NUROP Congress Paper
Jordan Canonical Forms of Linear Operators

Victor Tan\textsuperscript{1}, and Teo K.S.\textsuperscript{2}

Department of Mathematics, Faculty of Science, National University of Singapore
10 Kent Ridge Road, Singapore 117546

ABSTRACT

Any linear transformation can be represented by its matrix representation. In an ideal situation, all linear operators can be represented by a diagonal matrix. However, in the real world, there exist many linear operators that are not diagonalizable. This gives rise to the need for developing a system to provide a beautiful matrix representation for a linear operator that is not diagonalizable.

In this report, a new concept, known as the Jordan canonical form, will be introduced. A Jordan canonical form is a ‘almost-diagonal’ matrix, which can be used to represent a linear operator if the characteristic polynomial of the linear operator splits over the field concerned. In particular, the Jordan canonical form can always represent a linear operator preformed over the complex field.

The main objective of this report is focused on demonstrating and substantiating the procedure in constructing the Jordan canonical forms of linear operators, and its associated Jordan canonical basis.

There will be three main chapters in the report. The first chapter will introduce the essential concepts and theorems required. These theorem are essential to understand the principles behind the selection of the Jordan canonical basis, and for the subsequent chapters. The second chapter will introduce a method to deduce possible Jordan canonical forms. This is made possible thru an appreciation of the minimum polynomial associated with the linear operator. Finally, the last chapter in the report outlines a method which will completely determine the Jordan canonical form and its Jordan canonical basis.

\textsuperscript{1} Assistant Professor \textsuperscript{2} Student
CHAPTER ONE: FUNDAMENTAL THEOREMS AND DEFINITIONS

Chapter One provides the fundamental theorems and foundation for the required procedure that will be devised in the chapters that follows. It introduces the concept of generalised eigenspaces, which is a subspace that contains generalised eigenvectors that will eventually constitute the required Jordan canonical basis.

This chapter first establishes the fact that a finite dimensional vector space \( V \) is a direct sum of all its generalised eigenspaces. Because of this result, it is possible to use the union of the bases for all the generalised eigenspaces as a basis for the vector space \( V \).

The second half of the chapter deals with the concept of cycles of generalised eigenvectors. It is proven that every generalised eigenspace has an ordered basis consisting of a union of disjoint cycles of generalised eigenvectors. Hence the union of these cycles can be used as a basis for the vector space \( V \) by the previous result.

It is then further proven that the union of these cycles is precisely the required Jordan canonical basis that will lead to the Jordan canonical representation for the linear operator.

The chapter concluded by justifying that any linear operator has a Jordan canonical form if its characteristic polynomial splits. In particular, a linear operator performed over the complex field always has a Jordan canonical form.

MINIMUM POLYNOMIAL AND JORDAN CANONICAL FORMS

The first chapter has provided a rigorous foundation for the entire report. However, there is a need for a procedure and method to construct the Jordan canonical form of a linear operator. Chapter Two outlines one method of deducing the possible Jordan canonical forms by determining the unique minimum polynomial of the linear operator.

The last theorem, Theorem 2.6, in Chapter Two proves that the minimum polynomial of a linear operator imposes certain restrictions on the size of the Jordan blocks that constitute the Jordan canonical form. This is useful as it might eliminate other possibilities of Jordan canonical forms that do not satisfy the requirements as governed by the minimum polynomial of the linear operator.

This method, however, may not fully determine the exact Jordan canonical form. There exist many linear operators that have the same minimum polynomial, but different Jordan canonical forms. Furthermore, the minimum polynomial does not allow us to determine the Jordan canonical basis. This gives rises to the need for a more in-depth and powerful approach, discussed in Chapter Three, which introduces the ‘dot-diagram’
method. This method always works for any linear operator performed over the complex field.

**DETERMINING THE JORDAN CANONICAL FORM AND BASIS**

Chapter Three is developed extensively on the fundamental theorems discussed in the first chapter. Having proven, in the first chapter, that the cycles of generalised eigenvectors forms the required Jordan canonical basis, Chapter Three progressed to determine the vectors that belong to the cycles, such that the vectors will constitute the Jordan canonical basis.

It introduces the method of ‘dot diagram’, which is an array of dots in which each dot represents one unique element of the Jordan canonical basis. The dot diagram is arranged into rows and columns such that each column will represent a cycle of the generalised eigenspace. The first dot in each column represents the first vector (also known as the initial vector) in that cycle of generalised eigenvectors. The second dot corresponds to the second vector and so on. Hence the number of rows in the dot diagram is equal to the length of the largest cycle contained in the generalised eigenspace.

Each generalised eigenspaces will have its own dot diagram. All the dot diagrams, when pieced together, will exactly determine the unique Jordan canonical form of a linear operator.

This chapter outlines the procedure in deriving the dot diagram for each generalised eigenspaces. It uses the fact that the dot diagram is fully determined by the linear operator and the eigenvalue to provide the system of calculating the number of dots in every row of the dot diagram. Using a formula given by Theorem 3.2, a dot diagram for any generalised eigenspaces can be completely determined given the linear operator and the eigenvalue. This dot diagram will alone give the exact Jordan canonical form of the linear operator.

The number of dots in each column of the dot diagram will also provide information on the Jordan canonical basis, since the number of dots in each column is exactly the length of the corresponding cycle. Using the definition of cycles of generalised eigenvectors, the last example in Chapter Three outlines the method in obtaining the Jordan canonical basis.

In closing, the chapter highlights one useful application of the Jordan canonical form, that is, two matrices are similar if and only if they have the same Jordan canonical form. This application is exemplified in the last example of the chapter.
CONCLUSION

Through the understanding of Jordan canonical forms and the proofs for the fundamental theorems, a pleasant matrix representation for any linear operator, performed over the complex field, can be obtained. The simplicity of the Jordan canonical matrix representation grants great advantage in real life applications.

REFERENCES


3. http://www.ma.iup.edu/projects/CalcDEMma/JCF/jcf0.html
