Question 1. Let \( a_n = (-1)^n \frac{\cos n}{2^n} \). Then \( |a_n| \leq \frac{1}{2^n} \). Since \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) is convergent by the geometric series, the series \( \sum_{n=1}^{\infty} \left| (-1)^n \frac{\cos n}{2^n} \right| \) is convergent by the comparison test and so the series \( \sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{2^n} \) is absolutely convergent. \( \square \)

Question 2 (i). Let \( a_n = \ln \frac{n}{\sqrt{n}} \). Then \( a_n \geq 0 \). We show that \( a_n \) is eventually monotone decreasing. Let \( f(x) = \ln \frac{x}{\sqrt{x}} \). Then

\[
f'(x) = \frac{1}{x} \sqrt{x} - \ln x \frac{1}{2x^{3/2}} = \frac{2 - \ln x}{2x^{3/2}} \leq 0
\]

for \( x \geq e^2 \) and so \( \{a_n\} \) is monotone decreasing for \( n \geq 9 \). Since \( \lim_{n \to \infty} a_n = 0 \), the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}} \) is convergent by the alternating series test. \( \square \)

Question 2 (ii). Since \( \left| (-1)^{n+1} \frac{\ln n}{\sqrt{n}} \right| \geq \frac{1}{n^2} \) for \( n \geq 3 \) and the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is divergent by the \( p \)-series, the series \( \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{\ln n}{\sqrt{n}} \right| \) is divergent by the comparison test.

By (i), the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}} \) is conditionally convergent. \( \square \)

Question 3 (a). This series is conditionally convergent because it is convergent by the alternating series test and the series \( \sum_{n=1}^{\infty} \left| (-1)^n \frac{3}{2n+1} \right| \) is divergent by the limit comparison test with the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \). \( \square \)

Question 3 (b). Let \( a_n = (-1)^n \frac{n}{4n+3} \). Then \( \lim_{n \to \infty} a_{2n-1} = -\frac{1}{4} \) and \( \lim_{n \to \infty} a_{2n} = \frac{1}{4} \). Thus the limit of \( (-1)^n \frac{n}{4n+3} \) does not exist and so the series \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{4n+3} \) is divergent by the divergence test. \( \square \)
Question 3 (c). Let \( a_n = (-1)^n \left( \frac{1 + 2n}{3 + 4n} \right)^n \). Then
\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{1 + 2n}{3 + 4n} = \frac{2}{4} = \frac{1}{2} < 1.
\]
Thus the positive series \( \sum_{n=1}^{\infty} |a_n| \) is convergent by the simplified root test and so the series \( \sum_{n=1}^{\infty} (-1)^n \left( \frac{1 + 2n}{3 + 4n} \right)^n \) is absolutely convergent.

\( \square \)

Question 3 (d). This series is conditionally convergent because it is convergent by the alternating series test but the series \( \sum_{n=2}^{\infty} \left| \frac{1}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is divergent by the integral test.

\( \square \)

Question 4. We use integral test estimation for solving this question. Let \( f(x) = \frac{1}{x^5} \).
From \( \int_{1}^{\infty} f(x) \, dx = \left. \frac{1}{-4} x^{-4} \right|_{1}^{\infty} = \frac{1}{4n^4} < 0.001 \), we have \( n > \sqrt[4]{\frac{1000}{4}} = 3.976 \) or \( n \geq 4 \). Thus
\[
\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.036
\]
with the error less than 0.001.

\( \square \)

Question 5. We use alternating series test estimation for solving this question. Let \( a_n = \frac{1}{n^5} \). From \( a_{n+1} = \frac{1}{(n+1)^5} < 0.001 \), we have \( n + 1 > \sqrt[5]{1000} \) or \( n \geq 3 \) and so
\[
\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^5} \approx 1 - \frac{1}{2^5} + \frac{1}{3^5} \approx 0.9729
\]
with error less than 0.001.

\( \square \)

Question 6 (a). We use the ratio test for general series for solving this question. Let \( a_n = \frac{x^n}{n^2} \). From \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x|^{n+1} n^2}{(n+1)^2 |x|} = |x| < 1 \),
we have \( -1 < x < 1 \). We check the ending points \( x = \pm 1 \). When \( x = -1 \), the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) is convergent by the alternating series test. When \( x = 1 \), then the series...
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] is convergent by the \( p \)-series. Thus the domain of the function \( f(x) \) is the closed internal \([-1, 1]\).

**Question 6 (b).** We use the ratio test for general series for solving this question. Let \( a_n = \frac{(-1)^n(x - 1)^n}{2n + 1} \). From

\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x - 1|^{n+1}(2n + 1)}{(2n + 3)|x - 1|^n} = |x - 1| < 1,
\]

we have \( 0 < x < 2 \). We check the ending points \( x = 0 \) or \( 2 \). When \( x = 0 \), the series \( \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{2n + 1} = \sum_{n=1}^{\infty} \frac{1}{2n + 1} \) is divergent by the limit comparison test with the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \). When \( x = 2 \), then the series \( \sum_{n=1}^{\infty} \frac{(-1)^n(2 - 1)^n}{2n + 1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n + 1} \) is convergent by the alternating series test. Thus the domain of the function \( g(x) \) is the internal \((0, 2]\).