Some Problems on Linear Preservers

Wang F.¹ and Tan V.²

Department of Mathematics, National University of Singapore,
Singapore 117543

INTRODUCTION

Let \( \mathbf{M}(\mathcal{F}) \) be a space of matrices over the field \( \mathcal{F} \) and \( T: \mathbf{M}(\mathcal{F}) \to \mathbf{M}(\mathcal{F}) \) be a linear operator. A common problem considered in linear algebra is called a preserver problem. That is, characterize the linear operators which preserve a function or a set. We say \( T \) preserves a function \( f: \mathbf{M}(\mathcal{F}) \to \mathcal{F} \) if \( f(A) = f(T(A)) \) for all \( A \in \mathbf{M}(\mathcal{F}) \). We say \( T \) preserves a subset \( K \in \mathbf{M}(\mathcal{F}) \) if \( T(A) \in K \) for all \( A \in K \). In this report, we will review the history of linear preserver, and give two new theorem on determinant preservers.

LINEAR RANK-1 PRESERVERS ON \( \mathbf{M}_n(\mathbb{C}) \)

Linear Rank-1 Preservers

Let \( T \) be a linear operator on \( \mathbf{M}_n(\mathbb{C}) \). \( T \) is called a rank-1 preserver if \( \text{rank}(A) = 1 \) whenever \( \text{rank}(T(A)) = 1 \). In 1959, Marcus and Moyls characterized the general form of it: Suppose \( T \) is a linear rank-1 preserver on \( \mathbf{M}_n(\mathbb{C}) \). Then there exist invertible matrices \( P \) and \( Q \), such that either \( T(A) = PAQ \) for all \( A \in \mathbf{M}_n(\mathbb{C}) \), or \( T(A) = PA^TQ \) for all \( A \in \mathbf{M}_n(\mathbb{C}) \). This is one of the most powerful theorem on preserver problems, and we will use it to prove some interesting results of linear preservers on \( \mathbf{M}_n(\mathbb{C}) \)

Linear Determinant Preservers

Let \( T \) be a linear operator on \( \mathbf{M}_n(\mathbb{C}) \). \( T \) is called a determinant preserver if \( \det(A) = \det(T(A)) \) for all \( A \in \mathbf{M}_n(\mathbb{C}) \). In fact, this is the first problem on preserver problems, which is proved by Ferdinand Georg Frobenius (1849-1917) on 1897. He proved that: Let \( T \) be a linear determinant preserver on \( \mathbf{M}_n(\mathbb{C}) \). Then there exist invertible matrix \( P \) and \( Q \), with \( \det(PQ) = 1 \), such that either \( T(A) = PAQ \) for all \( A \in \mathbf{M}_n(\mathbb{C}) \), or \( T(A) = PA^TQ \) for all \( A \in \mathbf{M}_n(\mathbb{C}) \). In this section, we claimed that a linear determinant is also a rank-1 preserver, and then proved it by using the theorem of Marcus and Moyls on linear rank-1 preserver. We also proved that a linear operator \( T \) on \( \mathbf{M}_n(\mathbb{C}) \)

¹ Student
² Assistant Professor
is a determinant-trace preserver if and only if it is a characteristic polynomial preserver. Here, the linear operator \( T \) is called a determinant-trace preserver if \( A \) and \( T(A) \) have the same determinant and trace; and it is called a characteristic polynomial preserver if \( A \) and \( T(A) \) have the same characteristic polynomial. Frobenius is also the first mathematician who proved this theorem: Suppose \( T \) is a linear characteristic polynomial preserver on \( \mathbb{M}_n(\mathbb{C}) \). Then there exists an invertible matrix \( P \) such that, either \( T(A) = P A P^{-1} \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \), or \( T(A) = P^T A P^{-1} \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \).

**Linear Preservers of Nonnegative Matrices**

The theorem of linear rank-1 preserver can be used for classification on some linear preserver problem. In this section, we introduced two of them:

1. Suppose \( T \) is a linear determinant-trace preserver on \( \mathbb{M}_n(\mathbb{C}) \) which maps nonnegative matrices into nonnegative matrices. Then there exists a generalized permutation matrix \( P \) such that either \( T(A) = P A P^{-1} \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \), or \( T(A) = P^T A P^{-1} \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \).

2. Suppose \( T \) is a linear determinant-trace preserver on \( \mathbb{M}_n(\mathbb{C}) \) which maps nonnegative integer matrices into nonnegative integer matrices. Then there exists a permutation matrix \( P \) such that either \( T(A) = P A P^{-1} \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \), or \( T(A) = P^T A P^{-1} \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \).

**DETERMINANT PRESERVERS WITH** \( \det(A + \lambda B) = \det(T(A) + \lambda T(B)) \)

**Determinant Preservers on** \( \mathbb{M}_n(\mathbb{C}) \) with \( \det(A+\lambda B)=\det(T(A)+\lambda T(B)) \)

In the theorem of linear determinant preservers by Frobenius, it requires that the operator \( T \) is linear. However, the linearity is very strong as we use it heavily throughout the proof. An interesting result is that “can we replace the ‘linearity’ by a weaker condition?” In 2002, Dolinar and Šemrl proved that if \( T \) is a surjective operator on \( \mathbb{M}_n(\mathbb{C}) \) satisfying \( \det(A + \lambda B) = \det(T(A) + \lambda T(B)) \) for all \( A, B \in \mathbb{M}_n(\mathbb{C}) \) and \( \lambda \in \mathbb{C} \), then \( T \) is linear. In this section, we weaken this condition again. We proved that we can remove that “surjective” assumption. That is, if \( T \) is an operator on \( \mathbb{M}_n(\mathbb{C}) \) satisfying \( \det(A + \lambda B) = \det(T(A) + \lambda T(B)) \) for all \( A, B \in \mathbb{M}_n(\mathbb{C}) \) and \( \lambda \in \mathbb{C} \), then \( T \) is linear.

**Consequences of Theorem 2.3**

In the last section, we showed that the property \( \det(A + \lambda B) = \det(T(A) + \lambda T(B)) \) for all \( A, B \in \mathbb{M}_n(\mathbb{C}) \) and \( \lambda \in \mathbb{C} \) is equivalent to that “\( T \) is linear and preserves the determinant”. Therefore, by using the Frobenius’ theorem on linear determinant preserver stated in Chapter 1, we immediately can get some interesting consequences:
1. Let $T$ be an operator on $M_n(\mathbb{C})$ satisfying $\det(A + \lambda B) = \det(T(A) + \lambda T(B))$ for all $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Suppose $T$ is a trace preserver. Then there exists an invertible matrix $P$ such that, either $T(A) = PAP^{-1}$ for all $A \in M_n(\mathbb{C})$, or $T(A) = PA^TP^{-1}$ for all $A \in M_n(\mathbb{C})$.

2. Let $T$ be an operator on $M_n(\mathbb{C})$ satisfying $\det(A + \lambda B) = \det(T(A) + \lambda T(B))$ for all $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Suppose $T$ is a trace preserver, and $T$ maps nonnegative matrices into nonnegative matrices. Then there exists a generalized permutation matrix $P$ such that either $T(A) = PAP^{-1}$ for all $A \in M_n(\mathbb{C})$, or $T(A) = PA^TP^{-1}$ for all $A \in M_n(\mathbb{C})$.

3. Let $T$ be an operator on $M_n(\mathbb{C})$ satisfying $\det(A + \lambda B) = \det(T(A) + \lambda T(B))$ for all $A, B \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Suppose $T$ is a trace preserver, and $T$ maps nonnegative integer matrices into nonnegative integer matrices. Then there exists a permutation matrix $P$ such that either $T(A) = PAP^{-1}$ for all $A \in M_n(\mathbb{C})$, or $T(A) = PA^TP^{-1}$ for all $A \in M_n(\mathbb{C})$.

Determinant Preservers on $U_n$ with $\det(A + \lambda B) = \det(T(A) + \lambda T(B))$

It is known that if $T$ is a linear determinant preserver on $U_n$, then $[T(A)]_{ii} = c_i(A)_{\sigma(i)\sigma(i)}$ for some scalar $c_1, \cdots, c_n$ with $\prod_{i=1}^n c_i = 1$ and permutation $\sigma$ of $\{1, \cdots, n\}$. In this section, we generalized the theorem, and showed that if $T$ is an operator on $U_n$ satisfying $\det(A + \lambda B) = \det(T(A) + \lambda T(B))$ for all $A, B \in U_n$ and $\lambda \in \mathbb{C}$, then there exists scalar $c_1, \cdots, c_n$ with $\prod_{i=1}^n c_i = 1$ and permutation $\sigma$ of $\{1, \cdots, n\}$ such that $[T(A)]_{ii} = c_i(A)_{\sigma(i)\sigma(i)}$ for all $i = 1, \cdots, n$.

LINEAR RANK-1 PRESERVERS ON $\mathcal{H}_n$

Some Preliminary Lemmas

Not only for $M_n(\mathbb{C})$, we are also interested in classifying the general form of linear preservers on some other space. For example, $\mathcal{H}_n$, the space of hermitian matrices. In 1986, Johnson and Pierce proved that, if the linear invertible operation $T$ is a rank-1 preserver on $\mathcal{H}_n$, then either $T(A) = \varepsilon SAS_H$, $A \in \mathcal{H}_n$ or $T(A) = \varepsilon SA^TS_H$, $A \in \mathcal{H}_n$ for some invertible $S$ and $\varepsilon \in \{1, -1\}$. In this section, we proved some preliminary lemmas.
Linear Rank-1 Preservers on $\mathcal{H}_n$

In this section, we generalized the result by Johnson and Pierce: let $T$ be a linear rank-1 preserver on $\mathcal{H}_n$. Suppose there is a hermitian matrix whose image is invertible. Then there exists an invertible matrix $S$ and $\varepsilon \in \{1, -1\}$ such that either $T(A) = \varepsilon SAS^H$ for all $A \in \mathcal{H}_n$ or $T(A) = \varepsilon SA^T S^H$ for all $A \in \mathcal{H}_n$. Then, we use this result to get a more general result, which is first proven by Raphael Loewy in 1987: let $T$ be a linear rank-1 preserver on $\mathcal{H}_n$ with $\text{rank}(T) \geq 2$. Then there exists an invertible matrix $S$ and $\varepsilon \in \{1, -1\}$ such that either $T(A) = \varepsilon SAS^H$ for all $A \in \mathcal{H}_n$ or $T(A) = \varepsilon SA^T S^H$ for all $A \in \mathcal{H}_n$.

REFERENCES