1 A Chebyshev spectral method - 3 credits

Consider the following boundary value problem
\[ u_{xx} + 4u_x + e^x u = \sin(8x) \]
\[ u(-1) = u(1) = 0. \]
Solve this problem using a Chebyshev spectral method. Show the solution for different values of \( N \) (the number of terms in the polynomial expansion). Also, comment on and present the accuracy of the numerical solution.

2 Legendre-Gauss points - 4 credits

a) Write a code to find the Legendre-Gauss points \( \{x_j\}_{j=0}^N \) and the corresponding weights \( \{\omega_j\}_{j=0}^N \) for a given \( N > 0 \). Use the fact that the zeros (roots) of an orthogonal polynomial \( p_{N+1} \) generated by the recurrence relation
\[ p_{k+1} = (a_k x - b_k) p_k - c_k p_{k-1}, \quad k \geq 0 \]
can be computed as the eigenvalues of a symmetric, tridiagonal matrix, see lecture notes from Lecture 4. Present the computed values of \( \{x_j\}_{j=0}^N \) and \( \{\omega_j\}_{j=0}^N \) for some values of \( N \).

b) The proposed method in a) is not always stable and can suffer from round-off errors as \( N \) gets large. As an alternative way of finding the Legendre-Gauss points/roots, we can use Newton’s method. That is, to find the roots of \( p_{N+1} \) we use the algorithm:
Given \( x_j^0 \)
\[ x_j^{n+1} = x_j^n - \frac{p(x_j^n)}{p'(x_j^n)}, \quad n = 0, 1, 2 \ldots \]
for each of the \( N + 1 \) roots.
Write a code to compute the Legendre-Gauss points \( \{x_j\}_{j=0}^N \) using Newton’s method. Compare to your results in a) to make sure that your code is working. You need to think about how to find the starting values, \( x_j^0 \), for Newton’s method. Present your code/algorithm and the computed values of \( \{x_j\}_{j=0}^N \) for some values of \( N \).

c) Modify your code in b) such that you can use the code to compute the Legendre-Gauss-Lobatto points, \( \{x_j\}_{j=0}^N \), and the corresponding weights, \( \{\omega_j\}_{j=0}^N \), instead.
Verify that your result is correct by testing that any polynomial of degree \( k \leq 2N - 1 \) is integrated exactly by the quadrature rule
\[ \int_{-1}^{1} p_k(x) dx = \sum_{j=0}^{N} p_k(x_j) \omega_j \]
Present the test as well as the computed values of \( \{x_j\}^N_{j=0} \) and \( \{\omega_j\}^N_{j=0} \) for some values of \( N \).

3 Legendre-Galerkin method - 4 credits

Solve the following boundary value problem with a Legendre-Galerkin method.
\[
\begin{align*}
    u_{xx} + u &= x^2 + x, \quad -1 < x < 1 \\
    u(-1) &= u(1) = 0
\end{align*}
\]
Show the solution for different values of \( N \) (the number of terms in the polynomial expansion). Also, comment on and present the accuracy of the numerical solution by e.g. comparing the approximate solution to the exact solution.

4 Helmholtz equation in 2D - 4 credits

Solve the Helmholtz equation
\[
\begin{align*}
    u_{xx} + u_{yy} + k^2u &= f(x, y), \quad -1 < x < 1, \quad -1 < y < 1 \\
    u(-1, y) &= u(1, y) = u(x, -1) = u(x, 1) = 0,
\end{align*}
\]
where \( k = 9 \) and \( f(x, y) = e^{-10[(y-1)^2+(x-0.5)^2]} \) using a spectral method. Present a plot of the solution and comment on the accuracy.