1. Using the substitution method, or otherwise, find the following integrals.

Solution.

a) \[\int x^\frac{1}{2} \sin(x^\frac{3}{2} + 1) \, dx = \int \sin(x^\frac{3}{2} + 1) \cdot \frac{2}{3} \, d(x^\frac{3}{2} + 1) = -\frac{2}{3} \cos(x^\frac{3}{2} + 1) + C.\]

b) \[\int \csc^2 2t \cot 2t \, dt = -\frac{1}{2} \int \cot 2t \, d(\cot 2t) = -\frac{1}{4} \cot^2 2t + C.\]

c) \[\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} \, d\theta = \int \sin \frac{1}{\theta} \cos \frac{1}{\theta} \cdot (-1) \, d\left(\frac{1}{\theta}\right) = -\int \sin \frac{1}{\theta} \, d\left(\sin \frac{1}{\theta}\right) = -\frac{1}{2} \sin^2 \frac{1}{\theta} + C.\]

d) \[\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)} \, dx = \int \frac{18 \tan^2 x \, d(\tan x)}{(2 + \tan^3 x)} = \int \frac{6 \, d(\tan^3 x + 2)}{(2 + \tan^3 x)} = 6 \ln |\tan^3 x + 2| + C.\]

e) \[\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} \, d\theta = -2 \int (\cos \sqrt{\theta})^{-3} \, d(\cos \sqrt{\theta}) = (\cos \sqrt{\theta})^{-2} + C = \sec^2 \sqrt{\theta} + C.\]

2. Applying the method of integration by parts, or otherwise, find the following integrals.

Solution.

\[(a) \quad \int x \sin \left(\frac{x}{2}\right) \, dx = -2 \int x \, d \left[ \cos \left(\frac{x}{2}\right) \right] = -2 \left[ x \cos \left(\frac{x}{2}\right) - \int \cos \left(\frac{x}{2}\right) \, dx \right] + C \]
\[= -2 \left[ x \cos \left(\frac{x}{2}\right) - 2 \int \cos \left(\frac{x}{2}\right) \, d \left(\frac{x}{2}\right) \right] + C \]
\[= -2 \left[ x \cos \left(\frac{x}{2}\right) - 2 \sin \left(\frac{x}{2}\right) \right] + C.\]

\[(b) \quad \int t^2 e^{4t} \, dt = \frac{1}{4} \int t^2 \, d(e^{4t}) = \frac{1}{4} \left[ t^2 e^{4t} - 2 \int t e^{4t} \, dt \right] + C = \frac{1}{4} \left[ t^2 e^{4t} - \frac{1}{2} e^{4t} \right] + C \]
\[= \frac{1}{4} \left[ t^2 e^{4t} - \frac{1}{2} \left( t e^{4t} - \int e^{4t} \, dt \right) \right] + C \]
\[= \frac{1}{4} \left[ t^2 e^{4t} - \frac{1}{2} \left( t e^{4t} - \frac{e^{4t}}{4} \right) \right] + C \quad \text{(continue to simplify)}.\]
(c) \[ \int e^{-y} \cos y \, dy = \int e^{-y} \, d(\sin y) = e^{-y} \sin y + \int e^{-y} \sin y \, dy + C \]

\[ = e^{-y} \sin y - \int e^{-y} \, d(\cos y) + C = e^{-y} \sin y - e^{-y} \cos y - \int e^{-y} \cos y \, dy \]

\[ \Rightarrow \int e^{-y} \cos y \, dy = \frac{e^{-y}}{2} (\sin y - \cos y) + C. \]

(There is no harm to rename \( C/2 \) as \( C \).)

(d) \[ \int \theta^2 \sin(2\theta) \, d\theta = -\frac{1}{2} \int \theta^2 d(\cos(2\theta)) = -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - 2 \int \theta \cos(2\theta) \, d\theta \right] + C \]

\[ = -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - \int \theta \, d[\sin(2\theta)] \right] + C \]

\[ = -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - \theta \sin(2\theta) + \int \sin(2\theta) \, d\theta \right] + C \]

\[ = -\frac{1}{2} \left[ \theta^2 \cos(2\theta) - \theta \sin(2\theta) - \frac{1}{2} \cos(2\theta) \right] + C. \]

(e) \[ \int z(\ln z)^2 \, dz = \frac{1}{2} \int (\ln z)^2 \, d(z^2) = \frac{1}{2} \left[ z^2 (\ln z)^2 - 2 \int z(\ln z) \, dz \right] + C \]

\[ = \frac{1}{2} \left[ z^2 (\ln z)^2 - \int (\ln z) \, d(z^2) \right] + C \]

\[ = \frac{1}{2} \left[ z^2 (\ln z)^2 - z^2 (\ln z) + \int z \, dz \right] + C \]

\[ = \frac{1}{2} \left[ z^2 (\ln z)^2 - z^2 (\ln z) + \frac{z^2}{2} \right] + C. \]

3. a) What values of \( a \) and \( b \) with \( a < b \) maximize the value of

\[ \int_a^b (x - x^2) \, dx ? \]

b) What values of \( a \) and \( b \) (\( a < b \)) minimize the value of

\[ \int_a^b (x^4 - 2x^2) \, dx ? \]

**Solution.** a) The integrand \( x - x^2 = -x(x-1) \) is positive for \( x \) in \((0,1)\), and is non-positive otherwise. With integral \( \int_a^b f(x) \, dx \) interpreted as algebraic area of the region under the graph of \( f(x) \) from \( x = a \) to \( x = b \):

\[
\text{integral} = \text{area of subregion above x-axis} - \text{area of subregion below x-axis},
\]

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the integral gets its largest value if it is taken over \((0, 1)\), i.e. with \(a = 0\) and \(b = 1\).
b) Likewise the integrand \( x^4 - 2x^2 = x^2(x^2 - 2) = x^2(x + \sqrt{2})(x - \sqrt{2}) \) is negative or 0 when \( x \) is in \( [-\sqrt{2}, \sqrt{2}] \), but is positive otherwise. By the same line of reasoning as in a), the integral is the least (algebraically) if \( a = -\sqrt{2} \) and \( b = \sqrt{2} \).

4. Evaluate the following integrals: (see the solution).

Solution.

a) \[
\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ~ ds = \int_1^{\sqrt{2}} \left( 1 + s^{-3/2} \right) ~ ds = \left[ \sqrt{2} - 1 \right] - 2s^{-1/2} \bigg|_1^{\sqrt{2}} = (\sqrt{2} - 1) - \frac{2}{\sqrt{2}} + 2 = 1 + \sqrt{2} - 2^{3/4}.
\]

b) \[
\int_{-4}^{4} |x| ~ dx = \int_0^{4} x ~ dx + \int_{-4}^{0} (-x) ~ dx = \frac{1}{2} 4^2 + \frac{1}{2} 4^2 = 16.
\]

c) \[
\int_0^{\pi} \frac{1}{2} (\cos x + |\cos x|) ~ dx = \int_0^{\pi/2} \frac{1}{2} (\cos x + |\cos x|) ~ dx + \int_{\pi/2}^{\pi} \frac{1}{2} (\cos x + |\cos x|) ~ dx
\]
\[
= \int_0^{\pi/2} \cos x ~ dx + 0 = \sin x \bigg|_0^{\pi/2} = 1.
\]
d) \[ \int_0^\pi \sin^2\left(1 + \frac{\theta}{2}\right) \, d\theta = \int_0^\pi \frac{1}{2} \left[1 - \cos(2 + \theta)\right] \, d\theta = \frac{1}{2} \pi - \frac{1}{2} \sin(2 + \theta)|_0^\pi = \frac{1}{2} \pi - \frac{1}{2} [\sin(2 + \pi) - \sin 2] = \frac{1}{2} \pi + \sin 2. \]

5. The Fundamental Theorem of Calculus (I) says that

\[ \frac{d}{du} \int_a^u f(t) \, dt = f(u) \]

for a continuous function \( f \). Here \( a \) is a fixed number. It is a sort of chain rule to find

\[ \frac{d}{dx} \int_a^{g(x)} f(t) \, dt. \]

To see this, let

\[ F(u) = \int_a^u f(t) \, dt \quad \text{and} \quad u = g(x). \]

It follows that

\[ \frac{dF}{du} = \frac{d}{du} \int_a^u f(t) \, dt = f(u). \]

Furthermore,

\[ F \circ g(x) = F(g(x)) = \int_a^{g(x)} f(t) \, dt. \]

By the chain rule, we have

\[ \frac{dF(g(x))}{dx} = \frac{dF}{du} \frac{dg(x)}{dx} = f(u) g'(x) = f(g(x)) g'(x). \]

a) \( y = \int_0^{\sqrt{x}} \cos t \, dt; \quad \text{Solution.} \quad \cos \sqrt{x} \cdot \frac{d}{dx} \sqrt{x} = \frac{\cos \sqrt{x}}{2\sqrt{x}}. \)

b) \( y = \int_0^{x^2} \cos \sqrt{t} \, dt; \quad \text{Solution.} \quad \cos \sqrt{x^2} \cdot 2x = 2x \cos |x| = 2x \cos x. \)

c) \( y = \int_0^{\sin x} \frac{dt}{\sqrt{1 - t^2}}, \quad |x| < \frac{\pi}{2}. \quad \text{Solution.} \quad \frac{1}{\sqrt{1 - \sin^2 x}} \frac{d}{dx} \sin x = \frac{1}{\cos x} \cos x = 1. \)

6. For \( x > 0 \), the error function

\[ E(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt, \]

important in the theories of heat flow, signal transmission and probability, is required to be evaluated numerically because no elementary expression for the antiderivative of \( e^{-t^2} \) has been found. Apply Simpson’s Rule with \( n = 10 \) to estimate \( E(1) \). (\( 2/\sqrt{\pi} \approx 1.128379167 \).)
Solution. \( n = 10, \ h = 0.1 \) so
\[
t_0 = 0, t_1 = 0.1, t_2 = 0.2, \ldots, t_9 = 0.9, t_{10} = 1,
\]
i.e. \( t_i = i/10 \). Let
\[
y_i = y(t_i) = \frac{2}{\sqrt{\pi}} e^{-\frac{t_i^2}{100}}
\]
\[
y_0 = 1.128379167, y_1 = 1.117151606, y_2 = 1.084134787, y_3 = 1.031260909,
y_4 = 0.9615412988, y_5 = 0.8787825789, y_6 = 0.7872434317,
y_7 = 0.6912748604, y_8 = 0.5949857862, y_9 = 0.5019685742,
y_{10} = 0.4151074974.
\]
\[
S = \frac{1}{3} \times 0.1 \times (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + 2y_8 + 4y_9 + y_{10}) \approx 0.8427017130.
\]
Remark The computer shows that the above answer is correct up to 5 decimal places.

7. Find the area of the following region:
   a) The region bounded between \( y = \frac{1}{2} \sec^2 x, \ y = -4\sin^2 x, \ x = -\frac{\pi}{3} \) and \( x = \frac{\pi}{3} \).
   b) The region in the first quadrant bounded by \( y = x, \ y = \frac{1}{4}x^2 \) and \( y = 1 \).
   c) The region bounded by \( y = 4 - x^2, \ y = 2 - x, \ x = -2 \) and \( x = 3 \).

Ans. a) \( \frac{4}{3} \pi \), b) \( \frac{5}{6} \), c) \( \frac{49}{6} \).

Solution. a) Observe that \( \sec^2 x > 0 \) and \( -4\sin^2 x \leq 0 \) on \( [-\pi/3, \pi/3] \).
Area \[= \int_{-\pi/3}^{\pi/3} \left[ \frac{1}{2} \sec^2 x - (-4\sin^2 x) \right] dx \]
\[= \left[ \frac{1}{2} \tan x + \int (2 - 2\cos 2x) \right]_{-\pi/3}^{\pi/3} dx \]
\[= \tan \frac{\pi}{3} + (2x - \sin 2x) \bigg|_{-\pi/3}^{\pi/3} \]
\[= \sqrt{3} + \frac{4}{3} - 2\sin \frac{\pi}{3} = \frac{4}{3}\pi. \]

b) (There is some ambiguity in the question.)
The points of intersection: \[x = x^2/4\] implies \[x = 0\] or \[x = 4\]. Hence the points of intersection are (0, 0) and (4, 4).

We compute both the areas above and below the line \(y = 1\). Note that \(y = x^2/4 \iff x = 2\sqrt{y}\).

Area above the line \[= \int_{1}^{4} [2\sqrt{y} - (y)] dy = \left[ \frac{4}{3} y^{3/2} - \frac{1}{2} y^2 \right]_1^4 = \frac{8}{3} - \frac{5}{6} = \frac{11}{6}. \]

Area below the line \[= \int_{0}^{1} [2\sqrt{y} - (y)] dy = \left[ \frac{4}{3} y^{3/2} - \frac{1}{2} y^2 \right]_0^1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}. \]
c) We have that 
\[(2 - x) - (4 - x^2) = x^2 - x - 2 = (x + 1)(x - 2)\]
is negative if and only if \(x \in (-1, 2)\).

Hence
\[
\text{Area} = \int_{-2}^{3} \left| (2 - x) - (4 - x^2) \right| \, dx \\
= \left[ \int_{-2}^{-1} + \int_{1}^{3} \right] (x^2 - x - 2) \, dx + \int_{-1}^{2} -(x^2 - x - 2) \, dx \\
= \left[ \int_{-2}^{3} -2 \int_{-1}^{2} \right] (x^2 - x - 2) \, dx \\
= \left[ \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \right]_{-2}^{3} - 2 \left[ \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \right]_{-1}^{2} \\
= \frac{1}{3} \left[ (27 + 8) - 2(8 + 1) \right] - \frac{1}{2} \left[ (9 - 4) - 2(4 - 1) \right] - 2 \left[ 5 - 2(3) \right] \\
= \frac{1}{3} \left[ 17 + 1 \right] + \frac{1}{2} + 2 = \frac{49}{6}.
\]
8. a) Find the volume of the solid generated by revolving the region between the parabola \( x = y^2 + 1 \) and the line \( x = 3 \) about the line \( x = 3 \).

b) The region bounded by the parabola \( y = x^2 \) and the line \( y = 2x \) in the first quadrant is revolved about the \( y \)-axis to generate a solid. Find the volume of the solid.

Ans. a) \( \frac{64}{15} \sqrt{2} \pi \), b) \( \frac{8}{3} \pi \).

Solution.

a) The parabola and the line meet at \((x, y)\) with \( 3 = y^2 + 1 \), i.e. at \((3, \pm \sqrt{2})\).

By formula,

\[
\text{Volume} = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [(y^2 + 1) - 3]^2 \, dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} [y^4 - 4y^2 + 4] \, dy
\]

\[
= \pi \left[ \frac{1}{5}y^5 - \frac{4}{3}y^3 + 4y \right]_{-\sqrt{2}}^{\sqrt{2}}
\]

\[
= \pi 2 \left[ \frac{1}{5} 4\sqrt{2} - \frac{4}{3} 2\sqrt{2} + 4\sqrt{2} \right] = \frac{64}{15} \sqrt{2} \pi.
\]
b) The parabola and the line meet at $(x, y)$ with $x^2 = 2x$, i.e. at $(0, 0)$ and $(2, 4)$.

Now $y = 2x \Leftrightarrow x = y/2$ and $y = x^2 \Leftrightarrow x = \sqrt{y}$,

while $\sqrt{y} - (y/2) = \sqrt{y}(1 - \sqrt{y}/2)$ is positive for $y \in (0, 4)$.

So $x = \sqrt{y}$ is the outer curve and $x = y/2$ is the inner curve. Hence, 

\[
\text{volume} = \text{volume of space enclosed by outer shell} - \text{volume of hole enclosed by inner shell} \\
= \int_{0}^{4} \pi \sqrt{y}^2 \, dy - \int_{0}^{4} \pi \left(\frac{y}{2}\right)^2 \, dy = \pi \left[ \frac{1}{2} (4^2 - 0^2) - \frac{1}{4} \frac{1}{3} (4^3 - 0^3) \right] = \frac{8}{3} \pi.
\]