Major Facilities for Mathematical Thinking and Understanding.

(5) Stimulus-response.

Common examples can be quoted:

\[18 \times 19 = ?\]
\[531 \div 3 = ?\]

\[(x + 1)^2 = x^2 + 2x + 1,\]
\[(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.\]
\[(a + b)^n = a^n + na^{n-1}b + \cdots + C(n,r)a^{n-r}b^r + \cdots + b^n,\]

where \[C(n,r) = \frac{n!}{r!(n-r)!}.\]

\[(a + b)^1 = a + b,\]
\[(a + b)^2 = a^2 + 2ab + b^2,\]
\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.\]
Srinivasa Ramanujan

1729

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$
We would like to see how these simple considerations can lead to interesting mathematics.

We say that an integer $p > 1$ is a *prime number*, or a *prime*, if $p$ is not divisible by any positive integer $d$ satisfying $1 < d < p$. e.g. 7, 11, 23, 29,....
We have the following result which is intuitive enough:

**For any positive integer** \( n > 1 \), \( n \) can be expressed as a product of primes. The expression is unique apart from rearrangements.

\[
n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}
\]

Here \( p_1, p_2, \cdots, p_r \) are distinct primes, and \( \alpha_1, \alpha_2, \cdots, \alpha_r \) are positive integers.

In general, given a large positive integer \( n \), to find all the factors of \( n \) is a very time consuming process.
The question asks: Given a positive integer \( n (=1, 2, 3, \ldots) \), does there exist a right angle triangle with sides \( a, b \) and \( c \) (\( c > a, \ b > 0 \)), all being rational numbers, and area equal to \( n \)?

Mathematically, it means:

(1) \( a, b, c \) are positive rational numbers.

(2) \( a^2 + b^2 = c^2 \).

(3) \( \frac{ab}{2} = n \) \quad (n \text{ is an integer}).
\[ a^2 + b^2 = c^2. \]

Here we want \( a, b \) and \( c \) to be rational numbers.

Example 1.

\[
\left( \frac{P^2 - Q^2}{M} \right)^2 + \left( \frac{2PQ}{M} \right)^2 = \left( \frac{P^2 + Q^2}{M} \right)^2.
\]

In the above, \( P, Q \) and \( M \) are integers.

Example 2.

\[
\left( \sqrt{a^2 + b} + \sqrt{a^2 - b} \right)^2 + \left( \sqrt{a^2 + b} - \sqrt{a^2 - b} \right)^2 = 4a^2,
\]

where

\( a, \sqrt{a^2 + b}, \) and \( \sqrt{a^2 - b} \)

are all rational numbers.
Recall the following argument by descent.

$$\sqrt{7} = 2.645751.....$$
$$\sqrt{7} - 2 = 0.645751.....$$

Suppose $\sqrt{7} = \frac{p}{q}$

$$= \frac{p \cdot (\sqrt{7} - 2)}{q \cdot (\sqrt{7} - 2)} \leftarrow \text{smaller than } p$$
$$= \frac{p \cdot \sqrt{7} - 2p}{q \cdot \sqrt{7} - 2q} \leftarrow \text{smaller than } q$$
$$= \frac{1}{\sqrt{7}} \cdot \frac{p - 2q}{p - 2q}$$
$$= \frac{7q - 2p}{(p - 2q)} \leftarrow \text{an integer smaller than } p$$
$$\leftarrow \text{an integer smaller than } q$$
Consider the case when $n = 1$.

(1) $a$, $b$, $c$ are positive *rational* numbers.

(2) $a^2 + b^2 = c^2$.

(3) $\frac{ab}{2} = 1$
\[ \text{Area} = \frac{1}{2} a \cdot b = 1. \]

**Rescaling:**

\[
\text{Area} = \frac{1}{2} (aqq') \cdot (bqq') = \left( \frac{1}{2} a \cdot b \right) \cdot (qq')^2
\]

\[= 1 \cdot (\text{integer})^2.\]
The question for $n = 1$ is the same as to find a *right angle* triangle with sides $A$, $B$ and $C$, such that

1. $A$, $B$, $C$ are positive integers.
3. $Area = \frac{1}{2} \cdot AB = (\text{integer})^2$.

Fermat shows that it is impossible!
Let $A$, $B$, and $C$ be three positive integers with

\[ A^2 + B^2 = C^2. \]

Suppose

\[ A = k \cdot a \quad \text{and} \quad B = k \cdot b, \]

where $a$, $b$ and $k (> 1)$ are positive integers.

Then we also have

\[ C = k \cdot c. \]

Thus, without loss of generality, we may assume that $A$ and $B$ have no common factors ($> 1$).
For any positive integers $P > Q$, clearly
\[
(P^2 - Q^2)^2 + (2PQ)^2 = (P^2 + Q^2)^2
\]

We now consider the "mirror reflection" of the above.

Let $A$, $B$, and $C$ be three positive integers with
\[
\]
Suppose $A$ and $B$ have no common factor (>1), then

\[
A = P^2 - Q^2, \quad B = 2PQ
\]

and
\[
C = P^2 + Q^2.
\]

Here $P$ and $Q$ are positive integers without any common factor (>1).
A and B cannot be both even.

A and B cannot be both odd.

\[ A = 2n + 1, \quad B = 2m + 1 \]
\[ \Rightarrow \quad C^2 = A^2 + B^2 = 4(n^2 + m^2 + n + m) + 2. \]

But this shows that C is neither even nor odd.

Space for you to write down notes.
Hence we may assume that A is odd and B is even. That is,

\[ A = 2n + 1, \quad B = 2m \]

\[ \Rightarrow C^2 = A^2 + B^2 = 4(n^2 + m^2 + n) + 1 \]

\[ \Rightarrow C \text{ is odd.} \]

Since

\[ B^2 = C^2 - A^2 = (C - A) \cdot (C + A) \]

\[ \Rightarrow \left( \frac{B}{2} \right)^2 = \left( \frac{C - A}{2} \right) \cdot \left( \frac{C + A}{2} \right) \]
As

\[ A^2 = P^2 - Q^2 = (P + Q)(P - Q), \]

Both \((P + Q)\) and \((P - Q)\) are odd integers.

Moreover, \(P, Q, (P + Q)\) and \((P - Q)\) do not have mutual common factor \((>1)\).
Area of the right angle $\triangle$

$$= \frac{1}{2} A \cdot B = \frac{1}{2} (P^2 - Q^2) \cdot (2PQ)$$

$$= P \cdot Q \cdot (P + Q) \cdot (P - Q)$$

$$= \text{(integer)}^2$$
\[ P = c^2, \]
\[ Q = d^2, \]
\[ P + Q = U^2, \]
\[ P - Q = v^2, \]
\[ \Rightarrow c^2 = P = \frac{U^2 + v^2}{2}, \]
\[ \text{and} \quad d^2 = P = \frac{U^2 - v^2}{2} = \frac{(U - v)(U + v)}{2}. \]
Recall that both \((P + Q)\) and \((P - Q)\) are odd integers, as \(A\) is chosen to be the odd integer and
\[
A^2 = P^2 - Q^2 = (P + Q)(P - Q),
\]

From
\[
P + Q = U^2, \quad P - Q = v^2,
\]
we infer that both \(U\) and \(v\) are odd integers.

Thus
\[
U + v \quad \text{and} \quad U - v
\]
are both even integer. We assert that except the common factor of 2, \((U + v)\) and \((U - v)\) has no other common factor > 1.
Assume the contrary happens, that is

\[ U + \nu = 2 \cdot k \cdot N_+ \]
\[ U - \nu = 2 \cdot k \cdot N_- \]

\[ \Rightarrow 2U = 2 \cdot k \cdot (N_+ + N_-) \]
and  \[ 2\nu = 2 \cdot k \cdot (N_+ - N_-) \]

\[ \Rightarrow U = k \cdot (N_+ + N_-) \]
and  \[ \nu = k \cdot (N_+ - N_-) \]

\[ \Rightarrow P + Q = U^2 = k^2 \cdot (N_+ + N_-)^2 \]
and  \[ P - Q = \nu^2 = k^2 \cdot (N_+ - N_-)^2 \]

Here  \( k > 1 \), \( N_+ \), and \( N_- \) are positive integers.

But this contradicts the fact that \( (P + Q) \) and \( (P - Q) \) have no common factors > 1.
For numbers $a^2 \geq b > 0$, clearly

\[
\left(\sqrt{a^2 + b} + \sqrt{a^2 - b}\right)^2 + \left(\sqrt{a^2 + b} - \sqrt{a^2 - b}\right)^2 = 4a^2
\]

Hence we have a right angle triangle with side A, B and C given by

\[
A = \left(\sqrt{a^2 + b} + \sqrt{a^2 - b}\right)
\]

\[
B = \left(\sqrt{a^2 + b} - \sqrt{a^2 - b}\right)
\]

\[
C = 2a.
\]

The area of this triangle is equal to the number b.
The equation

\[ X^4 + Y^4 = Z^4 \]

has no integer solutions.
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