Major Facilities for Mathematical Thinking and Understanding.

(2) Vision, spatial sense and kinesthetic (motion) sense.
Hear what you see.
See what you hear.

Visualization, especially in science and technology, has a special power to complement and enhance the written word in *depicting phenomena, elucidating concepts, revealing new insights, provoking thought and appealing to the imagination.*
Mobius Strip

http://www.metacafe.com/watch/331665/
no_magic_at_all_mobius_strip/
Vivianna Torun
– Band of Infinity.
The name "figure-eight knot" got the knot from its typical form. It is the most important stopper knot for sailors because of that fact that a line cannot loosen unintentional. It's easy, try it yourself.
People often find it easier to manipulate the larger image that they can *move around in*.
A Platonic solid is a (convex) polyhedron all of whose faces are congruent regular polygons, and where the same number of faces meet at every vertex.
Five Platonic (Regular) Solids.

Octahedron

Dodecahedron

Icosahedron
icosahedron

truncated icosahedron

European football
Not a Platonic Solid.

A *face* in a polyhedron is a (planar) polygonal region bounded by a (unbroken) circuit of edges.
<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>octahedron</td>
<td>6</td>
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</tr>
<tr>
<td>dodecahedron</td>
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<td>12</td>
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<tr>
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</tbody>
</table>
v – number of vertexes; 
e – number of edges; and, 
f – number of faces.

<table>
<thead>
<tr>
<th></th>
<th>f</th>
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</table>

A remarkable observation by Euler (1707-1783):

\[ v - e + f = 2 \]
Seven Bridges of Konigsberg.

Is it possible to walk with a route that crosses each bridge exactly once, and return to the starting point?
A polyhedron $P$ is said to be \textit{convex} if a line segment joining any two points in $P$ is itself totally in $P$.

$$v - e + f = 2$$

for any \textit{convex} polyhedron $P$ (not necessarily regular).

The formula may not hold for non-convex polyhedron.

(Think of drilling a hole through a cube.)
Choose a point $C$ inside $P$. Since $P$ is convex, the line segment joining $C$ to any point inside the polyhedron $P$, or on $P$ itself, lies entirely within $P$. Next, find a radius $R$ so large that the sphere with center $C$ and radius $R$ contains the polyhedron $P$. As size won’t affect the formula $V-E+F = 2$, by a rescaling, we may assume that $R = 1$. 
Segments of great circles
The Fields Medal shows Archimedes and the inscription reads in Latin: *Transire suum pectus mundoque potiri* (Rise above oneself and grasp the world).
DON'T DISTURB MY CIRCLES.
Archimedes’s theorem

The map is equi-areal. That is, it sends any region in the sphere to a region of the same area in the cylinder.
Area = 2\theta.
Counted 3 times.

\[ \text{Area of sphere (} R = 1) \]

Reflection Over counted 2 times.

Archimede’s Theorem.

\[ 2 \times (2 \angle A + 2 \angle B + 2 \angle C) \]

\[ = 4\pi + 2 \times (2 \times \text{area } ABC) \]

\[ \pm \] directions.

Area of sphere (} R = 1).
⇒ area \ ABC

= \angle A + \angle B + \angle C - \pi.
Area of the spherical triangle

\[ \text{Area of the spherical triangle} = (\text{sum of the three interior angles of the triangle}) - \pi. \]

First verified by Girard’s. Note that the spherical triangle is on the unit sphere. The edges of the spherical triangle are segments of great circles.

Area of the spherical n-polygon

\[ \text{Area of the spherical n-polygon} = [\text{sum of the } (n) \text{ interior angles of the polygon}] - (n - 2) \pi. \]
Area of the unit sphere

\[ = 4\pi \]

\[ = \sum (\text{spherical } n-\text{gons}) \]

\[ = \sum \left[ \sum (\text{interior } \text{angles}) - (n - 2)\pi \right] \]

\[ = 2\pi v + \sum \left[ -n\pi + 2\pi \right] \]

\[ = 2\pi v - 2\pi e + 2\pi f \]

\[ \Rightarrow v - e + f = 2. \]
\[
\sum_{\text{all faces}} [n \pi] = \sum_{\text{all faces}} [\text{number of edges of the face}] \cdot \pi = 2e \pi
\]
\[
\sum_{\text{all faces}} \left[ \sum (\text{interior angles of the face}) \right] = 2\pi \cdot v
\]
5 Platonic solids?

Same height & length

\[ \theta_1 > \theta. \]
x - number of polygons meeting at a vertex,
  • y - number of vertices of each polygon,
  • f the number of faces
  • e the number of edges
  • v the number of vertices

As each edge is counted twice and vertex x times, we have

\[ v = \frac{yf}{x}, \quad e = \frac{yf}{2}. \]

Here we apply Euler’s formula

\[ f - e + v = 2 \]

\[ \Rightarrow f[1 - \frac{y}{2} + \frac{y}{x}] = 2. \]
When \( y = 3 \) (triangle), we can have \( x = 3, 4, 5 \); while if \( y = 4 \) and \( 5 \), \( x \) has to be 3. Using the above relations, one can determine \( f \), \( e \) and \( v \) uniquely in each case.

Once we understand that there is only one way to form the combinations, we conclude that, there are only five Platonic polyhedrons as listed in the table.

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>( x )</th>
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<th>( e )</th>
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If we stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface (we call it simply connected).

On the other hand, if we imagine that the same rubber band has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut.
Poincaré, almost a hundred years ago, knew that a two dimensional sphere is essentially characterized by simply connectedness.

“Consider a 3-dimensional polyhedron P. Suppose that every rubble band in P can be shrunk to a point. Then it is the `same’ as the 3-sphere.”

(A naïve version of the Poincare Conjecture. Solving the formal version could also claim U.S.$ 1,000,000.)
Continuous deformation of Geometric objects, a study of Topology.

See Youtube “The Poincare Conjecture”.
~1300, Dante described a universe in which the concentric terraces of hell nesting down to the center of the earth, are mirrored by concentric celestial spheres rising and converging to a single luminous point.
• Grigori Perelman

In 1994, Perelman sequestered himself away to tackle the problem, and for the following 8 years gave no signs of life.

In May 2003, he announced that he had solved the Poincare Conjecture and the Thurston Geometrization Conjecture.