Q1. Show that a square matrix $U$ is integer and unimodular if and only if its inverse $U^{-1}$ is integer and unimodular. \([UU^{-1} = U^{-1}U = I]\)

Q2. Show that
\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]
is totally unimodular.

Q3. Is the following matrix
\[
A = \begin{bmatrix}
1 & 1 & -1 & 1 \\
-1 & 0 & 1 & -1 \\
0 & 1 & 0 & -1
\end{bmatrix}
\]
totally unimodular? Why?

Q4. Suppose that $A \in \mathbb{R}^{m \times n}$ is of full row rank ($m \leq n$), $c, d \in \mathbb{Z}^m$ and $c < d$. Show that any extreme point to the polyhedron
\[
S = \{ x \in \mathbb{R}^n \mid Ax = b, c \leq x \leq d \}
\]
is an integer if $A$ is totally unimodular and $b \in \mathbb{Z}^m$.

Q5. Suppose that there are $n$ people and $m$ jobs, where $n \geq m$. Each job must be assigned to exactly one person, and each person can do at most one job. The cost of person $j$ doing job $i$ is $c_{ij}$. Then the Assignment Problem can be formulated as
\[
\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = 1, i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} x_{ij} \leq 1, j = 1, \ldots, n \quad \text{for } x \in B^{mn}.
\]
Show that the above problem can be reformulated as

\[
\text{minimize } \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to } \sum_{j=1}^{n} x_{ij} = 1, \ i = 1, \ldots, m \\
\sum_{i=1}^{m} x_{ij} \leq 1, \ j = 1, \ldots, n \\
x \geq 0.
\]

Q6. Let \( V = \{1, \ldots, m\} \). Suppose that \( V_1 = \{1\}, V_3 = \{m\}, V'(1) = \emptyset, V(m) = \emptyset \). Then the maximum flow problem is to maximize the total flow into vertex \( m \) under the capacity constraints

\[
\text{maximize } v \\
\text{subject to } \sum_{i \in V(1)} x_{1i} = v \\
\sum_{j \in V(i)} x_{ij} - \sum_{j \in V'(i)} x_{ji} = 0, \ i \in V_2 = \{2, \ldots, m - 1\} \\
\sum_{i \in V'(m)} x_{im} = v \\
0 \leq x_{ij} \leq d_{ij}, \ (i, j) \in E.
\]

(i) Write down the dual of the maximum flow problem.

(ii) Show that every basic feasible solution to the dual problem is an integer provided that all \( d_{ij} \) are integer.