1(a) \[ y'' + 6y' + 9y = 0 \] Set \[ y = e^{\lambda t} \]
\[ \lambda^2 + 6\lambda + 9 = 0 \implies \lambda = -3 \text{ (double root)} \]
\[ \implies y = (A + Bx)e^{-3x} \implies y' = Be^{-3x} - 3(A + Bx)e^{-3x} \]
\[ y(0) = 1 \implies A = 1 \quad y'(0) = -1 \implies B - 3A = -1 \]
\[ \implies B = 2 \implies y = (1 + 2x)e^{-3x}. \]

(b) \[ \lambda^2 - 2\lambda + (1 + 4\pi^2) = 0 \implies \lambda = 1 \pm 2\pi i \]
\[ \implies y = e^x[A \cos 2\pi x + B \sin 2\pi x] \]
\[ \implies y' = y + e^x[-2\pi A \sin 2\pi x + 2\pi B \cos 2\pi x] \]
\[ y(0) = -2 \implies A = -2 \]
\[ y'(0) = 2(3\pi - 1) \implies 2(3\pi - 1) = -2 + 2\pi B \]
\[ \implies B = 3 \implies y_p = e^x[-2 \cos 2\pi x + 3 \sin 2\pi x]. \]

2(a) Try \[ y = Ax^2 + Bx + c. \]
\[ y'' + 2y' + 10y = 2A + 2(2Ax + B) + 10(Ax^2 + Bx + C) \]
\[ = 25x^2 + 3 \]
\[ \implies 10A = 25, 4A + 10B = 0, 2A + 2B + 10C = 3 \]
\[ \implies A = 5/2; \quad B = -1; \quad C = 0 \]
\[ \implies y = \frac{5}{2}x^2 - x. \]

(b) Try \[ y = (Ax^2 + Bx + c)e^{3x}. \]
\[ y' = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + c)e^{3x} \]
\[ y'' = 2Ae^{3x} + 3(2Ax + B)e^{3x} + 3(2Ax + B)e^{3x} + 9(Ax^2 + Bx + c)e^{3x} \]
\[ \implies 9A - 18A + 8A = 1 \implies A = -1; \]
\[ 6A + 6A + 9B - 12A - 18B + 8B = 0 \implies B = 0; \]
\[ 2A + 3B + 3B + 9C - 6B - 18C + 8C = 0 \implies C = -2 \]
\[ \implies y_p = (-x^2 - 2)e^{3x}. \]
(c) $y'' - y = 2x \text{ Im } (e^{ix})$ (Im = imaginary part).

If we can solve the complex equation $z'' - z = 2xe^{ix}$, then $\text{Im } z$ satisfies the equation at top. Try $z = (Ax + B)e^{ix}$:

\[
z' = Ae^{ix} + i(Ax + B)e^{ix} \\
z'' = Aie^{ix} + iAxe^{ix} - (Ax + B)e^{ix}.
\]

\[
z'' - z = (2Ai - Ax - B - Ax - B)e^{ix} = 2xe^{ix}
\]

$\implies A = -1, -2i - 2B = 0 \implies B = -i$

$\implies z = (-x - i)e^{ix} = ix \cos x + x + i[-\cos x - x \sin x]$

$\text{Im } z = -\cos x - x \sin x \implies y_p = -\cos x - x \sin x$.

(d) $y'' + 4y = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2} \text{ Re } (e^{2ix})$.

(The equation $\lambda^2 + 4 = 0$ has roots $\pm 2i$.) Try $z = A + Bxe^{2ix}$.

\[
z'' = -4Bxe^{2ix} + 4iBe^{2ix} \implies z'' + 4z = -4Bxe^{2ix} + 4A + 4Bxe^{2ix} + 4iBe^{2ix}
\]

$\implies 4A = \frac{1}{2} \implies A = \frac{1}{8}, -\frac{1}{2} = 4iB \implies B = \frac{1}{8}i$.

\[
z = \frac{1}{8} + \frac{1}{8}ixe^{2ix} = \frac{1}{8}[1 + x(i \cos 2ix - \sin 2ix)].
\]

$y_p = \text{Re } z = \frac{1}{8} - \frac{1}{8}x \sin 2x$.

3(a) Variation of parameters. First solve $y'' + 4y = 0 \implies \lambda = \pm 2i$

$\implies y = A \cos 2x + B \sin 2x$. “Promote” $A$ and $B$ to functions $A(x)$ and $B(x)$.

Then $A(x) \cos(2x)$ is a solution of $y'' + 4y = \frac{1}{2}(1 - \cos 2x)$ if $A(x)$ and $B(x)$ are chosen to satisfy

\[
A' = -[\frac{1}{2} (1 - \cos 2x)] \sin 2x \\
B' = +[\frac{1}{2} (1 - \cos 2x)] \cos 2x
\]

where $W = (\cos 2x) \times (\sin 2x)' - (\cos 2x)'(\sin 2x) = 2$. It follows that

\[
A' = -\frac{1}{4} \sin 2x + \frac{1}{4} \cos 2x \sin 2x = -\frac{1}{4} \sin 2x + \frac{1}{8} \sin 4x
\]

\[
B' = \frac{1}{4} \cos 2x - \frac{1}{4} \cos^2(2x) = \frac{1}{4} \cos 2x - \frac{1}{8} (\cos 4x + 1)
\]

$\implies A = \frac{1}{8} \cos 2x - \frac{1}{32} \cos 4x$

\[
B = \frac{1}{8} \sin 2x - \frac{1}{32} \sin 4x - \frac{x}{8}.
\]

So

\[
y_p = A \cos 2x + B \sin 2x = \frac{1}{8} \left[(\cos 2x - \frac{1}{4} \cos 4x) \cos 2x + (\sin 2x - \frac{1}{4} \sin 4x - x) \sin 2x\right].
\]
Remark. The above particular solution is “basically” the same as in (2d), since

\[
\frac{1}{8} \left[ (\cos 2x - \frac{1}{4} \cos 4x) \cos 2x + (\sin 2x - \frac{1}{4} \sin 4x - x) \sin 2x \right]
\]

\[
= \frac{1}{8} (\cos^2 2x - \frac{1}{4} \cos 4x \cos 2x + \sin^2 2x - \frac{1}{4} \sin 4x \sin 2x) - \frac{x}{8} \sin 2x
\]

\[
= \frac{1}{8} (1 - \frac{1}{4} \cos 2x) - \frac{x}{8} \sin 2x.
\]

Observe that the extra term \(-\frac{1}{32} \cos 2x\) can be “absorbed” into the (homogeneous) solution \(y_h = C \cos 2x + D \sin 2x\), with \(C\) and \(D\) being arbitrary constants. Thus the general solution \(y = y_h + y_p\) remains the same form.

(b) \(y_h = C \cos x + D \sin(x)\), where \(C\) and \(D\) are constants. Apply the method of variation of parameters: \(A(x) \cos x + B(x) \sin(x)\), where

\[
A' = \frac{[-\sec(x)] \sin x}{w} \quad B' = \frac{[\sec x] \cos x}{w}.
\]

Here \(W = (\cos x)(\sin x)' - (\cos x)'(\sin x) = +1\).

It follows that \(A = -\int \frac{\sin x}{\cos x} = \ell n |\cos x|\), \(B = x\)

\[
\implies y_p = (\cos x) \ell n |\cos x| + x \sin x.
\]

4) \(\frac{dR}{dt} = aJ\), \(\frac{dJ}{dt} = -bR\)

\[
\implies \frac{d^2R}{dt^2} = a \frac{dJ}{dt} = -abR \implies R'' + (ab) R = 0.
\]

Hence \(\lambda = \pm \sqrt{ab}\)

\[
\implies R = A \cos(\sqrt{ab} t) + B \sin(\sqrt{ab} t). \quad \frac{dR}{dt} = -A\sqrt{ab} \sin(\sqrt{ab} t) + B\sqrt{ab} \cos(\sqrt{ab} t).
\]

\(R(0) = L, \frac{dR}{dt}(0) = 0\) since \(R\) is at its maximum at that time. So

\(R(0) = L = A \implies A = L;\)

\(\frac{dR}{dt}(0) = 0 = B\sqrt{ab} \implies B = 0 \implies R(t) = L \cos(\sqrt{ab} t);\)

\(J(t) = \frac{1}{a} \frac{dR}{dt} = -\frac{L}{a} \sqrt{ab} \sin(\sqrt{ab} t) = -L \sqrt{\frac{b}{a}} \sin(\sqrt{ab} t).\)

In the first cycle (and hence in every cycle as \(R\) and \(J\) have period \(2\pi/\sqrt{ab}\)) \(R > 0\) and \(J > 0\) only when \(\frac{3}{2} \pi < \sqrt{ab} t < 2\pi\), i.e. for only one quarter of a cycle. Clearly \(a\) measures the sensitivity of Romeo toward Juliet’s opinion of him; while \(b\) measures Juliet’s sensitivity. It can be seen that \(J_{\text{max}}\) (the maximal level of Juliet’s love for Romeo) = \(J(\frac{3}{2}\pi/\sqrt{ab}) = L \sqrt{\frac{b}{a}}\), which is < \(L\) if Juliet is less sensitive than Romeo (i.e., \(b < a\), but > \(L\) if Juliet is more sensitive than Romeo (i.e., \(b > a\)). This makes sense as \(L\) represents the initial and maximal level of Romeo’s love for Juliet. (Compare with the except on the next page.) Note that the period of a cycle, \(2\pi/\sqrt{ab}\), is short if either or both Romeo and Juliet are extremely sensitive (i.e., \(ab \gg 1\)). Thus the model predicts that highly emotional people will have rapidly fluctuating love lives.
5) From question 3 of Tutorial 9, \( B/s = 376 \) and since \( B = 1.5 \) we have \( S = 1.5/376 \approx 0.004 \). Now the roots of \( BN - sN^2 - 80 = 0 \) are \( \approx 64 \) and \( \approx 311 \). One can sketch the graph of the quadratic function \( f(N) := BN - sN^2 - 80 \). (Note that this would be a graph of \( \frac{dN}{dt} = f(N) \) against \( N \), not of \( N(t) \) against \( t \).) Now the trick in this problem is that we are not told how many bugs we had when spraying started - all we know is that it must be a number between 200 (the initial number of bugs) and 376 (the asymptotic value before spraying). However, THAT DOES NOT MATTER! For if we had LESS than 311 bugs (but more than 64), the graph is ABOVE the \( N \) axis, hence \( N' > 0 \) so the number will increase asymptotically towards 311. If we had MORE than 311 at the time of spraying, the graph is BELOW the \( N \) axis, hence \( N' < 0 \) so the number tends to 311 from above. (If we had 311 at that time, then \( N(t) = \text{constant} = 311 \).) Hence the answer is that we will have 311 bugs in the long run. We say that 311 is a STABLE equilibrium. (64 is also an equilibrium value, but it is UNSTABLE in the sense that a small drop below 64 will cause the number of bugs dropped to zero in finite time.)

---

*Below is a highlight from Romeo and Juliet, taken place when their mutual love was at or near the peak. Romeo had been banished to Mantua and was about to leave Juliet. This would be the last time they saw each other both “alive”. Observe the refined structure and the reverse symmetry - an underlining motif: death to love, and love to death.*

JUL. Wilt thou be gone? It is not yet near day.... Believe me, love, it was the nightingale.

ROM. It was the lark, the herald of the morn, no nightingale: look, love,... night candles are burnt out, and jocund day stands tiptoe on the misty mountain tops: I must be gone and live, or stay and die.

JUL. Yond light is not day-light, I know it, I: it is some meteor that the sun exhales, to be to thee this night a torch-bearer, and light thee on the way to Mantua: therefore stay yet; thou need'st not to be gone.

ROM. Let me be taken, let me be put to death; I am content, so thou wilt have it so.... I have more care to stay than will to go: Come death, and welcome! Juliet wills it so.