1. (10 marks) Find an equation of the plane through the point \((5, 1, -2)\) perpendicular to the line
\[
\frac{x}{2} = \frac{y - 1}{4} = \frac{z + 2}{3}.
\]
Find also the distance of the point \((2, 5, 1)\) from the plane.

**Solution.** The vector \(\langle 2, 4, 3 \rangle\) is parallel to the line, and hence a normal to the plane. An equation of the plane is given by
\[
2x + 4y + 3z = C \quad \text{and} \quad 2(5) + 4(1) + 3(-2) = C \implies C = 8.
\]
Denote by \(P\) the point with position vector \(\langle 5, 1, -2 \rangle\) and \(A\) the point with position vector \(\langle 2, 5, 1 \rangle\). Then \(\overrightarrow{AP} = \langle 5, 1, -2 \rangle - \langle 2, 5, 1 \rangle = \langle 3, -4, -3 \rangle\). The distance \(d\) is given by
\[
d = \frac{|\langle 3, -4, -3 \rangle \cdot \langle 2, 4, 3 \rangle|}{\sqrt{2^2 + 4^2 + 3^2}} = \frac{19}{\sqrt{29}}.
\]
2. (8 marks) Graph the points $z = x + iy$ that satisfy the condition

$$|z + i| \geq |z|.$$ 

Here $x$ and $y$ are real numbers. Give reasons for your answers.

**Solution.** We have

$$|z + i| \geq |z|$$

$$\iff |x + i(y + 1)| \geq |x + iy|$$

$$\iff \sqrt{x^2 + (y + 1)^2} \geq \sqrt{x^2 + y^2}$$

$$\iff x^2 + (y + 1)^2 \geq x^2 + y^2$$

$$\iff 2y + 1 \geq 0$$

$$\iff y \geq -\frac{1}{2}.$$ 

It is the region $\{(x, y) \mid y \geq -1/2\}$. 

*Figure not available.*
3. (12 marks) Find the following limits. Give reasons for your answers. You may assume that 
\[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]
(Note that you are NOT allowed to apply L’Hopital’s rule.)

(a) \[ \lim_{x \to 0} \frac{\tan(2x)}{\tan(3x)}. \]
(b) \[ \lim_{x \to \infty} \left( \sqrt{x^2 + x + 3} - \sqrt{x^2 + x - 3} \right). \]
(c) \[ \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}. \]

Solution. We have
\[ \lim_{x \to 0} \frac{\tan(2x)}{\tan(3x)} = \left( \lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} \right) \left( \lim_{x \to 0} \frac{\cos(3x)}{\cos(2x)} \right) = \left( 2 \lim_{x \to 0} \frac{\sin(2x)}{2x} \right) \left( \frac{1}{3} \lim_{x \to 0} \frac{3x}{\sin(3x)} \right) = \frac{2}{3}. \]
\[ \lim_{x \to \infty} \left( \sqrt{x^2 + x + 3} - \sqrt{x^2 + x - 3} \right) = \lim_{x \to \infty} \frac{\left( \sqrt{x^2 + x + 3} - \sqrt{x^2 + x - 3} \right) \left( \sqrt{x^2 + x + 3} + \sqrt{x^2 + x - 3} \right)}{\sqrt{x^2 + x + 3} + \sqrt{x^2 + x - 3}} = \lim_{x \to \infty} \frac{x^2 + x + 3 - (x^2 + x - 3)}{6 \sqrt{x^2 + x + 3} + \sqrt{x^2 + x - 3}} = \lim_{x \to \infty} \frac{6}{6 \sqrt{x^2 + x + 3} + \sqrt{x^2 + x - 3}} = 0. \]

Using the equality \( t^3 - 1 = (t - 1)(t^2 + t + 1) \), we obtain
\[ \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1}{(x^{\frac{1}{3}} - 1)(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)} = \lim_{x \to 1} \frac{1}{x^{\frac{1}{3}} + x^{\frac{1}{3}} + 1} = \frac{1}{3}. \]
4. (20 marks)

(a) i. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function. Give the definition of the statement “\( f(x) \) is continuous at the point \( x = c \).”

ii. State without proof the Intermediate Value Theorem.

(b) Let \( a \) and \( b \) be real numbers and let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined by

\[
g(x) = \begin{cases} 
    |x| & \text{if } x \leq 0 \\
    ax + b & \text{if } 0 < x \leq 2 \\
    \frac{1 - \cos(x\pi)}{(x-2)^2} & \text{if } x > 2
\end{cases}
\]

Find all values of \( a \) and \( b \) so that the function \( g(x) \) is continuous on \( \mathbb{R} \). Give reasons for your answers. You may use the limit

\[
\lim_{t \to 0} \frac{1 - \cos t}{t^2} = \frac{1}{2}.
\]

For such values of \( a \) and \( b \), show that there exists a real number \( c \) such that \( 2 < c < 3 \) and \( g(c) = 3 \).

**Solution.**

(a) (i) \( f(x) \) is continuous at \( x = c \) if

\[
\lim_{x \to c} f(x) = f(c).
\]

(ii) The I.V.T. states that if \( f \) is a continuous function on the closed interval \([a, b]\) (where \( a \) and \( b \) are numbers and \( a < b \)), then for any number \( y_o \) between \( f(a) \) and \( f(b) \), there is a number \( x_o \in [a, b] \) such that \( f(x_o) = y_o \).

(b) In order for \( g \) to be continuous at \( x = 0 \), we must have

\[
0 = \lim_{x \to 0} g(x) = \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (ax + b) = b \implies b = 0.
\]

Consider the limit

\[
\lim_{x \to 2^+} \frac{1 - \cos(x\pi)}{(x-2)^2} = \lim_{x \to 2^+} \frac{1 - \cos[(x - 2)\pi]}{(x - 2)^2},
\]

as \( \cos y = \cos(y - 2\pi) \) for \( y \in \mathbb{R} \). Let \( s = x - 2 \). When \( x \to 2^+ \), we have \( s \to 0^+ \). It follows that

\[
\lim_{x \to 2^+} \frac{1 - \cos[(x - 2)\pi]}{(x - 2)^2} = \lim_{s \to 0^+} \frac{1 - \cos(s\pi)}{s^2} = \pi^2 \lim_{s \to 0^+} \frac{1 - \cos(s\pi)}{(s\pi)^2} = \frac{\pi^2}{2}.
\]

For \( g \) to be continuous at \( x = 2 \), we must have

\[
2a = \lim_{x \to 2^-} g(x) = \lim_{x \to 0^+} g(x) = \frac{\pi^2}{2} \implies a = \frac{\pi^2}{4}.
\]

As \( g(2) = \pi^2/2 > 3 \) and \( g(3) = [1 - (-1)]/1 = 2 \), by the I.V.T., there is a point \( c \in [2, 3] \) such that \( g(c) = 3 \), \( c \neq 2 \) or \( 3 \) because \( g(2) \neq 3 \) and \( g(3) \neq 3 \).