Major Facilities for Mathematical Thinking and Understanding.

(4) Intuition, Metaphor and Association.

People have amazing facilities for sensing something without knowing where it comes from (intuition).

Think about, for examples, speculation and conjecture, sometime it is hard to explain why we guess in such a way.
People find out long ago that some phenomenon or situation or object is *like* something else (association).

Smoking – fume – lighting.

We also process the ability to build connections and comparisons, holding two things in mind at the same time (metaphor).

Consider poetry, where people use metaphors and associations constantly and freely.

Take an example from *Romeo and Juliet* by Shakespeare.
From the scene where Romeo and Juliet said goodbye to each other for the last time.

JULIET:

Wilt thou be gone? It is not yet near day:  
It was the nightingale, and not the lark,  
That pierced the fearful hollow of thine ear;  
Nightly she sings on yon pomegranate-tree:  
Believe me, it was the nightingale.

ROMEO:

It was the lark, the herald of the morn,  
No nightingale: look, love, what envious streaks  
Do lace the severing clouds in yonder east:  
Night's candles are burnt out, and jocund day  
Stands tiptoe on the misty mountain tops.  
I must be gone and live, or stay and die.
JULIET:
Yon light is not day-light, I know it, I:
It is some meteor that the sun exhales,
To be to thee this night a torch-bearer,
And light thee on thy way to Mantua:
Therefore stay yet; thou need'st not to be gone.

ROMEO:
Let me be taken, let me be put to death;
I am content, so thou wilt have it so.
...
I have more care to stay than will to go:
Come, death, and welcome! Juliet wills it so.
“A mathematician,” said old Weierstrass, “who is not at the same time a bit of a poet will never be a full mathematician.”

Thinking in terms of ideas requires us to apply intuition, association, and metaphor, which have the special power to complement and enhance the written record in depicting phenomena, elucidating concepts, revealing new insights, provoking thought and appealing to the imagination.
We use the cutting/dissection of polygonal regions to illustrate the ideas of metaphor and association.
Haberdasher's Problem (1907)

With four cuts, Dissect an Equilateral Triangle into a Square.
Can any two polygonal regions on the plane with the same area be `cut’ into same number of triangular regions, so that the corresponding triangular regions are congruent?

**Scissors Congruence.**

The answer is affirmative. This is widely known as the Polyai Theorem.
A triangle is scissors congruent to a rectangle having the same area and sharing the *longest* side.
2. SCISSORS-CONGRUENT POLYGONS

[Step 1] $K_1$

[Step 2] $\Delta_1 \rightarrow \square_1 \rightarrow \Delta_2 \rightarrow \square_2$

[Step 3] $\square_1 \rightarrow \square_2$

[Step 4] $\square_2 \rightarrow \square_3 \rightarrow \square_4$

$\Delta_3 \rightarrow \square_3$

$\Delta_4 \rightarrow \square_4$

$K'$

$m(K_1)$
`Filling up the can` – Any polygonal region is scissors congruent to a rectangular region of unit length base.

It follows from that two polygonal regions of the same area are scissors congruent (Polya's Theorem).
If the polygonal region $P$ is scissors congruent $Q$, and $Q$ to $R$, then $P$ and $R$ are scissor congruent.

An example to illustrate this:
Area as a number/

→ Unit and division.

Measure.

(1) If P is a polygon that can be cut into polygons P’ and P”, then

`measure’ P = `measure’ P’ + `measure’ P”

(2) If P and P’ are congruent, then

`measure’ P = `measure’ P’
The formula for the volume of a pyramid,

\[ V = \frac{1}{3} Ah, \]

had been known to Euclid, but all proofs of it involve some form of limiting process or calculus.

Gauss regretted this defect in two of his letters. This was the motivation for Hilbert: is it possible to prove the equality of volume using elementary "cut-and-glue" methods? (Association.)
Hilbert 3\textsuperscript{rd} problem: Given any two tetrahedra $T_1$ and $T_2$ with equal base area and equal height (and therefore equal volume), is it always possible to find a finite number of tetrahedra, so that when these tetrahedra are glued in some way to $T_1$ and also glued to $T_2$, the resulting polyhedra are congruent?

$\theta(e)$ \quad \text{Dihedral angle.}

$l(e)$ \quad \text{Length of the edge.}$
For two polyhedra A and B to be congruent upon subdivision, we must have

$$\sum_{i} m_i \theta_i = \sum_{j} n_j \phi_j + k\pi,$$

where $\theta_i$ and $\phi_j$ are the dihedral angles of A and B, respectively, $m, n$, are positive integers, and $k$ is an integer.
where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio number.

It is known that the dihedral angle of the Platonic solid tetrahedron is NOT a rational number times times pi.

Hence the Platonic solid tetrahedron and the cube are not congruent by subdivision.
Dehn’s invariant – a `number’ to handle the geometric intuition.

\[ 1 \otimes \frac{\pi}{2} + 1 \otimes \frac{\pi}{2} = 1 \otimes \pi. \]

“tensor” product

\[ D(P) = \sum_{e} l(e) \otimes (\theta(e) + \mathbb{Q}\pi) \]

A rational number times pi

The **cube** has Dehn’s invariant zero, while every regular **tetrahedron** has non-zero Dehn’s invariant.
In light of Dehn's theorem above, one might ask: "which polyhedra are scissors-congruent"?

The beautiful answer is due to Sydler (1965): two polyhedra are scissors-congruent if and only if they have the same volume and the same Dehn invariant.
In his 1995 Orsay lecture, Kontsevich introduced the `motivic measure’ for equations like

\[ x^2 + y^2 = 1. \]

The construction obtains its form from the scissors congruence, with its content distilled to bare simplicity.