Solutions and comments on Tutorial 4 Q8, 9 (prepared by V. Tan)

To find the radius and interval of convergence for a power series:

- Set $\lim |\frac{a_{n+1}}{a_n}| < 1$. Find the limit.

- If the limit is 0, the inequality becomes $0 < 1$; otherwise, the inequality simplifies to the form $|x - a| < b$. (Make sure the coefficient of $x$ is reduced to 1.)

- For the first case, the radius of convergence is $\infty$ and the interval of convergence is $(-\infty, \infty)$. For the second case, the radius of convergence is $b$ and the interval of convergence is $(a-b, a+b)$ with the possible inclusion of endpoints.

- For the second case, substitute the endpoints $a-b$ and $a+b$ into $x$ of the power series. They become ordinary series. Use various tests to check convergence of these two series. If the corresponding series converges, we include the endpoint in the interval of convergence. If the series diverges, then the endpoint is excluded.

8. $a_n = \frac{3^n x^n}{n}$. Set $\lim |\frac{a_{n+1}}{a_n}| < 1$.

$\lim |\frac{a_{n+1}}{a_n}| = \lim |\frac{n}{n+1} (3x - 2)| = |3x - 2|.$

So the inequality reduces to $|3x - 2| < 1$ or $|x - \frac{2}{3}| < \frac{1}{3}$ (to make the coeff. of $x$ equal 1).

So the radius of convergence is $\frac{1}{3}$ and the interval of convergence is $\left(\frac{2}{3} - \frac{1}{3}, \frac{2}{3} + \frac{1}{3}\right) = \left(\frac{1}{3}, 1\right)$ with the possible inclusion of the endpoints.

Now we test the endpoints:

Substitute $x = \frac{1}{3}$ into the power series we get $\sum \frac{(-1)^n}{n}$ which is convergence (use Alternating Series Test).

So this endpoint is included in the interval of convergence.

Substitute $x = 1$ into the power series we get $\sum \frac{1}{n}$ which is divergence (this is the well known Harmonic series). So this endpoint is excluded from the interval of convergence.

Therefore, the interval of convergence is $\left[\frac{1}{3}, 1\right)$.

9. $a_n = \frac{3^n x^n}{n}$. Set $\lim |\frac{a_{n+1}}{a_n}| < 1$.

$\lim |\frac{a_{n+1}}{a_n}| = \lim |\frac{3^n}{n+1}| = 0.$

So the inequality reduces to $0 < 1$.

So the radius of convergence is $\infty$ and the interval of convergence is $(-\infty, \infty)$. 