1). By a direct calculation or observe that
\[
\frac{x}{1-x} = x \left(\frac{1}{1-x}\right) = x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}.
\]

There are two ways to find the Maclaurin series for \( f(x) := \frac{1}{1-x} \). The first method, which is indirect, is to observe that the sum of the infinite geometric series:
\[
1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{for} \quad |x| < 1.
\]

On the other hand, it is not hand to see that
\[
f^{(n)}(x) = \left(\frac{1}{1-x}\right)^{(n)} \quad \text{(differentiating n times)} = \frac{n!}{(1-x)^{n+1}}.
\]

Here \( n = 1, 2, \cdots \). It follows that \( f^{(n)}(0) = n! \). So
\[
f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \cdots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \cdots = 1 + x + x^2 + \cdots + x^n + \cdots.
\]

2). Let \( f(x) = \frac{1}{x^2} \).

Then \( f'(x) = \frac{-2}{x^3} \),
\[
f''(x) = \frac{3 \cdot 2}{x^4},
\]
and in general \( f^{(n)}(x) = \frac{(-1)^n(n+1)!}{x^{n+2}} \).

\[
\therefore \quad f^{(n)}(1) = (-1)^n(n+1)!
\]

\[
\implies \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^{\infty} (-1)^n(n+1)(x-1)^n.
\]
3). \[
\frac{x}{1 + x} = \frac{(1 + x) - 1}{1 + x}
\]
\[
= 1 - \frac{1}{1 + x}
\]
\[
= 1 - \frac{1}{1 + (x + 2) - 2}
\]
\[
= 1 - \frac{1}{-1 + (x + 2)}
\]
\[
= 1 + \frac{1}{1 - (x + 2)}
\]
\[
= 1 + \frac{1}{1 - y} \quad (y = x + 2)
\]
\[
= 1 + \sum_{n=0}^{\infty} y^n \quad \text{(compare with Q. 1)}
\]
\[
= 1 + \sum_{n=0}^{\infty} (x + 2)^n
\]
\[
= 2 + \sum_{n=1}^{\infty} (x + 2)^n.
\]

4). Let \( f(x) = \ln(\cos x) \).

\[
\therefore \quad f'(x) = -\frac{\sin x}{\cos x} = -\tan x,
\]
\[
f''(x) = -\sec^2 x.
\]

\[
\therefore \quad f(0) = 0,
\]
\[
f'(0) = 0,
\]
\[
f''(0) = -1.
\]

\[
\therefore \quad f(x) \approx -\frac{1}{2}x^2.
\]
5). One may obtain some ideas by considering the function \( f(x) = \sin x \) together with
its graph. In this case \( T = 2 \pi \). Clearly

\[
\int_{-\pi}^{\pi} \sin x \, dx = 0.
\]

Also

\[
\int_{0}^{2\pi} \sin x \, dx = 0.
\]

Observe that

\[
g(t+2\pi) = \int_{0}^{t+2\pi} \sin x \, dx = \int_{0}^{2\pi} \sin x \, dx + \int_{2\pi}^{2\pi+t} \sin x \, dx = 0 + \int_{0}^{t} \sin x \, dx = g(t),
\]

where in the last step we use the periodicity of the sine function (observe also the
change of variables and integrations by substitution).

With this example as a guide, for any fixed \( t \), one begins with

\[
g(t + T) = \int_{0}^{t+T} f(u) \, du = \int_{0}^{T} f(u) \, du + \int_{T}^{t+T} f(u) \, du.
\]

We seek to demonstrate the followings. (Some students may find it useful to make
concrete what happens below by looking at \( f(u) = \sin u \).)

a) \( \int_{0}^{T} f(u) \, du = 0 \).

Demonstration: The key is to shift the region of integration from \([0, \pi]\) to \([-\pi/2, \pi/2]\),
and make use of the condition

\[
\int_{-\pi/2}^{\pi/2} f(u) \, du = 0.
\]

Introduce the change of variables \( u = T + v \).

\[
\therefore \int_{-\pi/2}^{\pi/2} f(u) \, dv = \int_{-\pi}^{0} f(T + v) \, dv = \int_{-\pi/2}^{0} f(v) \, dv = \int_{-\pi/2}^{\pi/2} f(u) \, du
\]

\[
\implies \int_{0}^{T} f(u) \, du = \int_{0}^{\pi/2} f(u) \, du + \int_{\pi/2}^{\pi} f(u) \, du
\]

\[
= \int_{0}^{\pi/2} f(u) \, du + \int_{0}^{\pi/2} f(u) \, du
\]

\[
= \int_{-\pi/2}^{\pi/2} f(u) \, du = 0.
\]
b) \[ \int_T^{t+T} f(u) \, du = \int_0^t f(u) \, du = g(t) \] (shifting backward).

Demonstration : Let \( U = T + v \).

\[ \therefore \int_T^{t+T} f(u) \, du = \int_0^t f(T+v) \, dv = \int_0^t f(v) \, dv \]

\[ = \int_0^t f(u) \, du . \]

\[ \therefore g(t+T) = g(t) \text{ for all } t. \]

6). We write \( f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \).

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{0} dx + \frac{1}{2\pi} \int_{0}^{\pi} 2 \, dx = \frac{3}{2} \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx , \]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} 2 \cos nx \, dx = 0, \quad n \geq 1 . \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} 2 \sin nx \, dx \]

\[ = \frac{1}{\pi} \left[ \left. \frac{-\cos nx}{n} \right|_{-\pi}^{0} + \frac{1}{\pi} \left[ \left. -\frac{2 \cos nx}{n} \right|_{0}^{\pi} \right] \right] \]

\[ = \frac{1}{\pi} \left( \frac{-1 + \cos n\pi}{n} \right) + \frac{1}{\pi} \left( \frac{-2 \cos n\pi + 2}{n} \right) \]

\[ = \frac{1}{\pi} \left( \frac{1 - \cos n\pi}{n} \right) = \left\{ \begin{array}{ll}
0 & \text{if } n = 2m ; \\
\frac{2}{(2m-1)\pi} & \text{if } n = 2m - 1 , \quad m \geq 1 .
\end{array} \right. \]

The Fourier series of \( f \) is

\[ \therefore f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin(2n-1)x . \]
7. \( a_0 = \frac{1}{2\pi} \int_{0}^{\pi} x^2 \, dx = \frac{\pi^2}{6}. \)

\[
a_n = \frac{1}{\pi} \int_{0}^{\pi} x^2 \cos nx \, dx = \frac{1}{\pi} \int_{0}^{\pi} x^2 \frac{d(\sin nx)}{n} \quad (d(\sin nx) = n \cos nx \, dx)
\]

\[
= \frac{1}{\pi} \left[ x^2 \frac{\sin nx}{n} \right]_0^\pi - \frac{1}{n\pi} \int_{0}^{\pi} \sin nx (2x) \, dx
\]

\[
= \frac{1}{\pi} \left[ x^2 \frac{\sin nx}{n} \right]_0^\pi + \frac{1}{n^2\pi} \int_{0}^{\pi} (2x) \, d(\cos nx)
\]

\[
= \frac{1}{\pi} \left[ x^2 \frac{\sin nx}{n} + \frac{2x}{n^2} \cos nx \right]_0^\pi - \frac{2}{n^2\pi} \int_{0}^{\pi} \cos nx \, dx
\]

\[
= \frac{1}{\pi} \left[ x^2 \sin nx + \frac{2x}{n^2} \cos nx - \frac{2 \sin nx}{n^3} \right]_0^\pi
\]

\[
= \frac{2}{n^2} \cos n\pi = \frac{2}{n^2} (-1)^n, \quad n \geq 1.
\]

\[
b_n = \frac{1}{\pi} \int_{0}^{\pi} x^2 \sin nx \, dx
\]

\[\cdots\text{(applying integration by parts two times as above)}\]

\[
= \frac{1}{\pi} \left[ \frac{-x^2}{n} \cos nx + \frac{2x}{n^2} \sin nx + \frac{2}{n^3} \cos nx \right]_0^\pi
\]

\[
= -\frac{\pi}{n} \cos n\pi + \frac{2}{\pi n^3} (\cos n\pi - 1)
\]

\[
= \left\{ \begin{array}{ll}
-\frac{\pi}{n} & \text{if } n = 2, 4, 6, \cdots ; \\
\frac{\pi}{n} - \frac{4}{\pi n^3} & \text{if } n = 1, 3, 5, \cdots ,
\end{array} \right.
\]

\[
= (-1)^{n+1} \frac{\pi}{n} - \left\{ \frac{2}{\pi n^3} - (-1)^n \frac{2}{\pi n^3} \right\}, \quad n \geq 1.
\]

\[
\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left\{ (-1)^n \frac{2}{n^2} \cos nx + \left[ (-1)^{n+1} \frac{\pi}{n} - \left( \frac{2}{\pi n^3} - (-1)^n \frac{2}{\pi n^3} \right) \right] \sin nx \right\}.
\]

At \( x = \pi \), the function experiences a jump discontinuity. It is known that the Fourier series converges to the “mid-point”. Using the periodicity of \( f \), we have, in particular,

\[
f(\pi +) = \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} f(x - 2\pi) = \lim_{y \to (-\pi)^+} f(y) = 0,
\]

where \( y = x - 2\pi \) so that \( x \to \pi^+ \Rightarrow y \to (-\pi)^+ \). It follows that

\[
\frac{1}{2} \{ f(\pi^-) + f(\pi^+) \} = \frac{1}{2} (\pi^2 + 0) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}.
\]

\[
\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \left( \frac{\pi^2}{2} - \frac{\pi^2}{6} \right) = \frac{\pi^2}{6}.
\]