§1. One-Tailed & Two-Tailed Tests.

A test of any statistical hypothesis, where the alternative is one-sided such as
\[ H_0 : \theta = \theta_0, \]
\[ H_1 : \theta > \theta_0, \]

or perhaps
\[ H_0 : \theta = \theta_0, \]
\[ H_1 : \theta < \theta_0, \]

is called a one-tailed test. The critical region for the alternative hypothesis \( \theta > \theta_0 \) lies entirely in the right tail of the distribution, while the critical region for the alternative hypothesis \( \theta < \theta_0 \) lies entirely in the left tail.

A test of any statistical hypothesis where the alternative is two-sided, such as
\[ H_0 : \theta = \theta_0, \]
\[ H_1 : \theta \neq \theta_0, \]

is called a two-tailed test, since the critical region is split into two parts having equal probabilities placed in each tail of the distribution of the test statistic. The alternative hypothesis \( \theta \neq \theta_0 \) states that either \( \theta < \theta_0 \) or \( \theta > \theta_0 \).

The null hypothesis, \( H_0 \), will always be stated using the equality sign so as to specify a single value. That is, the null hypothesis will always be a simple hypothesis. In this way the probability of committing a type I error can be controlled. Whether one sets up a one-tailed or two-tailed test will depend on the conclusion
to be drawn if \( H_0 \) is rejected. *The location of the critical region can be determined only after \( H_1 \) has been stated.*

The steps for testing a hypothesis concerning a population parameter \( \theta \) against some alternative hypothesis may be summarized as follows:

(i) State the null hypothesis \( H_0 \) that \( \theta = \theta_0 \).

(ii) Choose an appropriate alternative hypothesis \( H_1 \) from one of the alternatives \( \theta < \theta_0 \), \( \theta > \theta_0 \), or \( \theta \neq \theta_0 \).

(iii) Choose a significance level of size \( \alpha \). (It is customary to choose the value of \( \alpha \) to be 0.05 or 0.01.)

(iv) Select the appropriate test statistic and establish the critical region.

(v) Compute the value of the test statistic from the sample data.

(vi) Decision: Reject \( H_0 \) if the test statistic has a value in the critical region; otherwise accept \( H_0 \).

We will refer to the above as the **six-step procedure**.
**Example:**

A soft-drink machine is regulated so that the amount of drink dispensed is approximately normally distributed with a mean of 200 milliliters and a standard deviation of 15 milliliters. The machine is checked periodically by taking a sample of 9 drinks and computing the average content. Clearly, we should make the test

\[
\begin{cases}
H_0: & \mu = 200, \\
H_1: & \mu \neq 200.
\end{cases}
\]

As we are dealing with the population mean $\mu$, a natural choice of an appropriate test statistic will be $\bar{X}$. Suppose that we adopt the following decision procedure: if $\bar{x}$ falls in the interval

\[191 < \bar{x} < 209,
\]

the machine is thought to be operating satisfactorily; otherwise, we conclude that $\mu \neq 200$ milliliters.

In other words, the critical region is specified by the event $\{\bar{X} \leq 191\} \cup \{\bar{X} \geq 209\}$. (Equivalently, the acceptance region is $\{191 < \bar{X} < 209\}$.) Based on this test procedure, we can compute the probability of committing a type I error. Indeed,

\[
\alpha = \mathbb{P}\{\text{Type I error}\} = \mathbb{P}\{\text{Reject } H_0 \mid H_0 \text{ is true}\}
= \mathbb{P}\{\bar{X} \leq 191 \mid \mu = 200\} + \mathbb{P}\{209 \leq \bar{X} \mid \mu = 200\}
= 1 - \mathbb{P}\{191 < \bar{X} < 209 \mid \mu = 200\}
= 1 - \mathbb{P}\left\{ \frac{191 - 200}{15/\sqrt{9}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{209 - 200}{15/\sqrt{9}} \right\}
= 1 - \mathbb{P}\{-1.8 < Z < 1.8\} = 2\mathbb{P}\{Z > 1.8\}
= 2(0.0359) = 0.0718.
\]
One can also compute the probability of committing a type II error when $\mu = 215$ milliliters. It is given by

$$
\beta = \mathbb{P}\{\text{Type II error}\} = \mathbb{P}\{\text{Accept } H_0 \mid H_0 \text{ is false}\} \\
= \mathbb{P}\{191 < \overline{X} < 209 \mid \mu = 215\} \\
= \mathbb{P}\left\{ \frac{191 - 215}{15/\sqrt{9}} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < \frac{209 - 215}{15/\sqrt{9}} \right\} \\
= P\{-4.8 < Z < -1.2\} \approx P\{Z > 1.2\} \\
= 0.1151.
$$

Now let us control the probability of committing a type I error to be 0.05, i.e., $\alpha = 0.05$. In other words, we choose the significance level of size 0.05. In this case how should we establish the critical region?
Finding the Critical Region:

First, the test statistic is $\bar{X}$. Since the alternative hypothesis is a two-tailed one, the critical region should be of the form $|\bar{X} - \mu| > k$ for a suitable constant $k$. The value of $k$ can be determined by the following evaluation of the size of the critical region.

$$0.05 = \alpha = \Pr\{\text{Type I error}\} = \Pr\{\text{Reject } H_o \mid H_o \text{ is true}\}$$

$$= \Pr\{|\bar{X} - \mu| > k \mid \mu = 200\}$$

$$= 1 - \Pr\{|\bar{X} - \mu| < k \mid \mu = 200\}$$

$$= 1 - \Pr\left\{\left|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right| < \frac{k}{15/\sqrt{9}}\right\}$$

$$= 1 - \Pr\left\{-\frac{k}{15/\sqrt{9}} < Z < \frac{k}{15/\sqrt{9}}\right\}$$

$$= 2 \Pr\left\{Z > \frac{k}{15/\sqrt{9}}\right\}.$$

Thus, $\frac{k}{15/\sqrt{9}} = z_{0.025} = 1.96$, and in our example, the critical region of size 0.05 is given by

$$\{|\bar{x} - 200| > 9.8\} = \{\bar{x} < 190.2\} \cup \{\bar{x} > 209.8\}.$$

Carefully checking the derivation above, one would realize that, when the hypotheses to be tested are

$$\begin{cases} H_o : \mu = \mu_o, \\
H_1 : \mu \neq \mu_o, \end{cases}$$

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and the population distribution is assumed to be normal with known standard deviation \( \sigma \), the critical region (C.R.) takes the form

\[
\text{C.R.} = \{ |x - \mu_0| > z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \} = \{ x - \mu_0 < -z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \} \cup \{ x - \mu_0 > z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \}.
\]

Note that the “acceptance region” is determined by the corresponding \((1 - \alpha) 100\%\) confidence interval.

Equivalently, one can also normalize the test statistic first and obtain the following form for the critical region: let

\[
Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}},
\]

(where \( \sigma \) is known).

\[
\text{C.R.} = \{ |z| > z_{\alpha/2} \} = \{ z > -z_{\alpha/2} \} \cup \{ z > z_{\alpha/2} \}.
\]
§2. Tests Concerning Means.

Example: A manufacturer of sports equipment has developed a new synthetic fishing line that he claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that \( \mu = 8 \) kilograms against the alternative that \( \mu < 8 \) kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.05 level of significance.

**Sol.** Use the six-step procedure.

(i) \( H_0: \mu = 8(= \mu_0) \) kilograms.

(ii) \( H_1: \mu < 8(= \mu_0) \) kilograms.

(iii) \( \alpha = 0.05 \).

(iv) Use \( \bar{X} \) as the test statistic. In such a case, the C.R. is of the form \( \bar{x} - \mu_0 < k \) for a suitable constant \( k \) such that its size, \( P\{\bar{X} - \mu_0 < k | \mu = \mu_0\} = \alpha \). Clearly, such a constant \( k \) is given by \( -z_{\alpha} \frac{\sigma}{\sqrt{n}} \), and hence the C.R. is now given by \( \bar{x} < \mu_0 - z_{\alpha} \frac{\sigma}{\sqrt{n}} \). Equivalently, one can also use the normalized form, i.e., the critical region can be expressed as \( z < -z_{\alpha} \), where \( z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \).

(v) \( \bar{x} = 7.8, \alpha = 0.05, n = 50, \mu_0 = 8, z_{0.05} = 1.6449 \), and \( \sigma = 0.5 \). Now \( \mu_0 - z_{\alpha} \frac{\sigma}{\sqrt{n}} \approx 7.8837 \). Note that \( \bar{x} = 7.8 < 7.8837 \). Equivalently, we may compute the value of the normalized test statistic \( z = \frac{\bar{x} - 8}{0.5/\sqrt{50}} = -2.83 \).

(vi) Decision: Since \( \bar{x} = 7.8 < 7.8837 \) (or equivalently \( z = -2.83 < -z_{0.05} = -1.6449 \)), we reject \( H_0 \) at the 0.05 level of significance and conclude that the average breaking strength is less than 8 kilograms.