1(a) \( y' + (1 + \frac{1}{x})y = \frac{1}{x}e^{-x} \).

Integrating factor is \( \exp \int (1 + \frac{1}{x}) \, dx = \exp (x + \ln x) = xe^x \)

So \( \frac{d}{dx}(yxe^x) = x e^x \frac{1}{x} e^{-x} = 1 \implies yxe^x = x + c \implies y = e^{-x} + cx^{-1}e^{-x} \).

Here, and below, \( c \) is an integration constant.

(b) \( \exp \int -(1 + \frac{3}{x}) \, dx = \exp (-x - 3\ln |x|) = \frac{1}{x^3}e^{-x} \)

\( \implies \frac{d}{dx}(y \frac{1}{x^3} e^{-x}) = (x + 2) \frac{1}{x^3} e^{-x} \).

\( \frac{y}{x^3} e^{-x} = \int e^{-x} \frac{x^2}{x^3} \, dx + 2 \int e^{-x} \frac{1}{x^3} \, dx + c \)

\( = \int e^{-x} \frac{x^2}{x^3} \, dx + -\frac{e^{-x}}{x^2} - \int e^{-x} \frac{1}{x^2} \, dx + c \) (integration by parts)

\( = -\frac{e^{-x}}{x^2} + c \quad (x \neq 0) \).

It follows that \( y = -x + cx^3e^x \).

Since \( y(1) = e - 1 = -1 + ce \implies c = 1 \).

Ans.: \( y = -x + x^3e^x \).

(c) This is a Bernoulli type equation. Set

\( z = y^2 \quad z' = 2yy' \quad y' = z'/2y \).

\( \frac{z'}{2y} + y + \frac{x}{y} = 0 \implies \frac{1}{2}z' + z + x = 0 \)

\( \implies z' + 2z = -2x \implies \frac{d}{dx}(z e^{2x}) = -2xe^{2x} \).

\( \implies ze^{2x} = (-x + \frac{1}{2})e^{2x} + c \implies y^2 = \frac{1}{2} - x + ce^{-2x} \).

(d) Observe that \( 2yy' = (y^2)' \). Let \( Y = y^2 \). It follows that

\( xY' + (x - 1)Y = x^2 e^x \).

For \( x > 0 \), we have

\( Y' + (1 - \frac{1}{x})Y = xe^x \), \( \exp \int (1 - \frac{1}{x}) \, dx = \frac{1}{x}e^x \)

\( \implies \frac{d}{dx}(\frac{1}{x}e^x Y) = e^{2x} \implies \frac{1}{x}e^x Y = \frac{1}{2}e^{2x} + c \)

\( \implies y^2 = \frac{1}{2} x e^x + cx e^{-x} \).
2. Let \( N(t) \) be the Earth’s population, then

\[
\frac{dN}{dt} = \text{Birth rate} - \text{Death rate} - \text{Emigration rate} = BN - DN - Kt
\]

\[
\frac{dN}{dt} - (B - D)N = -Kt, \quad \exp \int (B - D) \, dt = e^{-(B-D)t}
\]

\[
\frac{d}{dt}(Ne^{-(B-D)t}) = -Kte^{-(B-D)t}
\]

\[
\implies Ne^{-(B-D)t} = -K \int te^{-(B-D)t} dt
\]

\[
= K \left[ \frac{t}{B - D} + \frac{1}{(B - D)^2} \right] e^{-(B-D)t} + c \quad \text{(integration by parts)}
\]

\[
\implies N = Ce^{(B-D)t} + K \left[ \frac{t}{B - D} + \frac{1}{(B - D)^2} \right].
\]

Let \( N_o \) be the population at \( t = 0 \) (year 2028).

\[
N_o = C + \frac{K}{(B-D)^2}
\]

\[
\implies N = \left( N_o - \frac{K}{(B-D)^2} \right) e^{(B-D)t} + K \left[ \frac{t}{B - D} + \frac{1}{(B - D)^2} \right].
\]

So the result depends on the relative values of \( N_o \) and \( \frac{K}{(B-D)^2} \). Consider the following three cases.

(I) \( N_o > \frac{K}{(B-D)^2} \implies \) population explosion not solved!

(II) \( N_o = \frac{K}{(B-D)^2} \) so that \( N = K \left[ \frac{t}{B - D} + \frac{1}{(B - D)^2} \right] \) Thus population increases linearly.

(III) \( N_o < \frac{K}{(B-D)^2} \). We know that \( \lim_{x \to \infty} e^x = \infty \). i.e. exponential defeats linear function eventually so ultimately the population will collapse. (Overdoing it! Too many emigrants!)

3. Substitute \( N = B/S \) and obtain \( 0 = \frac{dN}{dt} = B\frac{B}{S} - S\frac{B^2}{S^2} = 0 \). This is the equilibrium situation, with equal numbers of deaths and births per year. Clearly B/S is large when S is small and vice versa. So S measures how badly the animal is affected by overcrowding; tigers have large S, while rabbits have small S.

Using separation of variables we have

\[
\frac{dN}{BN - SN^2} = dt.
\]  \hspace{1cm} (3.1)

Now \( \frac{1}{BN - SN^2} = \frac{1}{N(B - SN)} = \frac{\alpha}{N} + \frac{\beta}{B - SN} \)
\[ 1 = \alpha (B - SN) + \beta (N) \]
\[ = \alpha B + (\beta - \alpha S)N \implies \alpha = \frac{1}{B}, \beta = \frac{S}{B}. \]

Integrating both sides of (3.1) we obtain
\[ t + c = \int dt = \int \frac{dN}{BN - SN^2} = \frac{1}{B} \ln N - \frac{1}{B} \ln |B - SN|. \quad (3.2) \]

In our case the population is growing, that is, \( \frac{dN}{dt} = N(B - SN) > 0 \).
Hence \( N \) must be less than its asymptotic value \( B/S \), i.e.
\[ N < B/S \implies B - SN > 0 \implies |B - SN| = B - SN. \]

It follows from (3.2) that
\[ \frac{1}{B} \ln \frac{N}{B - SN} = t + \text{constant}, \]

or \( \frac{N}{B - SN} = C e^{Bt} \implies \frac{N_o}{B - SN_o} = C \), where \( N_o = N(0) \). Solving for \( N \) we get
\[ N = \frac{B/S}{1 + \left( \frac{B/S}{N_o} - 1 \right) e^{-Bt}}. \]

We have \( N_o = 200 \), \( B = 1.5 \), and we know that \( N = 360 \) when \( t = 2 \). Hence
\[ 360 = \frac{B/S}{1 + \left( \frac{B/S}{200} - 1 \right) e^{-1.5 \times 2}} \]
\[ \implies \frac{B}{s} = \frac{360(1 - e^{-3})}{1 - \frac{360}{200} e^{-3}} \approx 376 \]
\[ \implies N = \frac{376}{1 + \left( \frac{376}{200} - 1 \right) e^{-1.5t}} \]
\[ \implies N(3) = \frac{376}{1 + \left( \frac{376}{200} - 1 \right) e^{-1.5 \times 3}} \approx 372. \]

That is, you will have 372 bugs after 3 days, and 376 in the long run (\( t \to \infty \)).
\[ \frac{dN}{dt} = BN - DN = BN - (SN + u \frac{dN}{dt})N = BN - SN^2 - uN \frac{dN}{dt} \]

\[ \Rightarrow (1 + uN) \frac{dN}{dt} = BN - SN^2 \]

\[ \Rightarrow \left( \frac{1 + uN}{BN - SN^2} \right) dN = dt \]

\[ \Rightarrow \left( \frac{1}{N(B - SN)} + \frac{u}{B - SN} \right) dN = dt \]

\[ \Rightarrow \frac{1}{BN} + \frac{s}{B} \left( \frac{1}{B - SN} \right) + \frac{u}{B - SN} = dt \]

\[ \ln N - \ln |B - SN| - \frac{Bu}{s} \ln |B - SN| = Bt + c \]

\[ \Rightarrow \frac{N}{|B - SN|^{(1 + \frac{Bu}{s})}} = Ce^{Bt}, \quad \text{where} \quad C = e^c. \]

Notice the singularity at \( N = B/S \) when the denominator becomes zero. We know that if the initial number of seals is \( > B/S \), their number will decline towards the critical population \( B/S \), whereas it increases if \( N < B/S \) initially. This is what would happen if there were no whales! The effect of the whales is to slow down the approach to equilibrium. For instances, try, if feasible, using a computer to graph \( \frac{y}{|1-y|} = e^x \) and \( \frac{z}{|1-z|^{10}} = 32e^x \). To give an idea how the feature in concerned are illustrated, consider \( x = 10, \)

\[ \frac{y}{|1-y|} = e^{10} \quad \Rightarrow \quad y \approx 0.9999546021; \quad \frac{z}{|1-z|^{10}} = 32e^{10} \quad \Rightarrow \quad z \approx 0.8959211467. \]

(Here we take only the solutions that are smaller than one.) Note that the second value is smaller, which means that for the value of \( x \), \( y \) is closer to the “critical value” \( 1 \) than \( z \). From this, we understand that for \( t \gg 1 \), \( N(t) \) with \( u = 0 \) is closer to the critical population \( B/S \) than \( N(t) \) with \( u > 0 \).