Question 1 (i). Let $a_n = \ln \frac{n+2}{n+3}$. Then the partial sum

$$S_n = a_1 + a_2 + \cdots + a_n = \ln \frac{1+2}{1+3} + \ln \frac{2+2}{2+3} + \ln \frac{3+2}{3+3} + \cdots + \ln \frac{n+2}{n+3}$$

$$= \ln \frac{(1+2)(2+2)(3+2)(4+2)\cdots(n-1+2)(n+2)}{(1+3)(2+3)(3+3)(4+3)\cdots(n-1+3)(n+3)}$$

$$= \frac{3 \cdot 4 \cdot 5 \cdot 6 \cdots (n+1) \cdot (n+2)}{4 \cdot 5 \cdot 6 \cdot 7 \cdots (n+2) \cdot (n+3)} = \ln \frac{3}{n+3} = \ln 3 - \ln(n+3).$$

Thus $\{S_n\}$ is divergent and so is the series $\sum_{n=1}^{\infty} \ln \frac{n+2}{n+3}$.

□

Question 1 (ii). Let $a_n = \frac{1}{n(n+2)}$. Observe that

$$\frac{1}{n(n+2)} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right).$$

The partial sum

$$S_n = \frac{1}{1 \cdot (1+2)} + \frac{1}{2 \cdot (2+2)} + \cdots + \frac{1}{n \cdot (n+2)}$$

$$= \frac{1}{2} \left( \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2} \right) \right)$$

$$= \frac{1}{2} \left( \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2} \left( 3 - \frac{2n+3}{(n+1)(n+2)} \right).$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \to \infty} S_n = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

□

Question 2 (a). Let $a_n = \frac{n^2 - 1}{2n^2 + n}$. Then $\lim_{n \to \infty} a_n = \frac{1}{2} \neq 0$ and so the series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n}$ is divergent by the divergence test.

□
Question 2 (b). Let \( a_n = \sin \frac{n\pi}{2} \). Then \( \lim_{n \to \infty} a_n \) does not exist and so the series \( \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \) is divergent by the divergence test.

Question 2 (c). Let \( a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) and let \( b_n = \frac{1}{n} \). Then
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \to \infty} \frac{\frac{n^2 + 1 + \frac{4}{n^3}}{1 + \frac{\ln n}{n^2}}}{1 + 0 + 0} = \frac{0 + 1 + 0}{1 + 0 + 0} = 1.
\]
Since \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent, so is \( \sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) by the limit comparison test.

Question 2 (d). Observe that
\[
3 + \sin \frac{n}{n^2} \leq \frac{4}{n^2}.
\]
Since \( \sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent by the \( p \)-series, the positive series \( \sum_{n=1}^{\infty} \frac{3 + \sin n}{n^2} \) is convergent by the comparison test.

Question 2 (e). Observe that
\[
\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1}} = \frac{2^{n+1}}{3^{n+1}} = \left( \frac{2}{3} \right)^{n+1}
\]
for \( n \geq 2 \). Since \( \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^{n+1} \) is convergent by the geometric series, the positive series \( \sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n} \) is convergent by the comparison test.

Question 2 (f). Let \( a_n = \frac{2}{n^{1+\frac{1}{n}}} \) and let \( b_n = \frac{1}{n} \). Observe that
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{n}}}{2} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}} \) is divergent by the limit comparison test.

Question 2 (g). Observe that
\[
\frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}.
\]
Since \( \sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n} \) is divergent by the comparison test. \( \square \)

Question 2 (h). Observe that

\[
\frac{1}{n(1 + \ln n)^p} = \frac{(1 + \ln n)^{-p}}{n} \geq \frac{1}{n}
\]

for \( p \leq 0 \). Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^p} \) is divergent for \( p \leq 0 \) by the comparison test. \( \square \)

Question 2 (i). Observe that

\[
\frac{n}{n^2 + 1} \geq \frac{n}{n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) is divergent by the comparison test. \( \square \)