1 Brownian motion: characterization

Brownian motion is named after the botanist Robert Brown, who first observed seemingly random motion of pollen grains suspended in fluids. A rigorous mathematical construction of Brownian motion was first given by Norbert Wiener, and for this reason, Brownian motion is also known as the Wiener process.

Theorem 1.1 [Characterization of Brownian motion] There exists a real-valued stochastic process \((B_t)_{t \geq 0}\), called the (one-dimensional) standard Brownian motion, whose law is uniquely characterized by the following properties:

(i) \(B\) has independent increments, i.e., for any \(t_0 < t_1 < \cdots < t_n < \infty\), \(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\) are independent.

(ii) \(B_0 = 0\), and for any \(0 \leq s < t\), \(B_t - B_s\) is distributed as a centered Gaussian random variable with variance \(t - s\).

(iii) Almost surely, \(B_t\) is continuous in \(t\).

Because \(B_t\) is almost surely continuous in \(t\), Brownian motion can also be regarded as a random variable taking values in the space of continuous functions \(C([0, \infty), \mathbb{R})\), which we equip with the topology of uniform convergence on compact sets, which in turn generates a Borel \(\sigma\)-algebra. On \(C([0, T], \mathbb{R})\) for finite \(T\), we simply use the sup-norm topology.

One may wonder why don’t we regard Brownian motion as a random variable taking values in the space of real-valued measurable functions defined on \([0, \infty)\), for which the natural \(\sigma\)-algebra is the one generated from the coordinate maps \(t : f \to f(t) \in \mathbb{R}\) for \(t \geq 0\)? The reason is that in such a \(\sigma\)-algebra, every measurable set is determined by the values of the functions at at most a countable number of coordinates, and the set of continuous functions is not even measurable!

Theorem 1.1 can be proved by first constructing \(B_t\) for \(t \in Q_2 := \{m2^{-n} : m, n \geq 0\}\), which is possible by Kolmogorov’s extension theorem. The hard part is then to show that \((B_t)_{t \in Q_2}\) is almost surely uniformly continuous on \([0, T] \cap Q_2\) for any \(0 < T < \infty\), which allows the extension of \(B_t\) from \(t \in Q_2\) to a continuous function on \([0, \infty)\). We will give a proof below, which will furthermore establish that Brownian motion is almost surely H"older continuous with exponent \(\gamma\) for any \(\gamma < 1/2\).

Because \(B\) has independent Gaussian increments, the finite-dimensional distributions of \(B\), i.e., the law of \((B_{t_1}, \ldots, B_{t_n})\) for any \(0 \leq t_1 < \cdots < t_n < \infty\), are multivariate normal distributions. This qualifies \(B\) as a Gaussian process. A continuous Gaussian process is uniquely determined by its finite-dimensional distributions, while multivariate normal distributions are uniquely determined by their covariance matrices — recall that if \(\vec{X} := (X_1, \ldots, X_d) \in \mathbb{R}^d\) has multivariate normal distribution with mean \(E[X_i] = 0\) and covariance matrix \(E[X_iX_j] = C_{ij}\) for \(1 \leq i, j \leq d\), then \(\vec{X}\) has distribution \(\frac{1}{(2\pi)^{d/2} \text{det}(C)^{1/2}} e^{-\frac{1}{2} \vec{x}^\top C^{-1} \vec{x}} d\vec{x}\) on \(\mathbb{R}^d\). Therefore we have the following alternative characterization of Brownian motion.
Theorem 1.2 [Brownian motion as a Gaussian process] The law of a standard Brownian motion \((B_t)_{t \geq 0}\) is uniquely determined by the following properties:

(i) \((B_t)_{t \geq 0}\) is a Gaussian process, i.e., its finite-dimensional distributions are multi-variate normal.

(ii) \(E[B_s] = 0\) and \(E[B_sB_t] = s\) for all \(0 \leq s \leq t < \infty\).

(iii) Almost surely, \(B_t\) is continuous in \(t\).

We now collect some properties of Brownian motion. First some invariance properties.

Theorem 1.3 [Invariance properties] Let \(B := (B_t)_{t \geq 0}\) be a standard Brownian motion. Then \(B\) satisfies the following invariance properties:

1. [Translation] For any \(t_0 \geq 0\), \((B_{t_0} + t - B_{t_0})_{t \geq 0}\) is equally distributed with \((B_t)_{t \geq 0}\).

2. [Diffusive scaling] For any \(a > 0\), \((B_{at}/\sqrt{a})_{t \geq 0}\) is equally distributed with \((B_t)_{t \geq 0}\).

3. [Time inversion] \(X_t := tB_1/\sqrt{t}\) with \(X_0 := 0\) is equally distributed with \((B_t)_{t \geq 0}\).

Proof. (1) and (2) follow easily by verifying that Properties (i)–(iii) in Theorem 1.1 are all preserved by translation and diffusive scaling. For (3), note that \((B_t)_{t > 0}\) and \((tB_1/\sqrt{t})_{t > 0}\) are both continuous Gaussian processes, whose laws are uniquely determined by their finite dimensional distributions, and in turn by their covariances. Since for all \(0 < s < t < \infty\),

\[
E[B_sB_t] = s, \\
E[sB_1/sB_1/t] = st - t^{-1} = s,
\]

\((tB_1/t)_{t > 0}\) and \((B_t)_{t > 0}\) must be equally distributed, and so are \((tB_1/t)_{t \geq 0}\) and \((B_t)_{t \geq 0}\). □

2 Brownian motion: path properties

Next we list some almost sure path properties for the standard Brownian motion.

Theorem 2.1 [Path properties] Let \((B_t)_{t \geq 0}\) be a standard Brownian motion. Then almost surely,

(i) \((B_t)_{t \in [0,1]}\) is Hölder continuous with exponent \(\gamma\) for any \(\gamma < 1/2\), i.e.,

\[
\sup_{0 \leq s < t \leq 1} \frac{|B_t - B_s|}{|t-s|^\gamma} < \infty. \tag{2.1}
\]

(ii) \((B_t)_{t \in [0,1]}\) is not Hölder continuous with exponent \(\gamma\) for any \(\gamma \geq 1/2\), i.e.,

\[
\sup_{0 \leq s < t \leq 1} \frac{|B_t - B_s|}{|t-s|^\gamma} = \infty. \tag{2.2}
\]

Furthermore, \((B_t)_{t \geq 0}\) is not Hölder continuous at any point \(t \geq 0\) with exponent \(\gamma\) for any \(\gamma > 1/2\), i.e.,

\[
\limsup_{s \to t} \frac{|B_s - B_t|}{|s-t|^\gamma} = \infty \quad \text{for all } t \geq 0 \text{ and } \gamma > \frac{1}{2}. \tag{2.3}
\]

In particular, \((B_t)_{t \geq 0}\) is almost surely nowhere differentiable.
(iii) [Law of the iterated logarithm]

\[
\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = \limsup_{t \to 0} \frac{B_t}{\sqrt{2t \log \log \frac{1}{t}}} = 1, \\
\liminf_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = \liminf_{t \to 0} \frac{B_t}{\sqrt{2t \log \log \frac{1}{t}}} = -1. 
\] (2.4)

In particular, \((B_t)_{t \geq 0}\) visits every point in \(\mathbb{R}\) infinitely often.

Remark 2.2 Although \((B_t)_{t \geq 0}\) is a.s. nowhere differentiable, its distributional derivative \(\dot{B}_t\) can be defined via

\[
\int_0^\infty \dot{B}_t f(t) dt := -\int_0^\infty B(t) f'(t) dt \quad \text{for all } f \in C_c^\infty([0, \infty), \mathbb{R}).
\]

This random distribution \(\dot{B}_t\) is called White Noise. The fact that \((B_t)_{t \geq 0}\) has stationary independent increments implies \(\dot{B}_t\) is translation invariant and independent in time. More precisely, for any \(f, g \in C_c^\infty((0, \infty), \mathbb{R})\), the random variable \(\int f(t) \dot{B}_t dt\) is equally distributed with \(\int f(t + x) \dot{B}_t dt\) for any \(x > 0\), and if \(g\) has disjoint support from \(f\), then \(\int f(t) \dot{B}_t dt\) and \(\int g(t) \dot{B}_t dt\) are independent. In fact \(\int f(t) \dot{B}_t dt\) is a Gaussian random variable with mean 0 and variance \(\int f(t)^2 dt\). The map \(f \mapsto \int f(t) \dot{B}_t dt\) is an isometry from \(L_2(\mathbb{R}) \to L_2(\Omega, \mathcal{F}, \mathbb{P})\), and hence we can construct the stochastic integral \(\int f(t) \dot{B}_t dt\) for a general \(f \in L_2(\mathbb{R})\) by first constructing it for \(f(t) = 1_{[a, b]}(t)\) and linear combinations of such indicator functions, and then take closure in \(L_2(\Omega, \mathcal{F}, \mathbb{P})\).

The key to proving the a.s. Hölder continuity of a Brownian motion is the following.

Theorem 2.3 [Kolmogorov’s Moment Criterion] Let \(X\) be a real-valued stochastic process indexed by \(\mathbb{Q}_2 = \{m2^{-n} : m, n \geq 0\}\). If for some \(\alpha, \beta > 0\),

\[
\mathbb{E}|X_t - X_s|^\beta \leq K|t - s|^{1 + \alpha} \quad \forall s, t \in \mathbb{Q}_2 \cap [0, 1], \tag{2.5}
\]

then for any \(0 < \gamma < \frac{\alpha}{2}\), almost surely there exists \(C := C(\omega) \in (0, \infty)\) such that

\[
|X_q - X_r| \leq C|q - r|^{\gamma} \quad \forall q, r \in \mathbb{Q}_2 \cap [0, 1]. \tag{2.6}
\]

In particular, \(X\) can be extended almost surely to a continuous process indexed by \([0, 1]\), such that (2.6) holds for any \(q, r \in [0, 1]\).

Proof. Fix \(0 < \gamma < \frac{\alpha}{2}\). Let \(D_n := \{i2^{-n} : 0 \leq i \leq 2^n\}\). For any \(s := i2^{-n} < t := j2^{-n} \in D_n\), by (2.5) and Chebychev inequality,

\[
\mathbb{P}(|X_t - X_s| > |t - s|^{\gamma}) \leq K|t - s|^{1 + \alpha - \beta \gamma} = K|i - j|^{1 + \alpha - \beta \gamma}2^{-n(1 + \alpha - \beta \gamma)}. \tag{2.7}
\]

If we let \(G_n := \{|X_{2^{i-n}} - X_{2^{j-n}}| \leq |j - i|^{\gamma}2^{-n\gamma} \forall 0 \leq i, j \leq 2^n, |i - j| \leq 2^{n\eta}\}\) for some fixed \(\eta \in (0, 1)\), then by (2.7),

\[
\mathbb{P}(G_n) \leq \sum_{0 \leq i < j \leq 2^n} K(j - i)^{1 + \alpha - \beta \gamma}2^{-n(1 + \alpha - \beta \gamma)} \leq K2^{-\gamma n}, \tag{2.8}
\]

where \(\epsilon = (1 - \eta)(\alpha - \beta \gamma) - 2\eta\) can be made positive if \(\eta > 0\) is sufficiently small. Fix such an \(\eta\). Then by Borel-Cantelli, almost surely there exists \(N_\omega \in \mathbb{N}\) such that \(\cap_{n \geq N_\omega} G_n\) occurs.
On the event $\cap_{n \geq N_\omega} G_n$, we will use the triangle inequality to deduce (2.6). Note that it suffices to show that (2.6) holds for some $C(\omega)$ for all $q, r \in \mathbb{Q}_2 \cap [0, 1]$ with $|q - r| \leq 2^{-N_\omega(1-\eta)}$, since for general $q, r \in \mathbb{Q}_2 \cap [0, 1]$, we only need to replace $C(\omega)$ by $C(\omega)2^{N_\omega(1-\eta)}$ by the triangle inequality. Now assume $0 \leq r < q \leq 1$ and $q - r < 2^{-N_\omega(1-\eta)}$. We can find an optimal scale $m \geq N_\omega$ such that

$$2^{-(m+1)(1-\eta)} \leq q - r < 2^{-m(1-\eta)}. \tag{2.9}$$

By binary expansion for $q$ and $r$, we can write

$$q = i2^{-m} + 2^{-q_1} + \cdots + 2^{-q_k}$$

$$r = j2^{-m} - 2^{-r_1} - \cdots - 2^{-r_l},$$

where $m < q_1 < \cdots < q_k$ and $m < r_1 < \cdots < r_l$. Since $q - r \geq (i - j)2^{-m}$ and $i - j \geq -1$, by (2.9), we have $|i - j| \leq 2^{m\eta}$. Since $\cap_{n \geq N_\omega} G_n$ occurs and $m \geq N_\omega$, we have

$$|X_{i2^{-m}} - X_{j2^{-m}}| \leq 2^{-m(1-\eta)\eta}. \tag{2.10}$$

By the triangle inequality,

$$|X_q - X_{i2^{-m}}| \leq \sum_{\sigma \geq m} 2^{-q_\sigma \gamma} \leq \sum_{\sigma > m} 2^{-\sigma \gamma} = C_\gamma 2^{-m \gamma}, \tag{2.11}$$

where $C_\gamma = \frac{1}{2^{\gamma - 1}}$. Similarly, the same bound also holds for $|X_r - X_{j2^{-m}}|$. Combining the above estimates and apply triangle inequality once more, we get

$$|X_q - X_r| \leq 2^{-(m(1-\eta)\gamma)} + 2C_\gamma 2^{-m \gamma} \leq (1 + 2C_\gamma)2^{(1-\eta)\gamma}2^{-m(1-\eta)\gamma} \leq C(\omega)|q - r|^\gamma$$

with $C(\omega) = (1 + 2C_\gamma)2^{(1-\eta)\gamma}$, and we are done.

**Remark 2.4 (Moment Criterion for Continuity and Tightness)** The proof of Theorem 2.3 can be easily modified to show that given a family of random continuous functions $\{(X^{(n)}_t)_{t \in [0, 1]}\}_{n \in \mathbb{N}}$, if (2.5) holds for some $K, \alpha, \beta$ uniformly in $n \in \mathbb{N}$, then for any $\epsilon > 0$, we can construct a compact set of Hölder continuous functions $H_t \subset C([0, 1], \mathbb{R})$, such that $P(X^{(n)}_t \in H_t) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$. This implies tightness for the family of $C([0, 1], \mathbb{R})$-valued random variables $\{X^{(n)}_t\}_{n \in \mathbb{N}}$. There is also an analogue of (2.5) in dimensions $d \geq 1$ with $1 + \alpha$ in (2.5) replaced by $d + \alpha$, when $(X_t)_{t \in [0, 1]}$ is $\mathbb{R}^d$-valued, see e.g. Kallenberg [K97, Corollary 14.9]. Garsia, Rodemich and Rumsey have given more refined continuity criterion in [GRR70], which was later extended to dimensions $d \geq 1$ by Garsia in [G72].

**Proof of (2.1).** Note that for any $0 \leq s < t \leq 1$, $B_t - B_s$ is a centered Gaussian random variable with variance $t - s$. Therefore

$$E|B_t - B_s|^\beta = |t - s|^\beta E|B_1|^\beta = f(\beta)|t - s|^\beta,$$

where $f(\beta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^\beta e^{-x^2/2} \, dx < \infty$.

Therefore (2.5) holds with $\alpha = \frac{\beta}{2} - 1$ for any $\beta > 2$. Since $\frac{\alpha}{\beta} \to \frac{1}{2}$ as $\beta \uparrow \infty$, by Theorem 2.3, $(B_t)_{0 \leq t \leq 1}$ is a.s. Hölder continuous with exponent $\gamma$ for all $\gamma \in (0, \frac{1}{2})$.

**Proof of (2.2).** It suffices to verify (2.2) for $\gamma = \frac{1}{2}$. Note that

$$K(\omega) := \sup_{0 \leq s < t \leq 1} \frac{|B_t - B_s|}{\sqrt{t - s}} \geq \max \left\{ \sup_{t \geq \frac{i-1}{n}} \sup_{\frac{i-1}{n} \leq s < t \leq \frac{i}{n}} \frac{|B_t - B_s|}{\sqrt{t - s}} : 1 \leq i \leq n \right\},$$
where \( K_n^{(i)}(\omega) := \sup_{\frac{i-1}{n} \leq s < t \leq \frac{i}{n}} \frac{|B_t - B_s|}{\sqrt{t-s}}, \) \( 1 \leq i \leq n, \) are independent because \((B_t)^{t \geq 0}\) has independent increments, and furthermore, \( K_n^{(i)} \) are equally distributed with \( K \) because

\[ \sqrt{n}(B_{i/n + t/n} - B_{i/n}) \rightarrow \sqrt{\gamma} \]

is equally distributed with \((B_t)_{0 \leq t \leq 1}\). Therefore for any \( M > 0, \)

\[ \mathbb{P}(K \leq M) \leq \mathbb{P}(K_n^{(1)} \leq M, \ldots, K_n^{(n)} \leq M) = \mathbb{P}(K \leq M^n) \quad \text{for all } n \in \mathbb{N}. \]

Since \( \mathbb{P}(K > M) > 0 \) for all \( M > 0, \) we must have \( \mathbb{P}(K \leq M) = 0 \) for all \( M > 0. \) Therefore \( K = \infty \text{ a.s.} \)

**Proof of (2.3).** The argument is based on time discretization and the observation that \( B_t \) is typically of order \( \sqrt{t}. \) In particular, for any \( \gamma > \frac{1}{2}, \) by the density of Gaussian distribution,

\[ \mathbb{P}(|B_t| < Ct^{\gamma}) = \mathbb{P}(|B_t| < Ct^{\gamma - \frac{1}{2}}) = \int_{-Ct^{\gamma - \frac{1}{2}}}^{Ct^{\gamma - \frac{1}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \sim \frac{2C}{\sqrt{2\pi}} t^{\gamma - \frac{1}{2}} \quad \text{as } t \downarrow 0. \quad (2.12) \]

It suffices to consider only \((B_t)_{t \in [0,1]}\). Fix \( \gamma > \frac{1}{2}, \) and fix \( k \in \mathbb{N} \) whose exact value will only depend on \( \gamma \) and will be determined later. Define

\[ A_n := \{ \exists t \in [0,1] \text{ s.t. } |B_{t+h} - B_t| \leq |h|^{\gamma} \forall |h| \leq \frac{2k}{n} \}. \]

Note that \( A_n \) is an increasing sequence of events, and

\[ \left\{ \exists t \in [0,1] \text{ s.t. } \limsup_{s \to t} \frac{|B_s - B_t|}{|s-t|^{\gamma}} < 1 \right\} \subset \bigcup_{n \in \mathbb{N}} A_n. \]

On the event \( A_n, \) let \( t \in \left[ \frac{m}{n}, \frac{m+1}{n} \right) \) for some \( 0 \leq m \leq n \) such that \( |B_{t+h} - B_t| \leq |h|^{\gamma} \) for all \( |h| \leq \frac{2k}{n} \). By the triangle inequality, we have the occurrence of the event

\[ F_m := \left\{ |B_{t+\frac{1}{n}} - B_{t+\frac{1}{n}}| \leq \frac{2}{n} \left( \frac{k}{n} \right)^{\gamma} \text{ \forall } j \in \{m, m+1, \ldots, m+k-1\} \right\}. \]

Therefore by the independent increments properties of \((B_t)_{t \geq 0},\)

\[ \mathbb{P}(A_n) \leq \sum_{m=0}^{n} \mathbb{P}(F_m) = (n+1)\mathbb{P}(F_0) = (n+1)\mathbb{P} \left( |B_t| \leq \frac{2k}{n} \right)^{\frac{1}{2}\gamma} \]

which by (2.12) is asymptotically of the order \( C_k n^{1-k(\gamma-\frac{1}{2})} \) for some \( C_k \) depending only on \( k, \) and it tends to 0 as \( n \to \infty \) if \( k > (\gamma - \frac{1}{2})^{-1}. \) For such a choice of \( k, \) we have \( \lim_{n \to \infty} \mathbb{P}(A_n) = 0. \) Since the sequence of events \( A_n \) increases in \( n, \) we have \( \mathbb{P}(A_n) = 0 \) for all \( n \in \mathbb{N}. \) In particular, we have shown that almost surely,

\[ \limsup_{s \to t} \frac{|B_t - B_s|}{|t-s|^{\frac{1}{2}\gamma}} \geq 1 \quad \text{for all } t \in [0,1]. \]

Since this is true for all \( \gamma > \frac{1}{2}, \) by slightly adjusting the values of \( \gamma, \) the above inequality implies (2.3).

**Proof of (2.4).** By symmetry, it suffices to prove the limsup statement in (2.4). We will prove the case \( t \to \infty, \) the statement for \( t \to 0 \) then follows from the fact that \((tB_t)_{t \geq 0}\) is equally distributed with \((B_t)_{t \geq 0}\).
First we recall that by the reflection principle and diffusive scaling invariance of the standard Brownian motion,

\[ \mathbb{P}_0 \left( \sup_{0 \leq s \leq t} B_s > a \right) = 2 \mathbb{P}_0(B_t > a) = 2 \mathbb{P}_0(B_1 > a/\sqrt{t}) = \sqrt{\frac{2}{\pi}} \int_{\frac{a}{\sqrt{t}}}^{\infty} e^{-x^2/2} \, dx. \quad (2.13) \]

Note that

\[ \int_y^{\infty} e^{-x^2/2} \, dx \leq y^{-1} e^{-y^2/2} \quad \text{for all } y > 0 \quad \text{and} \quad \int_y^{\infty} e^{-x^2/2} \, dx \sim y^{-1} e^{-y^2/2} \quad \text{as } y \to \infty. \quad (2.14) \]

Therefore for any \( f : \mathbb{N} \to (0, \infty) \) with \( \lim_{n \to \infty} f(n) = \infty \),

\[ \mathbb{P}_0 \left( \sup_{0 \leq t \leq n} B_t > \sqrt{nf(n)} \right) = \sqrt{\frac{2}{\pi}} \int_{2f(n)}^{\infty} e^{-x^2/2} \, dx \sim \sqrt{\frac{2}{\pi f(n)}} e^{-t(n)/2} \quad \text{as } n \to \infty. \]

If we take \( f(n) = (2 + \epsilon) \log n \) for any \( \epsilon > 0 \), then

\[ \sum_{n=1}^{\infty} \mathbb{P}_0 \left( \sup_{0 \leq t \leq n} B_t > \sqrt{nf(n)} \right) < \infty. \]

By Borel-Cantelli, a.s. \( \sup_{0 \leq t \leq n} B_t \leq (2 + \epsilon)n \log n \) for all \( n \) sufficiently large. Since \( \epsilon > 0 \) is arbitrary, this implies

\[ \lim_{t \to \infty} \sup_{0 \leq t \leq n} \frac{B_t}{\sqrt{2t \log t}} \leq 1. \]

To replace \( \log t \) by \( \log \log t \), we apply Borel-Cantelli along an exponentially increasing sequence of times \( t_n = \alpha^n \), instead of \( t_n = n \). With \( f(t) = 2a \log \log t \) for any \( \alpha > 1 \), we have

\[ \sum_{n=1}^{\infty} \mathbb{P}_0 \left( \sup_{0 \leq t \leq \alpha^n} B_t > \sqrt{\alpha^nf(\alpha^n)} \right) < \infty. \]

Therefore by Borel-Cantelli, a.s. \( \sup_{0 \leq t \leq \alpha^n} B_t \leq \sqrt{2\alpha^{n+1} \log \log \alpha^n} \) for all \( n \) large. For any \( t > 0 \), we can find \( n \) such that \( \alpha^n < t \leq \alpha^{n+1} \). Then a.s. for all \( t \) sufficiently large,

\[ \frac{B_t}{\sqrt{2t \log \log t}} \leq \frac{\sup_{0 \leq s \leq \alpha^{n+1}} B_s}{\sqrt{2\alpha^{n+2} \log \log \alpha^{n+1}}} \frac{\sqrt{2\alpha^{n+2} \log \log \alpha^{n+1}}}{\sqrt{2t \log \log t}} \leq \alpha \frac{\log \log \alpha^{n+1}}{\log \log \alpha^n} \leq \alpha^2. \]

Since \( \alpha > 1 \) can be chosen arbitrarily, this implies that

\[ \lim_{t \to \infty} \sup_{0 \leq t \leq \alpha^n} \frac{B_t}{\sqrt{2t \log \log t}} \leq 1 \quad \text{a.s.} \quad (2.15) \]

To prove the complimentary lower bound, we will find independent events and apply Borel-Cantelli. As before, let \( t_n = \alpha^n \), except \( \alpha > 1 \) will now be large. Note that

\[ \mathbb{P}_0(B_{t_{n+1}} - B_t > \sqrt{t_{n+1}f(t_{n+1})}) = \mathbb{P}_0(B_{t_{n+1}} - t_n > \sqrt{t_{n+1}f(t_{n+1})}) \sim \frac{e^{-\beta f(t_{n+1})}}{\sqrt{2\pi \beta f(t_{n+1})}} \quad \text{as } n \to \infty, \]

provided that \( \beta f(t_{n+1}) \uparrow \infty \), where \( \beta = \frac{t_{n+1}}{t_{n+1} - t_n} = \frac{\alpha}{\alpha - 1} \). Take \( f(t) = 2\beta^{-2} \log \log t \). Then \( f(t_n) \sim 2\beta^{-2} \log n \), and since \( \beta > 1 \), we have

\[ \sum_{n=1}^{\infty} \mathbb{P}_0(B_{t_{n+1}} - B_t > \sqrt{2\beta^{-2}t_{n+1} \log \log t_{n+1}}) = \infty. \]
Since \((B_{t_{n+1}} - B_{t_n})_{n \in \mathbb{N}}\) are independent, by Borel-Cantelli, a.s. the event \(B_{t_{n+1}} - B_{t_n} > \sqrt{2\beta^{-2} t_{n+1} \log \log t_{n+1}}\) occurs infinitely often. For such \(n\) sufficiently large, by (2.15) applied to \((-B_t)_{t \geq 0}\), we have

\[
B_{t_{n+1}} > B_{t_n} + \sqrt{2\beta^{-2} t_{n+1} \log \log t_{n+1}}
\]

\[
> \sqrt{2\beta^{-2} t_{n+1} \log \log t_{n+1}} - \sqrt{2(1 + \alpha^{-1}) t_n \log \log t_n}.
\]

Since \(\beta \downarrow 1\) as \(\alpha \uparrow \infty\), and \(\alpha > 1\) can be made arbitrarily large, this implies that

\[
\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} \geq \limsup_{n \to \infty} \frac{B_{t_n}}{\sqrt{2t_n \log \log t_n}} \geq 1 \quad \text{a.s.,}
\]

which concludes the proof of (2.4). \(\blacksquare\)

References

