FLEXIBLE HARVESTING

Derivatives, Rates of Change, and the Fishing Industry.

The fishing industry worldwide is suffering because fishermen are catching too many fish. Mathematicians are helping to advise fishermen about what to do to maximise their catch without driving the fish to extinction.

If there is no fishing, the number of fish \( N(t) \) [which is a function of time] satisfies the following equation:

\[
\frac{dN}{dt} = BN - sN^2.
\]  

(1)

Here \( B \) is the birth rate per capita, the number of baby fish hatched per year divided by the total number of fish. We take it to be a constant. The fish also die, of course, from disease, from being eaten by bigger fish, etc, and that is where the term \(-sN^2\) comes from. (Notice the minus sign!) The number \( s \) is assumed constant; if you think about it, you will see that \( s \) measures how “weak” the fish are. (Large \( s \) means that the fish die easily.) Usually we expect that, in the long run, the number of fish will be constant if there is no fishing; we call this the EQUILIBRIUM situation. From calculus we know that in that case \( dN/dt = 0 \). Solving, and rejecting the uninteresting case \( N = 0 \), we get

\[
N_e = \frac{B}{s}.
\]

(2)

We call \( N_e \) the EQUILIBRIUM POPULATION. If we know (by zoological research) the birth rate per capita \( B \) and the “weakness” of the fish, \( s \), then we can compute \( N_e \). Notice that if \( s \) is large, the equilibrium population is small — you don’t expect to have a large population of weak fish!

Now the fishermen come along. We advise them as follows. When there are many fish, you can catch many fish; but when fish are few, you should cut back and not catch so many. The simplest way to organise this is to say: each year we shall count the number of fish present (biologists can do this by doing surveys) and we suggest that you should catch a fraction \( \gamma \) of that number per year. That is, you should catch \( \gamma N \) fish per year.

Fisheries experts call this FLEXIBLE HARVESTING. Naturally \( \gamma \) has to be less than 1. Our equation for \( N \) is now

\[
\frac{dN}{dt} = BN - sN^2 - \gamma N.
\]

(3)

Again, notice the minus sign. The equilibrium population of fish is now given by

\[
N_e = \frac{[B - \gamma]}{s}.
\]

(4)

Notice that \( N_e \) is smaller than it was before — which makes sense, since we are eating fish. Of course it does not make sense for \( N_e \) to be negative; what this means is that if \( \gamma \) is larger than \( B \), then there is no equilibrium at all. If we write equation 3 as

\[
\frac{dN}{dt} = [B - \gamma]N - sN^2,
\]

(5)
we see at once that \( \frac{dN}{dt} \) is ALWAYS negative if \( B \) is less than \( \gamma \), and from calculus we know that this means that \( N \) will always decrease in that case — NOT what we want! It is important to remember that we control \( \gamma \), but not \( B \) — we cannot control the birth rate of the fish! So we have to take care to choose \( \gamma \) to be less than \( B \).

Now suppose that we have sensibly chosen \( \gamma \) to be smaller than \( B \), so that there is a non-zero equilibrium population given by equation 4. The number of fish we catch per year is always given by \( \gamma N \), so in the equilibrium situation the number of fish being caught per year is \( \gamma N_e \). This number is clearly a function of \( \gamma \), so let us call it \( F(\gamma) \). From equation 4 we have

\[
F(\gamma) = \gamma N_e = \left[ B\gamma - \gamma^2 \right]/s. \tag{6}
\]

Now \( \gamma \) is under our control — it measures how greedy we are. Question: how should we choose \( \gamma \)? Well, of course as fishermen we would like to MAXIMISE \( F(\gamma) \). So we differentiate and set the derivative equal to zero:

\[
\frac{dF}{d\gamma} = 0 = \left[ B - 2\gamma \right]/s. \tag{7}
\]

The second derivative is \(-2/s\) so we have a maximum here, as we wanted. So we should choose \( \gamma \) to be \( B/2 \) (which of course is less than \( B \), as above). If we are more greedy than this, THEN WE WILL ACTUALLY CATCH LESS FISH! The actual number of fish we should catch is given by evaluating \( F(\gamma) \) at this value of \( \gamma \), which is

\[
F(B/2) = \left[ B^2/2 - (B/2)^2 \right]/s = B^2/4s. \tag{8}
\]

Our job — not an easy one — is to persuade the fishermen that they will be better off in the long run if they are less greedy; they should aim to catch \( B^2/4s \) fish per year. Our other job is to do some biological research to find out the values of \( B \) and \( s \)! Notice that if we are comparing two species of fish, say cod and anchovies, and if the birth rate of anchovies is twice the birth rate of cod and they are equally weak, then we can allow not twice but FOUR times as many anchovies to be caught as cod — not something that you would have guessed. So calculus is useful!