is monotone with the discounted rate of return $\mu$, where a transformation (i.e. (4.1)) plays a crucial role.

Interestingly, a thorough characterization of free boundary shows that the bang-bang selling strategy heavily depends on the assumption that no time-vesting restrictions are imposed. However, time-vesting restrictions are not rare. For instance, the shares offered to a newly hired executive in an employee compensation plan are often restricted. One of the most common restrictions requires a certain length of time to pass (known as the vesting period). Strategic investors are also prohibited to sell their stakes within a given time vesting period. Our analysis reveals that given an initial time-vesting period, the optimal selling strategy can be a feedback one. Numerical results will be given as well.

The rest of the paper is organized as follows. In the subsequent section, we formulate problem (1.2) and (1.3) as variational inequality problems (also called obstacle problems, see Friedman (1982)). Section 3 and Section 4 are devoted to the geometric average case and to the arithmetic average case, respectively. Numerical results are presented in section 5. We conclude the paper in section 6.

2 PDE formulation

In this section, we will provide a PDE formulation for the optimal stopping problems (1.2) and (1.3). Let us begin with the buy case.

---

4Numerical results show that the free boundary seems to be monotone even in the sell case. See Remark 4.8.

5A typical example is that, in the runup to the planned initial public offering of Industrial and Commercial Bank of China (ICBC), on 28 April 2006, Goldman Sachs purchased a 5.75% stake for US$2.6 billion with three-year vesting period.

6By feedback strategy, we mean the strategy that depends on the state variables and/or time.
2.1 Buy case

As in (1.4), we denote by $A_t$ the running average over $[0, t]$. Then, we can write the value function associated with problem (1.2) as

$$\varphi(S_t, A_t, t) \equiv \min_{t \leq \nu \leq T} E_t \left( \frac{S_\nu}{A_T} \right) = \min_{t \leq \nu \leq T} E_t \left[ S_\nu \mathbb{E}_\nu \left( \frac{1}{A_T} \right) \right],$$

(2.1)

where $E_t(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_t)$.

Denote $\phi(S_t, A_t, t) \equiv E_t \left( \frac{1}{A_T} \right)$. Since

$$dA_t = \begin{cases} \frac{A_t}{T} \log S_t dt, & \text{geometric case} \\ \frac{S_t - A_t}{T} dt, & \text{arithmetic case} \end{cases},$$

it is easy to verify that $\phi$ satisfies

$$\begin{cases} L_0 \phi = 0, & 0 < S, A < \infty, \ t \in (0, T), \\ \phi(S, A, T) = \frac{1}{A}, \end{cases}$$

(2.2)

where $L_0 = -\frac{\partial}{\partial t} - \frac{1}{2} \sigma^2 S^2 \partial_{SS} - \mu S \partial_S - f(S, A, t) \partial_A$.

It follows from (2.1) that

$$\varphi(S_t, A_t, t) = \min_{t \leq \nu \leq T} E_t \left[ S_\nu \phi(S_\nu, A_\nu, \nu) \right],$$

(2.3)

which is the unique viscosity solution to the following HJB equation (or variational inequality equation)\footnote{2.2 resembles the well-known pricing model of a European-style Asian option, whereas (2.4) resembles that of an American-style Asian option. See, for example, Barles, Daher and Romano (1995), Wilmott, Dewynne and Howison (1995) or Jiang and Dai (2004).}

$$\begin{cases} \max \{ L_0 \phi, \varphi - S \phi \} = 0, & 0 < S, A < \infty, \ t \in (0, T), \\ \varphi(S, A, T) = \frac{S}{A} \end{cases}$$

(2.4)

We need to study (2.4) so as to examine the optimal strategies which depend on the stock price dynamics and the running average as well as time. Thanks to the GBM assumption (1.1), we can make use of the following transformation

$$z = \frac{A}{S}, \ \tau = T - t, \ V(z, \tau) = \varphi(S, A, t) \text{ and } \Phi(z, \tau) = S\phi(S, A, t)$$

(2.5)

and reduce (2.4) to

$$\begin{cases} \max \{ \mathcal{L}_1 V, V - \Phi \} = 0, & \text{in } D, \\ V(z, 0) = \frac{1}{z} \end{cases}$$

(2.6)

where $D = (0, \infty) \times (0, T)$,

$$\mathcal{L}_1 = \partial_\tau - \frac{1}{2} \sigma^2 z^2 \partial_{zz} - (\sigma^2 - \mu) z \partial_z - \overline{f}(z, \tau) \partial_z,$$

(2.7)

$$\overline{f}(z, \tau) = \begin{cases} -\frac{\tau}{T - \tau} \log z, & \text{geometric average}, \\ \frac{\tau}{T - \tau} (1 - z), & \text{arithmetic average}, \end{cases}$$
and $\Phi(z, \tau)$ satisfies
\[
\begin{aligned}
& L_1 \Phi = \sigma^2 z \Phi_z + (\sigma^2 - \mu) \Phi, \text{ in } D, \\
& \Phi(z, 0) = \frac{1}{z}.
\end{aligned}
\] (2.8)

In physics, problem (2.6) is known as an upper obstacle problem and $\Phi$ is the upper obstacle. From (2.6), we can see that the optimal strategies only depend on the ratio of the running average to the stock price as well as time.

### 2.2 Sell case

In a similar way, we introduce the value function associated with problem (1.3) as follows:
\[
\psi(S, A, t) = \max_{0 \leq \nu \leq T} E_t [S_\nu A_T] = \max_{0 \leq \nu \leq T} E_t [S_\nu \phi(S_\nu, A_\nu, \nu)]
\]

satisfying
\[
\begin{aligned}
& \min \{ L_0 \psi, \psi - S \phi \} = 0, \quad 0 < S, A < \infty, \ t \in (0, T), \\
& \psi(S, A, T) = \frac{S}{A},
\end{aligned}
\] (2.9)

where $L_0$ and $\phi$ are the same as in the buy case. In terms of the same transformation as in (2.5), we can deduce that $U(z, \tau) = \psi(S, A, t)$ satisfies the following lower obstacle problem:
\[
\begin{aligned}
& \min \{ L_1 U, U - \Phi \} = 0, \text{ in } D, \\
& U(z, 0) = \frac{1}{z},
\end{aligned}
\] (2.10)

where $L_1$ and $\Phi$ are as given in (2.5) and (2.7).

### 3 Geometric average case

In this section, we consider the geometric average case in which analytical solutions to the variational inequality problem (2.6) and (2.10) are available. Then we can explicitly work out the optimal strategies.

Let us first present the analytical expression of the obstacle function $\Phi$.

**Lemma 3.1** Let $\Phi$ be the obstacle function in the geometric average case, i.e., the solution to problem (2.8) with $f(z, \tau) = -\frac{z^2}{T} \log z$. Then
\[
\Phi(z, \tau) = z^{\frac{\tau^3}{3T^3}} \exp(g(\tau)),
\]

where $g(\tau) = \frac{\sigma^2 \tau^3}{6T^2} - (\mu - \frac{\sigma^2}{2}) \frac{\tau^2}{T}$.

**Proof.** Substituting into (2.8), it is easy to verify the result\(^8\).

---

\(^8\)The result can also be obtained by computing expectation. This comment applies to other results in this section.
3.1 Optimal buying strategy

Proposition 3.2 Let $V$ be the solution to problem (2.6) in the geometric average case. Then

$$V(z, \tau) = \begin{cases} 
\Phi(z, \tau) \exp(-\frac{\sigma^2}{2T} \tau^2 + \mu \tau), & \text{if } \mu \leq 0, \\
\Phi(z, \tau), & \text{if } 0 < \mu < \sigma^2, 0 \leq \tau \leq \frac{\mu}{\sigma^2} T, \\
\Phi(z, \tau) \exp(-\frac{\sigma^2}{2T} (\tau - \frac{\mu}{\sigma^2} T)^2), & \text{if } 0 < \mu < \sigma^2, \frac{\mu}{\sigma^2} T < \tau < T, \\
\Phi(z, \tau), & \text{if } \mu \geq \sigma^2,
\end{cases}$$

(3.1)

for any $(z, \tau) \in D$.

Proof. We postulate that $V(z, \tau)$ takes the form of $\Phi(z, \tau) \exp(b(\tau))$, namely,

$$V(z, \tau) = z^{\frac{T-\tau}{T}} \exp(g(\tau) + b(\tau)).$$

Substituting into (2.6), we get

$$\begin{cases} 
\max \left\{ b'(\tau) + \frac{\sigma^2}{T} \tau - \mu, b(\tau) \right\} = 0, \tau \in (0, T), \\
b(0) = 0,
\end{cases}$$

which has a unique solution

$$b(\tau) = \begin{cases} 
-\frac{\sigma^2}{2T} \tau^2 + \mu \tau, & \text{if } \mu \leq 0, \\
0, & \text{if } 0 < \mu < \sigma^2 \text{ and } 0 \leq \tau \leq \frac{\mu}{\sigma^2} T, \\
-\frac{\sigma^2}{2T} (\tau - \frac{\mu}{\sigma^2} T)^2, & \text{if } 0 < \mu < \sigma^2 \text{ and } \frac{\mu}{\sigma^2} T < \tau \leq T, \\
0, & \text{if } \mu \geq \sigma^2.
\end{cases}$$

This completes the proof.

Now let us define the buying region. It is worth pointing out that $\tau = T$ (i.e. $t = 0$) should be considered. Since $z = 1$ at $\tau = T$, we introduce $\hat{D} = D \cup \{(z, \tau) = (1, T)\}$. Then, the buying region can be defined as follows:

$$BR = \{(z, \tau) \in \hat{D} : V(z, \tau) = \Phi(z, \tau)\}.$$

(3.2)

Noting that (3.1) is also true for $(z, \tau) = (1, T)$, we immediately have the following corollary.

Corollary 3.3 Let $BR$ be the buying region as defined in (3.2).

i) If $\mu \leq 0$, then $BR = \emptyset$;

ii) If $0 < \mu < \sigma^2$, then $BR = \{(z, \tau) \in \hat{D} : 0 < \tau \leq \frac{\mu}{\sigma^2} T\}$;

iii) If $\mu \geq \sigma^2$, then $BR = \hat{D}$.

The corollary indicates that, given the geometric average as benchmark, one should never buy the stock before expiry if $\mu \leq 0$, and one should buy the stock immediately if $\mu \geq \sigma^2$. When $0 < \mu < \sigma^2$, one should not buy the stock until $\tau = \frac{\mu}{\sigma^2} T$. 

3.2 Optimal selling strategy

In a similar way, we can find the analytical solution of variational inequality (2.10) in the geometric average case.

Proposition 3.4 Let $U$ be the solution to problem (2.10) in the geometric average case. Then

$$
U(z, \tau) = \begin{cases} 
\Phi(z, \tau), & \text{if } \mu \leq 0, \\
\Phi(z, \tau), & \text{if } 0 < \mu \leq \frac{\sigma^2}{2T} \text{ and } \frac{2\mu}{\sigma^2} T \leq \tau \leq T, \\
\Phi(z, \tau) \exp(-\frac{\sigma^2}{2T} \tau^2 + \mu \tau), & \text{if } 0 < \mu \leq \frac{\sigma^2}{2} \text{ and } 0 < \tau < \frac{2\mu}{\sigma^2} T, \\
\Phi(z, \tau) \exp(-\frac{\sigma^2}{2T} \tau^2 + \mu \tau), & \text{if } \mu > \frac{\sigma^2}{2}.
\end{cases}
$$

(3.3)

for any $(z, \tau) \in \hat{D}$.

The proof resembles that of Proposition 3.2 and is omitted.

Similarly, we can define the corresponding selling region as follows:

$$SR = \{ (z, \tau) \in \hat{D} : U(z, \tau) = \Phi(z, \tau) \}.$$  

(3.4)

By Proposition 3.4, we obtain the following corollary.

Corollary 3.5 Let $SR$ be the selling region as defined in (3.4).

i) If $\mu \leq 0$, then $SR = \hat{D}$;

ii) If $0 < \mu \leq \frac{\sigma^2}{2}$, then $SR = \{ (z, \tau) \in \hat{D} : \frac{2\mu}{\sigma^2} T \leq \tau \leq T \}$;

iii) If $\mu > \frac{\sigma^2}{2}$, then $SR = \emptyset$.

Part ii) implies that we are initially in $SR$ for $0 < \mu \leq \frac{\sigma^2}{2}$, which, combined with part i) and iii), yields a bang-bang optimal selling strategy. That is, one should never sell the stock before expiry if $\mu > \frac{\sigma^2}{2}$, and one should immediately sell the stock at time 0 if $\mu \leq \frac{\sigma^2}{2}$.

We have seen that in the geometric average case, the optimal strategies are independent of the state variables $S$ and $A$. This fact heavily depends on the GBM assumption of stock price.

4 Arithmetic average case

Unlike the geometric average case, analytical solutions are no longer available in the arithmetic average case. We will make use of a PDE approach to investigate the optimal strategy.

First, let us make the following transformation which plays a critical role in our analysis:\footnote{Without the help of this transformation, it seems hard (at least for us) to prove the existence of the optimal selling boundary as a single function of time and its monotonicity in $\mu$.}

$$
x = \log((T - \tau)z), \quad F(x, \tau) = \log\left(\frac{\Phi(z, \tau)}{T}\right),
$$

$$
\nabla(x, \tau) = \log\left(\frac{V(z, \tau)}{\Phi(z, \tau)}\right), \quad \text{and } \nabla(x, \tau) = \log\left(\frac{U(z, \tau)}{\Phi(z, \tau)}\right).
$$

(4.1)
Then, (2.8), (2.6) and (2.10) reduce to
\[ \begin{cases} 
F_\tau - \frac{\sigma^2}{2}(F_{xx} + F_{x}^2) + (\mu - \frac{3}{2}\sigma^2 - e^{-x})F_x + (\mu - \sigma^2) = 0, \text{ in } \Omega, \\
F(x,0) = -x, 
\end{cases} \]  
(4.2)

\[ \begin{cases} 
\max \{ \mathcal{L} V + \sigma^2 F_x - (\mu - \sigma^2), V \} = 0, \text{ in } \Omega, \\
V(x,0) = 0, 
\end{cases} \]  
(4.3)

and
\[ \begin{cases} 
\min \{ \mathcal{L} U + \sigma^2 F_x - (\mu - \sigma^2), U \} = 0, \text{ in } \Omega, \\
U(x,0) = 0, 
\end{cases} \]  
(4.4)

respectively, where \( \Omega = (-\infty, \infty) \times (0, T) \) and

\[ \mathcal{L} = \partial_\tau - \frac{\sigma^2}{2} \left[ \partial_{xx} + (\partial_x)^2 + 2F_x \partial_x \right] + (\mu - \frac{\sigma^2}{2} - e^{-x}) \partial_x. \]

Correspondingly, we can define the selling and buying regions. Note that \( x = -\infty \) when \( \tau = T \). It is convenient to introduce \( \hat{\Omega} = \Omega \cup \{ x = -\infty \} \) and define

\[ BR_x = \{(x, \tau) \in \hat{\Omega} : V(x, \tau) = 0 \} \]  
(4.5)

and

\[ SR_x = \{(x, \tau) \in \hat{\Omega} : U(x, \tau) = 0 \}. \]  
(4.6)

4.1 Two lemmas

Let us introduce two lemmas which are useful for both buy and sell cases.

**Lemma 4.1** Let \( V(x, \tau) \) and \( U(x, \tau) \) be the solutions to problem (4.3) and (4.4), respectively. Then,

\[ V(x, \tau) < U(x, \tau) \text{ in } \Omega. \]

**Proof.** It is easy to see that \( \mathcal{L} V \leq \mathcal{L} U(x, \tau) \) in \( \Omega \). Applying the strong maximum principle gives the result.

**Lemma 4.2** Suppose \( F(x, \tau) \) is the solution to (4.2), then \( F(x, \tau) \) has the following properties.

i) \( -1 < F_x(x, \tau) < 0, \forall (x, \tau) \in \Omega; \)

ii) \( F_{xx}(x, \tau) < 0, \forall (x, \tau) \in \Omega; \)

iii) \( F_{xx}(x, \tau) \geq 0, \forall (x, \tau) \in \Omega; \)

iv) \( F_x(x, \tau; \mu + \delta, \sigma) < F_x(x, \tau; \mu, \sigma) + \frac{\delta}{\sigma^2}, \forall \delta > 0, (x, \tau) \in \Omega; \)

v) \( \lim_{x \to -\infty} F_x(x, \tau) = 0 \text{ and } \lim_{x \to \infty} F_x(x, \tau) = -1, \forall \tau \in (0, T]. \)

The proof of Lemma 4.2 is placed in Appendix A.
4.2 Optimal buying strategy

To begin with, we present the properties of $\bar{V}(x, \tau)$.

**Proposition 4.3** The variational inequality problem (4.3) has a unique solution $\bar{V}(x, \tau) \in W^{2,1}_p(\Omega_N), 1 < p < +\infty$, where $\Omega_N$ is any bounded set in $\Omega$. Moreover,

$$0 \leq \bar{V}_x \leq 1 \text{ and } \bar{V}_\tau \leq 0 \text{ in } \Omega.$$  

(4.7)

Proof. Using the penalized approach [cf. Friedman (1982)], it is not hard to show that (4.3) has a unique solution $\bar{V}(x, \tau) \in W^{2,1}_p(\Omega_N), 1 < p < +\infty$, where $\Omega_N$ is any bounded set in $\Omega$. To show $0 \leq \bar{V}_x \leq 1$, we only need to confine ourselves to the non-coincidence set $\Lambda = \{(x, \tau) \in \Omega : \bar{V} > 0\}$. Denote $w = \bar{V}_x$ and $v = \bar{V}_x - 1$, then $w$ and $v$ satisfy

$$\begin{cases}
  w_x - \frac{\sigma^2}{2}(w_{xx} + 2w w_x + 2F_{xx}w + 2F_xw_x) + (\mu - \frac{\sigma^2}{2} - e^{-x})w_x + e^{-x}w = -\sigma^2 F_{xx}, \text{ in } \Lambda \\
  w|_{\partial \Lambda} = 0,
\end{cases}$$

and

$$\begin{cases}
  v_x - \frac{\sigma^2}{2}(v_{xx} + 2v v_x + 2F_{xx}v + 2F_xv_x) + (\mu - \frac{3}{2}\sigma^2 - e^{-x})v_x + e^{-x}v = -e^{-x}, \text{ in } \Lambda \\
  v|_{\partial \Lambda} = -1,
\end{cases}$$

respectively. Since $-\sigma^2 F_{xx} > 0$ and $-e^{-x} \leq 0$, one can deduce $w \geq 0$ and $v \leq 0$ in $\Lambda$ by the maximum principle, which is desired.

At last, let us prove $\bar{V}_\tau \leq 0$. Denote $\tilde{V}(x, \tau) = \bar{V}(x, \tau + \delta)$. It suffices to show $Q(x, \tau) = \tilde{V}(x, \tau) - \bar{V}(x, \tau) \leq 0$ in $\Omega$, $\forall \delta > 0$. Suppose not, then

$$\Delta = \{(x, \tau) \in \Omega : Q(x, \tau) > 0\} \neq \emptyset.$$ 

It is easy to verify that $Q(x, \tau)$ satisfies

$$\begin{cases}
  Q_x - \frac{\sigma^2}{2} (Q_{xx} + (\tilde{V}_x + \bar{V}_x)Q_x + 2F_{xx}Q_x) + (\mu - \frac{\sigma^2}{2} - e^{-x})Q_x \\
  \leq \delta \sigma^2 F_{x\tau}(\cdot, \cdot)(\tilde{V}_x - 1) \leq 0, \text{ in } \Delta,
\end{cases}$$

where we have used part iii) of Lemma 4.2 and $\tilde{V}_x \leq 1$. Applying the maximum principle, we get $Q \leq 0$ in $\Delta$, which conflicts with the definition of $\Delta$. The proof is complete.

In terms of Lemma 4.2 and Proposition 4.3, we will show the following theorem.

**Theorem 4.4** Let $BR_x$ be the buying region as defined in (4.5).

i) If $\mu \leq 0$, then $BR_x = \emptyset$;

ii) If $0 < \mu < \sigma^2$, there is a monotonically increasing boundary $x_b^*(\tau) : (0, T) \to (-\infty, \infty)$ such that

$$BR_x = \{(x, \tau) \in \Omega : x \geq x_b^*(\tau), 0 < \tau < T\}.$$  

(4.8)

Moreover,

$$\lim_{\tau \to 0^+} x_b^*(\tau) = -\infty;$$  

(4.9)

iii) If $\mu \geq \sigma^2$, then $BR_x = \tilde{\Omega}$.
Proof. According to part i) in Lemma 4.2 and (4.3),
\[ \mathcal{L}V \leq -\sigma^2(F_x + 1) + \mu < 0, \] for \( \mu \leq 0. \)

Applying the strong maximum principle, we infer \( V < 0 \) in \( \Omega \) for \( \mu \leq 0. \) Due to (4.7), we infer \( \nabla V(-\infty, T) < 0. \) Part i) then follows.

If \( \mu \geq \sigma^2, \) part i) in Lemma 4.2 leads to
\[ \sigma^2 F_x - (\mu - \sigma^2) \leq 0. \]

So, \( V = 0 \) is a solution to (4.3), which implies part iii).

It remains to show part ii). Since \( V_x \geq 0, \) we can define a free boundary
\[ x_0^*(\tau) = \inf\{x \in (-\infty, \infty) : V(x, \tau) = 0\}, \] for any \( \tau \in (0, T). \)

Due to \( V_x \leq 0, \) we infer that \( x_0^*(\tau) \) is monotonically increasing with \( \tau. \) Let us prove \( x_0^*(\tau) > -\infty \) for all \( \tau. \) Suppose not, then there exists a \( \tau_0 > 0, \) such that \( x_0^*(\tau) = -\infty \) for all \( \tau \in [0, \tau_0]. \)

This leads to \( V(x, \tau) = 0, \) in \( (-\infty, \infty) \times [0, \tau_0]. \) By (4.3), we have \( \mathcal{L}V + \sigma^2 F_x - (\mu - \sigma^2) \leq 0 \) in \( (-\infty, \infty) \times (0, \tau_0], \)

So,
\[ \lim_{x \to -\infty} F_x(x, \tau) \leq \frac{\mu}{\sigma^2} - 1 < 0, \forall 0 < \tau \leq \tau_0, \]
which is in contradiction with part v) in Lemma 4.2. Further, it can be shown that \( x_0^*(\tau) < \infty \) in terms of the standard argument of Brezis and Friedman (1976) [cf. also the proof of Lemma 4.2 in Dai, Kwok and Wu (2004)], where \( \lim_{x \to -\infty} F_x(x, \tau) = -1 \) will be used. In addition, due to the monotonicity of \( x_0^*(\tau), \) we deduce that \( \{x = -\infty\} \notin BR_x. \) So, (4.8) follows.

At last, we need to prove (4.9). Assume contrary, i.e., \( \lim_{\tau \to 0^+} x_0^*(\tau) = x_0 > -\infty. \)

Then, we have
\[ \mathcal{L}V + \sigma^2 F_x - (\mu - \sigma^2) = 0, \forall x < x_0, 0 < \tau < T, \]
which, combined with \( V(x, 0) = 0 \) for all \( x, \) gives
\[ V(\tau|_{\tau=0} = -\sigma^2 F_x|_{\tau=0} + (\mu - \sigma^2) = \mu > 0, \forall x < x_0. \]

This conflicts with \( V_x \leq 0. \) The proof is complete.

4.3 Optimal selling strategy

Now let us look at the sell case. Using a similar argument as in the buy case, we can deal with the scenario of \( \mu \leq 0 \) and \( \mu \geq \sigma^2. \) However, the scenario of \( 0 < \mu < \sigma^2 \) is much challenging because we no longer have the monotonicity of \( U \) w.r.t. \( \tau. \) To overcome the difficulty, we introduce an auxiliary problem:

\[ \begin{cases} \mathcal{L}U + \sigma^2 F_x - (\mu - \sigma^2) = 0, \text{ in } \Omega, \\ U(x, 0) = 0. \end{cases} \] (4.10)
Lemma 4.5 Let $\bar{U}^*(x, \tau)$ be the solution to (4.10). Then for any $\tau \in (0, T]$,

$$
\begin{align*}
\lim_{x \to -\infty} \bar{U}^*(x, \tau) &> 0 \text{ if } \mu > \frac{\sigma^2}{2}, \\
\lim_{x \to -\infty} \bar{U}^*(x, \tau) &= 0 \text{ if } \mu = \frac{\sigma^2}{2}, \\
\lim_{x \to -\infty} \bar{U}^*(x, \tau) &< 0 \text{ if } \mu < \frac{\sigma^2}{2}.
\end{align*}
$$

We place the proof in Appendix B.

Proposition 4.6 The variational inequality problem (4.3) has a unique solution $\bar{U}(x, \tau) \in W^{2,1}_p(\Omega_N)$, $1 < p < +\infty$, where $\Omega_N$ is any bounded set in $\Omega$. Moreover, for any $(x, \tau) \in \Omega$,

i) $0 \leq \bar{U}_x \leq 1$;

ii) $\bar{U}(x, \tau; \mu) \leq \bar{U}(x, \tau; \mu + \delta)$ for $\delta > 0$;

iii) $\bar{U}(x, \tau) = \bar{U}^*(x, \tau) > 0$ for $\mu \geq \frac{\sigma^2}{2}$. And, for any $\tau \in (0, T]$,

$$
\begin{align*}
\lim_{x \to -\infty} \bar{U}(x, \tau) &> 0, \text{ if } \mu > \frac{\sigma^2}{2}, \\
\lim_{x \to -\infty} \bar{U}(x, \tau) &= 0, \text{ if } \mu = \frac{\sigma^2}{2}, \\
\lim_{x \to -\infty} \bar{U}(x, \tau) &< 0, \text{ if } \mu < \frac{\sigma^2}{2}.
\end{align*}
$$

Proof: The proof of part i) is the same as that of Proposition 4.3. Now let us prove part ii). Suppose not, then

$\mathcal{O} = \{(x, \tau) \in \Omega : H(x, \tau) < 0\} \neq \emptyset$,

where $H(x, \tau) = \bar{U}(x, \tau; \mu + \delta) - \bar{U}(x, \tau)$. Denote $F^\delta_x(x, \tau) = F_x(x, \tau; \mu + \delta)$. It can be verified that

$$
\begin{align*}
H_x - \frac{\sigma^2}{2}(H_{xx} + H^2_x + 2\bar{U}_x H_x + 2F^\delta_x H_x) + (\mu + \delta - \frac{\sigma^2}{2} - e^{-x})H_x \\
\geq -\sigma^2 F^\delta_x - \sigma^2 F_x - \delta(1 - \bar{U}_x)
\end{align*}
$$

in $\mathcal{O}$, $\mathcal{O}$.\]

By part iv) in Lemma 4.2, $F^\delta_x < F_x + \frac{\delta}{\sigma^2}$, which, combines with $\bar{U}_x \leq 1$, gives

$$
-(\sigma^2 F^\delta_x - \sigma^2 F_x - \delta)(1 - \bar{U}_x) \geq 0.
$$

Again applying the maximum principle, we get $H \geq 0$, in $\mathcal{O}$, a contradiction with the definition of $\mathcal{O}$.

To show part iii), it is easy to see $\bar{U}^*_x(x, \tau) > 0$ in $\Omega$ by virtue of $F_{xx} < 0$ and the strong maximum principle. Combining with Lemma 4.5, we infer $\bar{U}^*(x, \tau) > 0$ in $\Omega$ when $\mu \geq \frac{\sigma^2}{2}$. So, $\bar{U}^*(x, \tau)$ must be the solution to (4.4), which yields $\bar{U}(x, \tau) = \bar{U}^*(x, \tau)$ for $\mu \geq \frac{\sigma^2}{2}$. Then (4.11) and (4.12) follow. To show (4.13), apparently we have $\lim_{x \to -\infty} \bar{U}(x, \tau) \geq 0$. Thanks to part ii) and (4.12), we infer $\lim_{x \to -\infty} \bar{U}(x, \tau) \leq 0$ for $\mu < \frac{\sigma^2}{2}$, which leads to (4.13). This completes the proof.

We then study the optimal selling region.
Theorem 4.7 Let $SR_x$ be the optimal selling region as defined in (4.6).

i) If $\mu > \frac{\sigma^2}{2}$, then $SR_x = \emptyset$.

ii) If $\mu = \frac{\sigma^2}{2}$, then $SR_x = \{x = -\infty\}$.

iii) If $0 < \mu < \frac{\sigma^2}{2}$, then $\{x = -\infty\} \subset SR_x$. Moreover, there is a free boundary $x^*_s(\tau) : (0, T] \rightarrow (-\infty, +\infty) \cup \{-\infty\}$ such that

$$SR_x = \{(x, \tau) \in \hat{\Omega} : x \leq x^*_s(\tau)\}.$$ 

iv) If $\mu \leq 0$, then $SR_x = \hat{\Omega}$.

Proof: Part i) and ii) follow by part iii) of Proposition 4.6. The proof of part iv) is similar to that of part i) in Theorem 4.4. Now let us prove part iii). Thanks to (4.13), we immediately get $\{x = -\infty\} \subset SR_x$. Combining with $\bar{U}_x \geq 0$, we can define

$$x^*_s(\tau) = \sup\{x \in (-\infty, +\infty) : \bar{U}(x, \tau) = 0\}, \text{ for any } \tau \in (0, T].$$

We only need to show that $x^*_s(\tau) < \infty$. Let $x^*_s(\tau)$ be the free boundary as given in part ii) of Theorem 4.4. Due to Lemma 4.1, we infer $x^*_s(\tau) \leq x^*_b(\tau)$, which, combined with $x^*_b(\tau) < \infty$, yields the desired result. The proof is complete.

Remark 4.8 Numerical results show that $x^*_s(\tau)$ is always monotonically increasing, which implies that the selling region shrinks as time to maturity decreases. Also, it is verified by numerical results that $x^*_s(\tau) > -\infty$ when $0 < \mu < \frac{\sigma^2}{2}$, which indicates that the selling region is non-empty even excluding $\{x = -\infty\}$. But currently we cannot prove these results.

Since the averaging period starts from time 0, we have $x = -\infty$ at time 0 (i.e. $\tau = T$). By Theorem 4.7, we have $\{x = -\infty\} \subset SR_x$ if $\mu \leq \frac{\sigma^2}{2}$, and $SR_x = \emptyset$ if $\mu > \frac{\sigma^2}{2}$. We then obtain the bang-bang selling strategy as follows.

Corollary 4.9 It is optimal to sell the stock immediately at time 0 if $\mu \leq \frac{\sigma^2}{2}$, and to hold the stock until expiry $T$ if $\mu > \frac{\sigma^2}{2}$.

Let us revisit part iii) of Theorem 4.7. When $0 < \mu < \frac{\sigma^2}{2}$, a time-varying no-trading region does exist, but the initial position is in the selling region. That is why we have the bang-bang selling strategy. However, it is not the case if there is an initial time-vesting period. Indeed, assume that selling stock is not permitted for $t \in [0, T_0)$, and our goal is

$$\max_{T_0 \leq \nu \leq T} \mathbb{E} \left( \frac{S_\nu}{A_T} \right).$$

Then we can obtain the same PDE models and almost the same results as above, but need to restrict attention to $t \in [T_0, T)$ or $\tau \in (0, T - T_0]$. Note that at the end of vesting period $t = T_0$ or $\tau = T - T_0$, the position has a chance to belong to the no-trading region, which leads to a feedback strategy. We will use a numerical example to illustrate the phenomenon in next section. The same argument applies to the case of maximum price benchmark.\(^{10}\)

\(^{10}\)If we consider the case of geometric average with initial time vesting period $[0, T_0]$, we will have the same analytical solutions of value function restricted in $\tau \in [0, T - T_0]$ and then derive the corresponding optimal strategies. It is easy to show that the optimal selling strategy at the end of vesting period $t = T_0$ is still bang-bang.
5 Numerical results

Since the case of arithmetic average is more realistic but lacks analytical solutions, we resort to numerical solutions so as to investigate the optimal strategies. The penalty method, widely used to solve variational inequality equations in finance (see Forsyth and Vetzal (2002), Dai, Kwok and You (2007)), is employed to find numerical solutions to (4.3) and (4.4), where $F(x, \tau)$, as the solution to (4.2), can be obtained by using the standard finite difference method.

Figure 1 presents the optimal buying boundary $x_b^*(\tau)$ in the arithmetic average case, where the parameter values used are $T = 2$, $\mu = 0.06$, $\sigma = 0.4$ ($0 < \mu < \sigma^2$). As shown in part ii) of Theorem 4.4, the buying region $BR_x$ is above the optimal buying boundary, and the latter is monotonically increasing in time to maturity $\tau$ and tends to $-\infty$ as $\tau$ goes to zero. These indicate that it is more likely to buy the stock when time gets closer to the maturity.

In Figure 2, we present the optimal selling boundary $x_s^*(\tau)$ in the arithmetic average case, where default parameter values used are $T = 2$, $\sigma = 0.4$. To investigate the effect of $\mu$ on optimal strategy, we take $\mu = 0.03$, 0.06, respectively, and both satisfy $0 < \mu < \sigma^2$. It can be observed that the selling region $SR_x$ expands as $\mu$ increases, which coincides with the intuition that the smaller the expected return rate of stock, the higher the chance to sell stock. Note that the selling region $SR_x$ is below the optimal selling boundary, and the initial position $(x = -\infty, \tau = 2)$ must be in $SR_x$, which leads to an immediate selling strategy if there is no time vesting. However, given a time-vesting period, say, selling is permitted only when $\tau \leq \tau_0$, then the position $(x_{\tau_0}, \tau_0)$ is likely to be in the no trading region and the selling strategy would be a feedback one.

In Figure 3, we present an example to further explain the optimal selling strategy when time-vesting is involved. The parameter values used are $T = 5$, $\mu = 0.03$, $\sigma = 0.4$, and the vesting period is $[0, 3]$, that is, selling is permitted only when the calendar time $t \geq 3$. The horizontal axis now stands for the calendar time $t$ and the vertical axis the ratio of the spot price to the running average, i.e., $S_t/A_t$. The curve in dotted line refers to the optimal selling boundary above which is the selling region $SR_x$ (bear in mind $x = \log((T-\tau)t^{\frac{1}{2}})$, $t = T-\tau$). We carry out simulations to obtain three typical sample paths of $S_t/A_t$. Path A corresponds to an immediate selling after the vesting period. Path B corresponds to the situation that one should not sell the stock until $S_t/A_t$ hits the optimal selling boundary after the vesting period. Path C corresponds to the case that one should never sell the stock until expiry. These indicate that the optimal selling strategy is a feedback one.

6 Conclusion

Assuming the geometric Brownian motion of stock price, we examine the optimal decision to buy or sell a stock over a given time horizon with reference to the ultimate average price of stock, where the average is either arithmetic or geometric. This is an optimal stopping problem which can be formulated as a variational inequality problem. We make use of a PDE approach to study the optimal selling/buying strategy. It turns out that the optimal selling strategy is bang-bang, while the optimal buying strategy can be a feedback one subject to the type of average and parameter values. More precisely, for the sell case, if $\mu > \sigma^2$, it is optimal to hold the stock until expiry; if $\mu \leq \sigma^2$, it is optimal to sell the stock immediately at time 0. For the buy case, if $\mu \geq \sigma^2$, one should buy the
Figure 1: The optimal buying boundary $x_b^*(\tau)$ in the arithmetic average case. $\tau$ is time to maturity, and $x = \log((T - \tau) \frac{A}{S})$. Parameter values used: $\mu = 0.06$, $\sigma = 0.4$, $T = 2$.

Figure 2: The optimal selling boundary $x_s^*(\tau)$ in the arithmetic average case. $\tau$ is time to maturity, and $x = \log((T - \tau) \frac{A}{S})$. Default parameter values used: $\sigma = 0.4$, $T = 2$. 
Figure 3: The trading strategy under three sample paths with time-vesting period 3 years. \( t \) is the calendar time, and the vesting period is \([0, 3]\). Parameter values used: \( \mu = 0.03, \sigma = 0.4, T = 5 \).

stock immediately; if \( \mu \leq 0 \), one should never buy the stock before expiry; if \( 0 < \mu < \sigma^2 \), there is an optimal buying boundary and one should buy the stock once the boundary is reached. In addition, we show that optimal strategy only depends on the time to expiry for the geometric average case, and on the ratio of stock price to the running average in addition to the time to expiry for the arithmetic average case.

It is worth pointing out that the bang-bang strategy for the sell case is the same as that obtained by Shiryaev, Xu and Zhou (2008b) taking the ultimate maximum as benchmark. This, from another angle, justifies the robustness of the bang-bang selling strategy.

Nevertheless, we highlight that the bang-bang selling strategy heavily depends on the assumption that there are no time-vesting restrictions. However, the time-vesting restrictions are not rare and are often imposed in employee incentive plans or IPOs of strategic investors. Both of our theoretical analysis and numerical results reveal that given an initial time-vesting period, the optimal selling strategy can also be a feedback one. Moreover, in terms of a similar analysis developed by the present paper, the same phenomenon can be proved when taking the ultimate maximum as benchmark (see Dai et al. (2008)).

A Appendix: The proof of Lemma 4.2

Proof. Denote \( \tilde{F}(x, \tau) \equiv F_x(x, \tau) \), \( F^{xx}(x, \tau) \equiv F_{xx}(x, \tau) \) and \( F^{x\tau}(x, \tau) \equiv F_{x\tau}(x, \tau) \). It is easy to verify that \( \tilde{F}, F^{xx} \) and \( F^{x\tau} \) satisfy

\[
\begin{align*}
\left\{ \begin{array}{l}
\tilde{F}_\tau - \frac{\sigma^2}{2}(F^{xx} + 2F^{x\tau}) + (\mu - \frac{3}{2}\sigma^2 - e^{-x})F_x + e^{-x}\tilde{F} = 0, \quad \text{in } \Omega, \\
\tilde{F}(x, 0) = -1,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
F^{xx}_\tau - \frac{\sigma^2}{2}(F^{xx}_x + 2F_xF^{xx}) + (\mu - \frac{3}{2}\sigma^2 - e^{-x})F^{xx}_x + 2e^{-x}F^{x\tau} = e^{-x}F_x, \quad \text{in } \Omega, \\
F^{xx}(x, 0) = 0,
\end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
F^{x\tau}_\tau - \frac{\sigma^2}{2}(F^{x\tau}_x + 2F_xF^{x\tau}) + (\mu - \frac{3}{2}\sigma^2 - e^{-x})F^{x\tau}_x + e^{-x}F^{x\tau} = 0, \quad \text{in } \Omega, \\
F^{x\tau}(x, 0) = e^{-x},
\end{array} \right.
\end{align*}
\]
respectively. By virtue of the (strong) maximum principle\textsuperscript{11}, we obtain part i), ii) and iii).

Next we prove part iv). Denote $\tilde{F}^\delta(x, \tau) = F_x(x, \tau; \mu + \delta, \sigma)$ and $\hat{F}(x, \tau) = F_x(x, \tau; \mu, \sigma) + \frac{\delta}{\sigma^2}$. Let $P = \tilde{F}^\delta - \hat{F}$. Then, it suffices to show $P < 0$ in $\Omega$. It is easy to check that $\tilde{F}^\delta(x, \tau)$ and $\hat{F}(x, \tau)$ satisfy
\[
\begin{cases}
\tilde{F}_\tau^\delta - \frac{\sigma^2}{2} (\tilde{F}_{xx}^\delta + 2 \tilde{F}_x^\delta F_x^\delta) + (\mu + \delta - \frac{3}{2} \sigma^2 - e^{-x}) \tilde{F}_x^\delta + e^{-x} \hat{F}_x^\delta = 0, & \text{in } \Omega, \\
\tilde{F}^\delta(x, 0) = -1
\end{cases}
\] (A.1)

and
\[
\begin{cases}
\hat{F}_\tau - \frac{\sigma^2}{2} (\hat{F}_{xx} + 2 \hat{F}_x F_x) + (\mu + \delta - \frac{3}{2} \sigma^2 - e^{-x}) \hat{F}_x + e^{-x} \hat{F} = \frac{\delta}{\sigma^2} e^{-x}, & \text{in } \Omega, \\
\hat{F}(x, 0) = -1 + \frac{\delta}{\sigma^2},
\end{cases}
\] (A.2)

respectively. Subtracting (A.2) from (A.1), we obtain
\[
\begin{cases}
P_\tau - \frac{\sigma^2}{2} (P_{xx} + 2 P_x \hat{F}_x + \hat{F}_3 P) + (\mu + \delta - \frac{3}{2} \sigma^2 - e^{-x}) P_x + e^{-x} P = -\frac{\delta}{\sigma^2} e^{-x}, & \text{in } \Omega, \\
P(x, 0) = -\frac{\delta}{\sigma^2}.
\end{cases}
\]

Applying the maximum principle gives the desired result.

It remains to show part v). Note that
\[
F(x, \tau) = \log \frac{\Phi(z, \tau)}{T} = \log \left( \mathbb{E}_t \left( \frac{S_T}{T A_T} \right) \right) = \log \mathbb{E}_t \left[ \left( t A_t \frac{S_T}{S_t} + \int_t^T S_v d\nu \right)^{-1} \right]
\]
\[
= \log \mathbb{E} \left[ e^{x + \int_0^\tau e^{(\mu - \frac{\sigma^2}{2})s + \sigma B_s} ds} \right]^{-1}.
\]

It follows
\[
F_x(x, \tau) = -e^{x} \mathbb{E} \left[ \left( e^{x + \int_0^\tau e^{(\mu - \frac{\sigma^2}{2})s + \sigma B_s} ds} \right)^{-2} \right] \mathbb{E} \left[ \left( e^{x + \int_0^\tau e^{(\mu - \frac{\sigma^2}{2})s + \sigma B_s} ds} \right)^{-1} \right].
\]

We then get the desired results by letting $x \to -\infty$ and $\infty$.

\section*{B Appendix: The proof of Lemma 4.5}

Proof. Let
\[
\varphi^*(S_t, A_t, t) = \mathbb{E}_t \left( \frac{S_T}{A_T} \right),
\]

which represents the value function associated with a simple strategy: holding the stock until expiry $T$. Similar to the transformation (4.1), we consider
\[
U^*(z, \tau) = \varphi^*(S, A, t), \quad \overline{U}^*(x, \tau) = \log \left( \frac{U^*(z, \tau)}{\Phi(z, \tau)} \right),
\]

\footnote{Apparently the solutions we consider do not grow too fast as state variables go to infinity. So, we can use the maximum principle of unbounded domain.}
where the definitions of $x$, $z$ and $\tau$ are the same as earlier. It is easy to check that $U^*(x, \tau)$ is the solution to (4.10). So, we only need to show

$$U^*(0, \tau) > \Phi(0, \tau), \text{ if } \mu > \frac{\sigma^2}{2},$$

$$U^*(0, \tau) = \Phi(0, \tau), \text{ if } \mu = \frac{\sigma^2}{2},$$

$$U^*(0, \tau) < \Phi(0, \tau), \text{ if } \mu < \frac{\sigma^2}{2}.$$ 

Let us only consider the case of $\mu \geq \frac{\sigma^2}{2}$ since the case of $\mu < \frac{\sigma^2}{2}$ is similar. Note that

$$\Phi(0, \tau) = \lim_{z \to 0^+} \Phi(z, \tau) = \lim_{z \to 0^+} T \mathbb{E}_t \left( \frac{S_t}{T^Z} \right) = T \lim_{z \to 0^+} \mathbb{E}_t \left( tz + \int_t^T S_{\nu} d\nu \right)^{-1}$$

$$= T \mathbb{E} \left( \int_t^T \exp((\mu - \frac{\sigma^2}{2})(\nu - t) + \sigma B_{\nu-t}) d\nu \right)^{-1} \leq T \mathbb{E} \left( \int_0^T e^{\sigma B_{\nu}} d\nu \right)^{-1}, \text{ for } \tau \in (0, T).$$

In a similar way,

$$U^*(0, \tau) = \lim_{z \to 0^+} U^*(z, \tau) = \lim_{z \to 0^+} T \mathbb{E}_t \left( tz + \int_t^T S_{\nu} d\nu \right)^{-1}$$

$$= T \mathbb{E} \left( \int_t^T \exp \left( (\mu - \frac{\sigma^2}{2})(\nu - t) + \sigma B_{\nu-t} \right) d\nu \right)^{-1}$$

$$= T \mathbb{E} \left( \int_t^T \exp \left( (\mu - \frac{\sigma^2}{2})(T - t) + \sigma B_{T-t} \right) d\nu \right)^{-1}$$

$$\geq T \mathbb{E} \left( \int_0^T e^{\sigma B_{\nu}} d\nu \right)^{-1} \geq \Phi(0, \tau), \text{ for } \tau \in (0, T),$$

where $B_{\nu}^* = B_{(T-t)-\nu} - B_{T-t}$ is also a standard Brownian motion. In addition, we have the equality if and only if $\mu = \frac{\sigma^2}{2}$. The proof is complete.

References


